A HOMOLOGICAL INTERPRETATION OF THE TRANSVERSE QUIVER GRASSMANNIANS

GIOVANNI CERULLI IRELLI, GRÉGOIRE DUPONT, AND FRANCESCO ESPOSITO

Abstract. In recent articles, the investigation of "canonically positive bases" of cluster algebras associated to affine quivers led the second–named author to introduce a variety called *transverse quiver Grassmannian* and the first–named and third–named authors to consider the smooth part of quiver Grassmannians. In this paper, we prove that, for any affine quiver Q, the transverse quiver Grassmannian of an indecomposable representation M is the set of points N in the quiver Grassmannian of M such that $\mathrm{Ext}^1(N, M/N) = 0$. As a corollary we obtain that when the transverse quiver Grassmannian is non–empty, it coincides with the smooth locus of the quiver Grassmannian.

1. Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in order to design a combinatorial model for understanding positivity in algebraic groups and canonical bases in quantum groups [15, 16, 2, 17].

To any quiver Q with vertex set Q_0, without loops and 2–cycles, is associated a (coefficient–free) cluster algebra $\mathcal{A}_Q$ which is a certain commutative $\mathbb{Z}$–subalgebra of the field $\mathcal{F} = \mathbb{Q}(x_1, \cdots, x_n)$ of rational functions in $n = |Q_0|$ variables. The ring $\mathcal{A}_Q$ is equipped with a distinguished set of generators called *cluster variables* which are grouped into possibly overlapping free generating sets of the field $\mathcal{F}$ called *clusters*. In particular every element of $\mathcal{A}_Q$ can be expressed as a rational function in every cluster of $\mathcal{A}_Q$. By the *Laurent phenomenon* every element of $\mathcal{A}_Q$ can actually be expressed as a Laurent polynomial in every cluster of $\mathcal{A}_Q$ [15].

In [3, 4, 12], the authors provide closed formulas for the computation of the Laurent expansion of every cluster variable in every cluster. Let us recall this result in the particular case when the cluster is associated to an acyclic seed (we address the reader to the survey [18]). Let Q be a quiver without oriented cycles and let M be a finite-dimensional complex representation of Q with dimension vector $\mathbf{d} = (d_i)_{i\in Q_0}$. Following the notations of [7] we associate to M the following Laurent polynomial in the variables $\{x_i : i \in Q_0\}$

$$CC(M) = \frac{\sum_{\mathbf{e}\in\mathbb{N}^{Q_0}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) \prod_{i,j\in Q_0} \left(x_i^{d_j - e_j} x_j^{e_i}\right)^{b_{ij}}}{\prod_{i\in Q_0} x_i^{d_i}} \tag{1}$$

where b_{ij} denotes the number of arrows in Q from the vertex i to the vertex j, $\mathrm{Gr}_{\mathbf{e}}(M)$ denotes the complex projective variety of $\mathbf{e}$-dimensional sub–representations of M (see section 2 for details) and $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$ denotes its Euler-Poincaré characteristic. The following result, due to B. Keller and P. Caldero, provides the promised description of the cluster variables of $\mathcal{A}_Q$. Recall that a representation M of Q is called *rigid* if $\mathrm{Ext}^1(M, M) = 0$.

Theorem 1.1. [4] *Let $\mathcal{A}_Q$ be the (coefficient-free) cluster algebra with initial seed $(Q, \{x_i : i \in Q_0\})$. Then the correspondence $M \mapsto CC(M)$ is a bijection between the set of isomorphism classes of indecomposable rigid representations of the quiver Q, and the set of all cluster variables in $\mathcal{A}_Q$ not belonging to $\{x_i : i \in Q_0\}$.*

In the theory of cluster algebras, it is natural to consider the cone of positive elements, i.e. elements whose expansion in every cluster has positive coefficients. The study of this cone leads to look for a $\mathbb{Z}$–basis $\mathbf{B}$ of $\mathcal{A}_Q$ whose positive linear combinations coincide with this cone. The existence of such a basis was proved only in a few cases :

Theorem 1.2. [21][8] *Let Q be a quiver of type $\tilde{A}_{1,1}$ or of type $\tilde{A}_{2,1}$ and let $\mathcal{A}_Q$ be the associated cluster algebra. Then there exists a $\mathbb{Z}$–basis $\mathbf{B}$ of $\mathcal{A}_Q$ such that the cone of positive elements of $\mathcal{A}_Q$ coincides with the cone of positive linear combinations of elements of $\mathbf{B}$.*

In the same spirit as theorem 1.1, the authors of the present article provided independently in [14] and [9] two geometric realizations of the basis of theorem 1.2. Let us briefly recall these two constructions.

In [9] the first and third–named authors study the geometry of quiver Grassmannians associated to indecomposable representations of the Kronecker quiver. As a consequence of their results it is natural to consider the map which associates to a representation M of Q the Laurent polynomial $CC^{(0)}(M)$ defined by slightly modifying (1) as follows:

$$CC^{(0)}(M) = \frac{\sum_{\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)^{(0)}) \prod_{i,j\in Q_0} \left(x_i^{d_j-e_j} x_j^{e_i}\right)^{b_{ij}}}{\prod_{i\in Q_0} x_i^{d_i}}. \tag{2}$$

where $\mathrm{Gr}_{\mathbf{e}}(M)^{(0)}$ is the subvariety of $\mathrm{Gr}_{\mathbf{e}}(M)$ constituted of sub-representations N of M such that $\mathrm{Ext}^1(N, M/N) = 0$.

In [14] the second–named author defines a subvariety $\mathrm{Tr}_{\mathbf{e}}(M)$ of $\mathrm{Gr}_{\mathbf{e}}(M)$ called transverse quiver Grassmannian for any indecomposable representation M of an arbitrary affine quiver Q (see section 3). To any such representation, let $\theta(M)$ be the following slight modification of the right–hand side of (1):

$$\theta(M) = \frac{\sum_{\mathbf{e}} \chi(\mathrm{Tr}_{\mathbf{e}}(M)) \prod_{i,j\in Q_0} \left(x_i^{d_j-e_j} x_j^{e_i}\right)^{b_{ij}}}{\prod_{i\in Q_0} x_i^{d_i}}. \tag{3}$$

We define θ on direct sums by $\theta(M \oplus N) = \theta(M)\theta(N)$.

In [9, theorems 3.1 and 3.2] and [14, theorem 5.1] we provide a geometric realization of the basis $\mathbf{B}$ of theorem 1.2 as follows:

Theorem 1.3. [9][14] *Let Q be a quiver either of type $\tilde{A}_{1,1}$ or of type $\tilde{A}_{2,1}$. Then for every element b in the basis $\mathbf{B}$ of theorem 1.2 there exists a representation M_b of Q and a monomial $\boldsymbol{m}_b$ in the initial cluster variables such that*

$$b = \boldsymbol{m}_b . CC^{(0)}(M_b) = \boldsymbol{m}_b . \theta(M_b).$$

Theorem 1.3 shows that the maps $M \mapsto CC^{(0)}(M)$ and $M \mapsto \theta(M)$ coincide when evaluated on particular representations of particular affine quivers. It hence suggests a relation between the two subvarieties $\mathrm{Gr}_{\mathbf{e}}(M)^{(0)}$ and $\mathrm{Tr}_{\mathbf{e}}(M)$ of $\mathrm{Gr}_{\mathbf{e}}(M)$. The following theorem, which is our main result, states that they actually coincide :

Theorem 1.4. *Let Q be an affine quiver and let M be an indecomposable representation of Q. Then*

$$\mathrm{Tr}(M) = \{N \in \mathrm{Gr}(M) \,:\, \mathrm{Ext}^1(N, M/N) = 0\}. \tag{4}$$

The following corollary of theorem 1.4 provides a geometric interpretation of the transverse quiver Grassmannian where $\langle \cdot, \cdot \rangle$ denotes the Euler form of Q.

Corollary 1.5. *Let Q be an affine quiver and let M be an indecomposable representation of Q with dimension vector $\mathbf{d}$. For any dimension vector $\mathbf{e}$, the variety $\mathrm{Tr}_{\mathbf{e}}(M)$ is non–empty if and only if $\mathrm{Gr}_{\mathbf{e}}(M)$ is reduced and has dimension $\langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle$. In this case $\mathrm{Tr}_{\mathbf{e}}(M)$ coincides with the smooth part of $\mathrm{Gr}_{\mathbf{e}}(M)$.*

Theorem 1.4 is of double interest. On one hand, it provides a natural interpretation of the transverse quiver Grassmannian whose definition given in [14] was *ad hoc*. On the other hand, together with corollary 1.5, the definition of $\mathrm{Tr}(M)$ makes it possible to compute explicitly smooth parts of quiver Grassmannians associated to indecomposable representations.

The paper is organized as follows: in section 2 we recall some facts about representations of affine quivers and about the geometry of quiver Grassmannians. In section 3 we recall the definition of transverse quiver Grassmannians. In section 4 we prove theorem 1.4. In section 5 we prove corollary 1.5.

2. Quiver Grassmannians

In this section we recall some well-known facts about quiver Grassmannians. Such varieties are considered in several places in the literature; our references are [20, 11, 10, 6, 12, 3, 5, 4].

2.1. Representations of quivers.

A *quiver* $Q = (Q_0, Q_1)$ is an oriented graph where Q_0 denotes the set of vertices and Q_1 the set of arrows. We always assume that Q_0 and Q_1 are finite sets and that the underlying unoriented graph of Q is connected. A quiver is called *acyclic* if it does not contain any oriented cycle.

Let $\mathbb{C}$ denote the field of complex numbers. A (complex) *representation* M of Q consists of a collection of (finite dimensional complex) vector spaces $M(i)$, one for each vertex i in Q_0, together with a collection of $\mathbb{C}$-linear maps $M(a): M(i) \longrightarrow M(j)$, one for each arrow $a: i \longrightarrow j$ in Q_1. A *sub–representation* N of M consists of vector subspaces $N(i)$ of $M(i)$ such that $M(a)(N(i)) \subset N(j)$ for every arrow $a: i \to j$ in Q_1.

Given two representations M and M' of Q, a *morphism* of Q–representations $g: M \to M'$ is a Q_0–tuple $(g(i) : i \in Q_0)$ of $\mathbb{C}$-linear maps $g(i): M(i) \to M'(i)$ such that $M(a) \circ g(i) = g(j) \circ M'(a)$ for every arrow $a: i \to j$ in Q_1. We denote by $\mathrm{rep}(Q)$ the category of Q–representations. As usual, the category $\mathrm{rep}(Q)$ is identified with the category $\mathbb{C}Q$-mod of finitely generated left-modules over the path algebra $\mathbb{C}Q$ of Q. Thus, a Q–representation will sometimes be called a *module*. Note that throughout the article, unadorned Hom and Ext are taken in the category $\mathrm{rep}(Q)$.

By definition, the *dimension vector of a Q-representation M* is $\mathbf{dim}\,(M) = (\dim M(i) : i \in Q_0) \in \mathbb{N}^{Q_0}$. The *Euler form* on $\mathrm{rep}(Q)$ is the bilinear form given by

$$\langle M, N\rangle = \dim \mathrm{Hom}(M, N) - \dim \mathrm{Ext}^1(M, N)$$

which only depends on $\mathbf{dim}\, M$ and $\mathbf{dim}\, N$ so that it induces a $\mathbb{Z}$-bilinear form on $\mathbb{Z}^{Q_0}$ which is given by

$$\langle \mathbf{d}, \mathbf{d}'\rangle = \sum_{i \in Q_0} d_i d'_i - \sum_{a: i \to j \in Q_1} d_i d'_j$$

for any $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^{Q_0}$.

We denote by τ the Auslander-Reiten translation on $\mathbb{C}Q$-mod and by τ^{-1} its inverse (see e.g. [1]). An indecomposable representation M is called *preprojective* if it is in the τ^{-1}-orbit of an indecomposable projective module. An indecomposable representation M is called *preinjective* if it is in the τ-orbit of an indecomposable injective module. An indecomposable Q–representation M is called *regular* if it is neither preprojective nor preinjective. Direct sums of indecomposable preprojective (resp. regular, preinjective) representations are called preprojective (resp. regular, preinjective) representations. Any representation M can be uniquely decomposed into $M = M_P \oplus M_R \oplus M_I$ where M_P is preprojective, M_R is regular and M_I is preinjective.

Any two representations M and N satisfy the *Auslander-Reiten formula* :

$$\mathrm{Ext}^1(M,N) \simeq D\mathrm{Hom}(N,\tau M)$$

where $D = \mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})$ denotes the standard duality.

The indecomposable preprojective (resp. preinjective) representations form a connected component in the Auslander-Reiten quiver of Γ_Q of $\mathrm{rep}(Q)$. This component is called *preprojective* (resp. *preinjective*). All the other connected components are called *regular*.

From now on, we assume that Q is an affine quiver, that is, an acyclic orientation of an *euclidean* diagram of type $\tilde{A}, \tilde{D}$ or $\tilde{E}$. Note that these quivers are also called *extended Dynkin quiver* in the literature. The representation-theory of such quivers is well-known, we refer the reader to [19, 22] for details.

Regular components in Γ_Q form a $\mathbb{P}^1\mathbb{C}$-family and each component is a (stable) tube, that is of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^p)$ for some integer $p \geq 1$ called the *rank* of the tube. A tube is called *homogeneous* if $p = 1$ and *exceptional* otherwise. Slightly abusing notations, the full subcategory of $\mathrm{rep}(Q)$ formed by objects in a given tube $\mathcal{T}$ is still denoted by $\mathcal{T}$. Each tube $\mathcal{T}$ in Γ_Q is abelian, uniserial and standard, that is, isomorphic to the mesh category of the quiver $\mathcal{T}$. A module at the mouth of a tube is called *quasi-simple.* Let R be a quasi-simple module in a tube $\mathcal{T}$, then there exists a unique sequence of irreducible monomorphisms

$$R = R^{(1)} \longrightarrow R^{(2)} \longrightarrow \cdots \longrightarrow R^{(n)} \longrightarrow \cdots$$

called *ray* starting at R and a unique sequence of irreducible epimorphisms

$$\cdots \longrightarrow \tau^{n-1}R^{(n)} \longrightarrow \cdots \longrightarrow \tau R^{(2)} \longrightarrow R^{(1)} = R$$

called *coray* ending at R.

An indecomposable module M has a self-extension if and only if it is regular in a tube $\mathcal{T}$ of rank $p \geq 1$ and if $M \simeq R^{(n)}$ for some quasi-simple R in $\mathcal{T}$ and some integer $n \geq p$.

If P is any preprojective module, I is any preinjective module and R, R' are any regular modules taken in two distinct tubes, then

$$\mathrm{Hom}(R,P) = 0,\ \mathrm{Hom}(I,R) = 0, \mathrm{Hom}(I,P) = 0 \text{ and } \mathrm{Hom}(R,R') = 0,$$

or equivalently, using the Auslander-Reiten formula,

$$\mathrm{Ext}^1(P,R) = 0,\ \mathrm{Ext}^1(R,I) = 0, \mathrm{Ext}^1(P,I) = 0 \text{ and } \mathrm{Ext}^1(R',R) = 0. \tag{5}$$

2.2. **Quiver Grassmannians.** Let Q be any acyclic quiver and M be any representation of Q. To any $\mathbf{e} = (e_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$, the *quiver Grassmannian of* $\mathbf{e}$*–dimensional subrepresentations* is the set

$$\mathrm{Gr}_{\mathbf{e}}(M) = \{N \text{ sub–representation of } M \,:\, \mathbf{dim}\, N = \mathbf{e}\}.$$

This is a closed subset of the product $\prod_{i \in Q_0} \mathrm{Gr}_{e_i}(M(i))$ of usual Grassmannians of vector subspaces and it is hence a complex projective variety. The *quiver Grassmannian* is the (finite) union

$$\mathrm{Gr}(M) = \bigsqcup_{\mathbf{e} \in \mathbb{N}^{Q_0}} \mathrm{Gr}_{\mathbf{e}}(M).$$

In [20] (see also [6]) it is shown that the tangent space $T_N(\mathrm{Gr}_{\mathbf{e}}(M))$ at a point N of $\mathrm{Gr}_{\mathbf{e}}(M)$ equals:

$$T_N(\mathrm{Gr}_{\mathbf{e}}(M)) = \mathrm{Hom}(N, M/N). \tag{6}$$

Moreover the following inequalities hold :

(7) $\langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle \leq \dim \mathrm{Gr}_{\mathbf{e}}(M) \leq \dim T_N(\mathrm{Gr}_{\mathbf{e}}(M)) \leq \langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle + \dim \mathrm{Ext}^1(M,M)$

where $\mathbf{d} = \mathbf{dim}\, M$ so that $\mathbf{dim}\, M/N = \mathbf{d}-\mathbf{e}$. We recall that a point N in $\mathrm{Gr}_{\mathbf{e}}(M)$ is called *smooth* if $\dim T_N(\mathrm{Gr}_{\mathbf{e}}(M)) = \dim \mathrm{Gr}_{\mathbf{e}}(M)$ and $\mathrm{Gr}_{\mathbf{e}}(M)$ is called *smooth* if every point in $\mathrm{Gr}_{\mathbf{e}}(M)$ is smooth. It follows in particular that when M is rigid, $\mathrm{Gr}_{\mathbf{e}}(M)$ is smooth for any $\mathbf{e} \in \mathbb{N}^{Q_0}$.

Note that the inequalities (7) can be strict as shown in the following example.

Example 2.1. Consider the acyclic quiver Q of type $\tilde{A}_{2,1}$ and the indecomposable regular representation of Q :

$$M := \begin{tikzcd} & k^3 \arrow[dr, "J_3(0)"] & \\ k^3 \arrow[ur, "="] \arrow[rr, "="'] & & k^3 \end{tikzcd}$$

of dimension $(3,3,3)$ (here $J_3(0)$ denotes the 3×3 indecomposable nilpotent Jordan block). In the notation of section 2.1, M lies in a tube of rank two and has the form $M = R_0^{(6)}$ where R_0 is the quasi-simple (which is actually simple in this case) of dimension $(0,1,0)$. Moreover $\tau M = R_1^{(6)}$ where R_1 is the unique indecomposable representation of dimension $(1,0,1)$. It follows that $\dim \mathrm{Ext}^1(M,M) = \dim \mathrm{Hom}(R_0^{(6)}, R_1^{(6)}) = 3$.

Let $\mathbf{e} = (0,2,1)$. Then $\langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle = 0$. It is easy to verify that $\mathrm{Gr}_{\mathbf{e}}(M)$ is isomorphic to $\mathbb{P}^1$ and it is hence smooth of dimension 1 (see also remark 3.1). It thus follows that the inequalities in (7) are all strict except the middle one which is an equality.

3. Transverse quiver Grassmannians

Assume now that Q is an affine quiver. We recall the definition of the transverse quiver Grassmannian provided in [14]. Let M be an indecomposable $\mathbb{C}Q$-module. If M is rigid, we set $\mathrm{Tr}(M) = \mathrm{Gr}(M)$.

If M is non-rigid, it is regular and it is thus contained in a tube $\mathcal{T}$ of rank $p \geq 1$. We denote by $R_i, i \in \mathbb{Z}/p\mathbb{Z}$ the quasi-simple modules in $\mathcal{T}$ ordered such that $\tau R_i \simeq R_{i-1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. For any $n \geq 0$ and any $i \in \mathbb{Z}/p\mathbb{Z}$, we denote by $R_i^{(n)}$ the unique indecomposable $\mathbb{C}Q$-module in $\mathcal{T}$ with quasi-socle R_i and quasi-length n with the convention that $R_i^{(0)} = 0$. Up to relabeling the R_i's, there exist $l \geq 1$ and $0 \leq k \leq p-1$ such that $M \simeq R_0^{(lp+k)}$.

For any $m \geq n$, $\mathrm{Hom}(R_i^{(n)}, R_i^{(m)}) = \mathbb{C}\iota$ for some monomorphism ι so that we can identify $\mathrm{Gr}(R_i^{(n)})$ with a closed subset of $\mathrm{Gr}(R_i^{(m)})$. By definition, the *transverse quiver Grassmannian* of M is thus :

$$\mathrm{Tr}(R_0^{(lp+k)}) = \mathrm{Gr}(R_0^{(lp+k)}) \setminus \mathrm{Gr}^{R_0^{(k+1)}}(R_0^{(lp-1)})$$

where $\mathrm{Gr}^{R_0^{(k+1)}}(R_0^{(lp-1)})$ is the set of subrepresentations in $\mathrm{Gr}(R_0^{(lp-1)})$ containing $R_0^{(k+1)}$ as a subrepresentation. In other words,

$$\mathrm{Tr}(M) = \mathrm{Gr}(M) \setminus \left\{ N \in \mathrm{Gr}(M) \, : \, R_0^{(k+1)} \subset N \subset R_0^{(lp-1)} \right\}.$$

Note that

$$\mathrm{Gr}^{R_0^{(k+1)}}(R_0^{(lp-1)}) \simeq \mathrm{Gr}_{\mathbf{dim} R_0^{(lp-1)} - \mathbf{dim} R_0^{(k+1)}}(R_0^{(lp-1)}/R_0^{(k+1)}).$$

For any indecomposable $\mathbb{C}Q$-module M and any dimension vector $\mathbf{e}$, we set

$$\mathrm{Tr}_{\mathbf{e}}(M) = \mathrm{Tr}(M) \cap \mathrm{Gr}_{\mathbf{e}}(M).$$

As mentioned in the introduction, the motivation for the definition of the transverse quiver Grassmannian comes from the study of [13, Conjecture 7.10] concerning canonically positive bases in cluster algebras associated to affine quivers and more specifically from *higher difference properties* introduced and studied in [14].

Remark 3.1. Let Q be an affine quiver and M be an indecomposable representation of Q. According to the corollary 1.5, for any dimension vector $\mathbf{e} \in \mathbb{N}^{Q_0}$, when $\mathrm{Tr}_{\mathbf{e}}(M)$ is not empty it coincides with the set of smooth points in $\mathrm{Gr}_{\mathbf{e}}(M)$. Nevertheless, $\mathrm{Tr}_{\mathbf{e}}(M)$ does not coincide with the set of smooth points in $\mathrm{Gr}_{\mathbf{e}}(M)$ in general. Indeed, in example 2.1 it is shown a quiver Grassmannian $Gr_{\mathbf{e}}(M)$ which is smooth of dimension 1. Then one can check directly that for every $N \in \mathrm{Gr}_{\mathbf{e}}(M)$, $\dim \mathrm{Ext}^1(N, M/N) = 1$ (this follows from the fact that $N \simeq R_0^{(1)} \oplus P_2$ where P_2 is the projective of dimension $(0, 1, 1)$, from (5) and from the combinatorics of the tube of rank two containing M). But in view of theorem 1.4 this computation implies that $\mathrm{Tr}_{\mathbf{e}}(M)$ is empty. So $Gr_{\mathbf{e}}(M)^{(0)} = Gr_{\mathbf{e}}(M)$ but $Tr_{\mathbf{e}}(M)$ is empty.

4. Proof of Theorem 1.4

In this section we prove the main theorem of the article. We start by proving the following lemma :

Lemma 4.1. *For every subrepresentations N_1 and N_2 of M we have*

$$\dim \mathrm{Ext}^1(N_1, M/N_2) \leq \dim \mathrm{Ext}^1(M, M).$$

Proof. Applying $\mathrm{Ext}^1(N_1, -)$ to the exact sequence $M \longrightarrow M/N_2 \longrightarrow 0$, we get an epimorphism $\mathrm{Ext}^1(N_1, M) \twoheadrightarrow \mathrm{Ext}^1(N_1, M/N_2)$ since $\mathbb{C}Q$ is hereditary. Similarly, $\mathrm{Ext}^1(-, M)$ applied to the exact sequence $0 \longrightarrow N_1 \longrightarrow M$, we get an epimorphism $\mathrm{Ext}^1(M, M) \twoheadrightarrow \mathrm{Ext}^1(N_1, M)$. Composing the two epimorphisms one gets an epimorphism $\mathrm{Ext}^1(M, M) \twoheadrightarrow \mathrm{Ext}^1(N_1, M/N_2)$. This finishes the proof. $\square$

Fix now an affine quiver Q and an indecomposable representation M of Q. Then we need to prove that

$$\mathrm{Tr}(M) = \{N \in \mathrm{Gr}(M) \, : \, \mathrm{Ext}^1(N, M/N) = 0\}. \tag{8}$$

If M is rigid, i.e. $\mathrm{Ext}^1(M, M) = 0$, then $\mathrm{Tr}(M) = \mathrm{Gr}(M)$ by definition and it follows from Lemma 4.1 that $\mathrm{Ext}^1(N, M/N) = 0$ for any $N \in \mathrm{Gr}(M)$ and thus (8) holds.

We thus assume that M is not rigid. With notations of section 3, we write $M \simeq R_0^{(lp+k)}$ for some $l \geq 1$ and $0 \leq k \leq p - 1$ where $p \geq 1$ is the rank of the tube containing M.

In order to prove (8) we prove the following two facts:

$$\text{if } N \in \mathrm{Gr}_{\mathbf{e}}^{R_0^{(k+1)}}(R_0^{(lp-1)}) \text{ then } \mathrm{Ext}^1(N, M/N) \neq 0 \; ; \tag{9}$$

$$\text{if } N \in \mathrm{Tr}_{\mathbf{e}}(M) \text{ then } \mathrm{Ext}^1(N, M/N) = 0. \tag{10}$$

Since M is regular, any submodule N of M is of the form $N = N_R \oplus N_P$ where N_R is regular indecomposable (or zero) and N_P is preprojective. Moreover, the quotient M/N is of the form $M/N = (M/N)_R \oplus (M/N)_I$ where $(M/N)_R$ is regular indecomposable (or zero) and $(M/N)_I$ is preinjective. It follows that

$$\mathrm{Ext}^1(N, M/N) = \mathrm{Ext}^1(N_P \oplus N_R, (M/N)_R \oplus (M/N)_I) \simeq \mathrm{Ext}^1(N_R, (M/N)_R).$$

Let us prove (9). Let $N \in \mathrm{Gr}^{R_0^{(k+1)}}(R_0^{(lp-1)})$. Since $\mathrm{Hom}(R_0^{(k+1)}, N_P) = 0$, it follows that $R_0^{(k+1)}$ is a submodule of N_R and thus

$$N_R = R_0^{(t)} \text{ for some } k + 1 \leq t \leq lp - 1.$$

$$\begin{tikzcd}
R_0 \arrow[dr, hook] & & & & & & & & & & & & R_{k-1} \\
& \cdots \arrow[dr, hook] & & & & & & & & & & \cdots \arrow[ur, two heads] & \\
& & R_0^{(k+1)} \arrow[dr, hook] & & & & & & & & R_{p-1}^{(k+1)} = \tau R_0^{(k+1)} \arrow[ur, two heads] \arrow[dr, hook] & & \\
& & & \cdots \arrow[dr, hook] & & & & & & \cdots \arrow[ur, two heads] & & \cdots \arrow[dr, hook] & \\
& & & & R_0^{(lp-1)} \arrow[dr, hook] & & & & R_{k+1}^{(lp-1)} \arrow[ur, two heads] & & & & \tau R_0^{(lp-1)} \\
& & & & & \cdots \arrow[dr, hook] & & \cdots \arrow[ur, two heads] & & & & & \\
& & & & & & R_0^{(lp+k)} \arrow[ur, two heads] & & & & & &
\end{tikzcd}$$

FIGURE 1. The ray and the coray passing through $M = R_0^{(lp+k)}$

Thus, in the tube containing M, the module N_R is contained between $R_0^{(k+1)}$ and $R_0^{(lp-1)}$ in the ray of the tube passing through M.

Moreover there are surjective morphisms:

$$R_{k+1}^{(lp-1)} \twoheadrightarrow (M/N)_R \twoheadrightarrow R_{p-1}^{(k+1)}$$

i.e. $(M/N)_R$ is contained between $R_{k+1}^{(lp-1)}$ and $R_{p-1}^{(k+1)}$ in the coray passing through M. Indeed there are surjective morphisms:

$$M/R_0^{(k+1)} \twoheadrightarrow (M/N)_R \twoheadrightarrow M/R_0^{(lp-1)}$$

and clearly

$$M/R_0^{(k+1)} \simeq R_{k+1}^{(lp+k-(k+1))} = R_{k+1}^{(lp-1)} \tag{11}$$

and

$$M/R_0^{(lp-1)} \simeq R_{lp-1}^{(lp+k-(lp-1))} = R_{p-1}^{(k+1)} \tag{12}$$

We claim that $\mathrm{Ext}^1(R, R') \neq 0$ if R is an indecomposable regular module contained between $R_0^{(k+1)}$ and $R_0^{(lp-1)}$ in the ray of the tube passing through M and R' is an indecomposable regular module contained between $R_{k+1}^{(lp-1)}$ and $R_{p-1}^{(k+1)}$ in the coray passing through M. According to the Auslander-Reiten formula, it is equivalent to prove that $\mathrm{Hom}(R', \tau R) \neq 0$. Since $\mathrm{Hom}(R_{k+1}^{(lp-1)}, R_{p-1}^{(k+1)}) = \mathbb{C}\pi$ where π is an epimorphism, this follows easily from the combinatorics of the tubes (see figure 1). This proves the claim and thus (9) holds.

We now prove (10). By definition $N \in \mathrm{Tr}(M)$ if either $R_0^{(k+1)}$ is not a subrepresentation of N or N is not a subrepresentation of $R_0^{(lp-1)}$. We claim that

$N \in \mathrm{Tr}(M)$ if either

$$N_R \simeq R_0^{(t)} \text{ for some } 0 \leq t \leq k \tag{13}$$

or there exists a surjective morphism

$$R_0^{(lp+k)}/R_0^{(lp)} \simeq R_{lp}^{(k)} \simeq R_0^{(k)} \twoheadrightarrow (M/N)_R\ . \tag{14}$$

(13) is clearly equivalent to the fact that $R_0^{(k+1)}$ is not a subrepresentation of N. We now prove that (14) is equivalent to the fact that $N \subset R_0^{(lp+k)}$ is not a subrepresentation of $R_0^{(lp-1)}$. Let a be the minimal integer such that $N \subset R_0^{(lp+a)}$ so that $0 \leq a \leq k$. Consider the composition $f \in \mathrm{Hom}(M, (M/N)_R)$ given by

$$M \to M/N \to (M/N)_R$$

which is surjective. Since M and $(M/N)_R$ are regular modules and that each tube is closed under kernels, $\ker f$ is a regular module. Since it is a submodule of M and $N \subset \ker f$, we get $\ker f \simeq R_0^{(lp+b)}$ with $a \leq b \leq k$. Thus, there exists an epimorphisms

$$R_0^{(lp+k)}/R_0^{(lp)} \to R_0^{(lp+k)}/R_0^{(lp+b)} = R_0^{(lp+k)}/\ker(f) \simeq (M/N)_R.$$

We now prove that if N satisfies either (13) or (14) then $\mathrm{Ext}^1(N, M/N) = 0$. If (13) holds, $N_R \simeq R_0^{(t)}$ for some $0 \leq t \leq k$ and then it is easy to see that $\mathrm{Ext}^1(N_R, R') = 0$ for any regular quotient R' of $R_0^{(lp+k)}$. In particular, we get $\mathrm{Ext}^1(N_R, (M/N)_R) = 0$. Similarly, if (14) holds, then $(M/N)_R \simeq R_j^{k-j}$ for some $0 \leq j \leq k$ and thus, it follows that $\mathrm{Ext}^1(R_0^{(s)}, (M/N)_R) = 0$ for any $s \geq 0$. In particular, $\mathrm{Ext}^1(N_R, (M/N)_R) = 0$. This finishes the proof of theorem 1.4. $\square$

5. Proof of Corollary 1.5

We conclude the article by proving corollary 1.5 which points out a geometric interpretation of the transverse quiver Grassmannians. We thus fix an affine quiver Q and an indecomposable representation of Q with dimension vector $\mathbf{d}$. Let $\mathbf{e}$ be a dimension vector. Then we need to prove that $\mathrm{Tr}_{\mathbf{e}}(M)$ is non–empty if and only if $\mathrm{Gr}_{\mathbf{e}}(M)$ is reduced and has dimension $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle$ and that in this case $\mathrm{Tr}_{\mathbf{e}}(M)$ coincides with the smooth part of $\mathrm{Gr}_{\mathbf{e}}(M)$.

Suppose that $\mathrm{Tr}_{\mathbf{e}}(M)$ is non empty and let N be an element of it. Then, in view of theorem 1.4, $\mathrm{Ext}^1(N, M/N) = 0$ and hence $\dim T_N(\mathrm{Gr}_{\mathbf{e}}(M)) = \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle$, in view of (6). Then, by (7), $\dim \mathrm{Gr}_{\mathbf{e}}(M) = \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle$ and hence $\mathrm{Gr}_{\mathbf{e}}(M)$ is reduced because $\dim T_N(\mathrm{Gr}_{\mathbf{e}}(M)) = \dim \mathrm{Gr}_{\mathbf{e}}(M)$ and N is smooth.

On the other hand if $\mathrm{Gr}_{\mathbf{e}}(M)$ is smooth then there exists a smooth point $N \in \mathrm{Gr}_{\mathbf{e}}(M)$, i.e. $\dim T_N(\mathrm{Gr}_{\mathbf{e}}(M)) = \dim \mathrm{Gr}_{\mathbf{e}}(M)$. By hypothesis $\dim T_N(\mathrm{Gr}_{\mathbf{e}}(M)) = \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle = \dim \mathrm{Gr}_{\mathbf{e}}(M)$ and hence, by (7), $\mathrm{Ext}^1(N, M/N) = 0$ and $N \in \mathrm{Tr}_{\mathbf{e}}(M)$ in view of theorem 1.4.

In this case the smooth part of $\mathrm{Gr}_{\mathbf{e}}(M)$ consists of subrepresentations N of M such that $\dim T_N(\mathrm{Gr}_{\mathbf{e}})(M)) = \dim \mathrm{Gr}_{\mathbf{e}}(M))$ which is equivalent to the equality $\dim T_N(\mathrm{Gr}_{\mathbf{e}})(M)) = \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle$ which is equivalent to $\mathrm{Ext}^1(N, M/N) = 0$ which is equivalent to $N \in \mathrm{Tr}_{\mathbf{e}}(M)$. This finishes the proof of corollary 1.5. $\square$

Acknowledgments

This paper was written while the second–named author was at the university of Sherbrooke as a CRM-ISM postdoctoral fellow under the supervision of Ibrahim Assem, Thomas Brüstle and Virginie Charette. The first–named author thanks C. De Concini for the financial support for this project, as part of a post–doctoral scholarship at the Department of Mathematics of “Sapienza Università di Roma”.

References

[1] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[2] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

[3] Philippe Caldero and Frédéric Chapoton. Cluster algebras as Hall algebras of quiver representations. *Comment. Math. Helv.*, 81(3):595–616, 2006.

[4] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras. II. *Ann. Sci. École Norm. Sup. (4)*, 39(6):983–1009, 2006.

[5] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras. *Invent. Math.*, 172(1):169–211, 2008.

[6] Philippe Caldero and Markus Reineke. On the quiver Grassmannian in the acyclic case. *J. Pure Appl. Algebra*, 212(11):2369–2380, 2008.

[7] Philippe Caldero and Andrei Zelevinsky. Laurent expansions in cluster algebras via quiver representations. *Mosc. Math. J.*, 6(3):411–429, 2006.

[8] Giovanni Cerulli Irelli. Canonically positive basis of cluster algebras of type $A_2^{(1)}$, 2009. arXiv.org:0904.2543.

[9] Giovanni Cerulli Irelli and Francesco Esposito. Geometry of quiver Grassmannians of Kronecker type and canonical basis of cluster algebras. *arXiv:1003.3037v2 [math.RT]*, 2010.

[10] William Crawley-Boevey. Subrepresentations of general representations of quivers. *Bull. London Math. Soc.*, 28(4):363–366, 1996.

[11] Harm Derksen, Aidan Schofield, and Jerzy Weyman. On the number of subrepresentations of a general quiver representation. *J. Lond. Math. Soc. (2)*, 76(1):135–147, 2007.

[12] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. Quivers with potentials and their representations II: Applications to cluster algebras, 2009.

[13] Grégoire Dupont. Quantized Chebyshev polynomials and cluster characters with coefficients. *Journal of Algebraic Combinatorics*, 31(4):501–532, june 2010.

[14] Grégoire Dupont. Transverse quiver grassmannians and bases in affine cluster algebras. *Algebra and Number Theory*, (to appear) 2010. arXiv:0910.5494v2 [math.RT].

[15] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[16] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.

[17] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. IV. Coefficients. *Compos. Math.*, 143(1):112–164, 2007.

[18] Bernhard Keller. Cluster algebras, quiver representations and triangulated categories, 2008. arXiv.org:0807.1960.

[19] C.M. Ringel. Tame algebras and integral quadratic forms. *Lecture Notes in Mathematics*, 1099:1–376, 1984. MR0774589 (87f:16027).

[20] Aidan Schofield. General representations of quivers. *Proc. London Math. Soc.*, 65(3):46–64, 1992. MR1162487 (93d:16014).

[21] Paul Sherman and Andrei Zelevinsky. Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. *Mosc. Math. J.*, 4(4):947–974, 982, 2004.

[22] Daniel Simson and Andrzej Skowroński. *Elements of the Representation Theory of Associative Algebras, Volume 2: Tubes and Concealed Algebras of Euclidean type*, volume 71 of *London Mathematical Society Student Texts*. Cambridge University Press, 2007.

Sapienza Università di Roma, Dipartimento di Matematica. Piazzale Aldo Moro 2, 00185, Roma (ITALY)

E-mail address: `cerulli@mat.uniroma1.it`

Université de Sherbrooke, Département de Mathématiques. 2500, Boul. de l'Université, J1K 2R1, Sherbrooke QC (CANADA).

E-mail address: `gregoire.dupont@usherbrooke.ca`

Università degli studi di Padova, Dipartimento di Matematica Pura ed Applicata. Via Trieste 63, 35121, Padova (ITALY)

E-mail address: `esposito@math.unipd.it`