GENERIC CLUSTER CHARACTERS

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Abstract. Let $\mathcal{C}$ be a Hom-finite triangulated 2-Calabi-Yau category with constructible cones and let T be a cluster-tilting object in $\mathcal{C}$. We introduce a set $\mathcal{G}^T(\mathcal{C})$ of generic values of the cluster character associated to T parameterized by the Grothendieck group $K_0(\operatorname{add} T)$. We prove that the set $\mathcal{G}^T(\mathcal{C})$ naturally contains the cluster monomials of the cluster algebra associated to the Gabriel quiver of the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$.

When $\mathcal{C} = \mathcal{C}_Q$ is the cluster category of an acyclic quiver Q and $T = \mathbf{k}Q$ is the canonical cluster-tilting object, we prove that $\mathcal{G}^T(\mathcal{C})$ satisfies multiplicative properties compatible with the virtual generic decomposition introduced by Igusa-Orr-Todorov-Weyman. As a consequence, we prove that this set coincides with the set of generic variables previously introduced by the author in the context of acyclic cluster algebras.

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1. Introduction and main results

1.1. **Cluster algebras.** Cluster algebras were introduced in [FZ02] in order to design a combinatorial framework for studying total positivity in algebraic groups and canonical bases in quantum groups. It is expected, and proved in some cases, that a cluster algebra has distinguished linear bases providing combinatorial models for canonical or semicanonical bases in quantum groups [FZ02, GLS10, Nak09]. Besides this question, cluster algebras have shown interactions with various areas of mathematics like Lie theory, combinatorics, Teichmüller theory, Poisson geometry or quiver representations. The connection with representation theory was particularly fruitful for the construction of linear bases in a large class of cluster algebras [CZ06, CK08, Cer09, Dup08, DXX09, GLS10].

In full generality, a (coefficient-free) *cluster algebra* can be associated to any pair $(Q, \mathbf{x})$ where $Q = (Q_0, Q_1)$ is a quiver and $\mathbf{x} = (x_i | i \in Q_0)$ is a Q_0-tuple of indeterminates over $\mathbb{Z}$. By a quiver $Q = (Q_0, Q_1)$, we always mean an oriented graph such that Q_0 is a finite set of vertices and Q_1 is a finite set of arrows. Moreover, we always assume that a quiver Q does not contain any loops or 2-cycles. Given such a pair $(Q, \mathbf{x})$, we denote by $\mathcal{A}(Q, \mathbf{x})$ the corresponding cluster algebra.

$\mathcal{A}(Q, \mathbf{x})$ is a subalgebra of the algebra $\mathbb{Z}[\mathbf{x}^{\pm 1}]$ of Laurent polynomials in the x_i with $i \in Q_0$. It is equipped with a distinguished set of generators called *cluster variables*, gathered into possibly overlapping sets of fixed cardinality called *clusters*, generated by a recursive process called *mutation*. We denote by $\mathrm{Cl}(Q, \mathbf{x})$ the set of cluster variables in $\mathcal{A}(Q, \mathbf{x})$. Monomials in variables belonging to a same cluster are called *cluster monomials* and we denote by $\mathcal{M}(Q, \mathbf{x})$ the set of cluster monomials in $\mathcal{A}(Q, \mathbf{x})$.

A $\mathbb{Z}$-*basis* in the cluster algebra $\mathcal{A}(Q, \mathbf{x})$ is a free generating set of $\mathcal{A}(Q, \mathbf{x})$ viewed as a $\mathbb{Z}$-module. If $\mathcal{A}(Q, \mathbf{x})$ has finitely many cluster variables, the set $\mathcal{M}(Q, \mathbf{x})$ of cluster monomials is a $\mathbb{Z}$-basis in $\mathcal{A}(Q, \mathbf{x})$ [CK08]. It is conjectured, and proved in most cases, that cluster monomials are always linearly independent over $\mathbb{Z}$. Nevertheless, it was observed in [SZ04] that they are not sufficient to generate $\mathcal{A}(Q, \mathbf{x})$ as a $\mathbb{Z}$-module if $\mathcal{A}(Q, \mathbf{x})$ has infinitely many cluster variables. The aim of this article is to provide, for a wide class of cluster algebras, a general construction of a distinguished family of elements in $\mathbb{Z}[\mathbf{x}^{\pm 1}]$ containing naturally the cluster monomials in $\mathcal{A}(Q, \mathbf{x})$ and which is expected to constitute a $\mathbb{Z}$-basis in the cluster algebra $\mathcal{A}(Q, \mathbf{x})$.

1.2. **Triangulated 2-Calabi-Yau realizations.** Given a triangulated 2-Calabi-Yau category $\mathcal{C}$ over an algebraically closed field $\mathbf{k}$ with a cluster-tilting object T, the set of cluster-tilting objects in $\mathcal{C}$ reachable from T is in bijection with the set of cluster variables in $\mathcal{A}(Q_T, \mathbf{x})$ where Q_T is the Gabriel quiver of the algebra $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ [BIRS09] (see Section 2 for details). This bijection can be made explicit using the so-called *cluster character*

$$X_?^T : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathbb{Z}[\mathbf{x}^{\pm 1}]$$

introduced in [Pal08] whose definition is recalled in Section 2.

When $\mathcal{C} = \mathcal{C}_Q$ is the cluster category of an acyclic quiver and $T = \mathbf{k}Q$ is the canonical cluster-tilting object, the cluster character X^T coincides with the Caldero-Chapoton map CC introduced in [CC06, CK06]. In [Dup08], the author introduced and studied generic values of the Caldero-Chapoton map under the name of *generic variables*. It is known that these generic variables form a $\mathbb{Z}$-basis in the cluster algebra $\mathcal{A}(Q, \mathbf{x})$ if Q is a tame quiver [Dup08, DXX09] and a $\mathbb{C}$-basis of the $\mathbb{C}$-vector space $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}(Q, \mathbf{x})$ if Q is acyclic [GLS10, Sch09]. In this article, we generalize this construction to a much wider context and conjecture that it still provides a $\mathbb{Z}$-linear basis of the corresponding cluster algebra.

1.3. **Main results.** If $\mathbf{k}$ is the field of complex numbers and $\mathcal{C}$ has constructible cones in the sense of [Pal09], we associate to any element $\gamma \in K_0(\mathrm{add}\, T)$ a Laurent polynomial $X(\gamma)$ by taking the character of the cone of a generic morphism in $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$-mod with δ-vector γ in the sense of [DF09] (see Section 3 for details). The set

$$\mathcal{G}^T(\mathcal{C}) = \{X(\gamma) | \gamma \in K_0(\mathrm{add}\, T)\}$$

is called the set of *generic characters*.

Theorem 4.1 states that the set $\mathcal{G}^T(\mathcal{C})$ of generic characters contains naturally the set of cluster monomials (and thus of cluster variables) in the cluster algebra $\mathcal{M}(Q_T, \mathbf{x})$.

When $\mathcal{C} = \mathcal{C}_Q$ is the cluster category of an acyclic quiver Q and $T = \mathbf{k}Q$ is the canonical cluster-tilting object, we prove that $\mathcal{G}^T(\mathcal{C})$ satisfies multiplicative properties compatible with the virtual generic decomposition of [IOTW09] (Theorem 6.1).

In this case, we also prove that the set of $\mathcal{G}^T(\mathcal{C})$ coincides with the set $\mathcal{G}(Q)$ of generic variables introduced in [Dup08]. In particular, it provides a linear basis in the acyclic cluster algebra $\mathcal{A}(Q, \mathbf{x})$ (Theorem 7.3 and Corollary 7.5).

1.4. **Organization of the paper.** In Section 2, we recall necessary background concerning 2-Calabi-Yau triangulated realizations of cluster algebras. In Section 3, we define generic cluster characters in full generality and connect this construction to general presentations of modules introduces in [DF09]. In Section 4 we prove that generic cluster characters naturally contain cluster monomials of the corresponding cluster algebra. In the sequel, we focus on the case of cluster categories associated to acyclic quivers. In Section 5, we relate indices and dimension vectors in cluster categories in order to obtain a parameterization of generic characters by their **g**-vectors. In Section 6, we prove the compatibility of generic characters with the virtual generic decomposition of [IOTW09]. Finally, in Section 7, we prove that this construction indeed generalizes the construction of generic variables provided in [Dup08].

2. Preliminaries

2.1. **Triangulated 2-Calabi-Yau realizations.** Let $\mathbf{k}$ be the field of complex numbers. Throughout the paper, $\mathcal{C}$ will be a Hom-finite triangulated Krull-Schmidt $\mathbf{k}$-category satisfying the 2-Calabi-Yau property. It means that there is a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(M, N) \simeq D\mathrm{Hom}_{\mathcal{C}}(N, M[2])$$

for any two objects M, N in $\mathcal{C}$ where $D = \mathrm{Hom}_{\mathbf{k}}(-, \mathbf{k})$ is the standard duality and $[1]$ is the suspension functor in $\mathcal{C}$.

A *cluster-tilting object* in $\mathcal{C}$ is a basic rigid (that is, without self-extension) object T such that $\mathrm{Ext}^1_{\mathcal{C}}(T, X) = 0$ implies $X \in \mathrm{add}\, T$. We always assume that $\mathcal{C}$ contains a cluster-tilting object $T = T_1 \oplus \cdots \oplus T_n$. We denote by Q_T the Gabriel quiver of the algebra $B_T = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ and we fix a n-tuple $\mathbf{x}$ of indeterminates over $\mathbb{Z}$. We denote by $\mathcal{T}$ the additive category $\mathrm{add}\, T$. We recall that a cluster tilting-object T' in $\mathcal{C}$ is called *reachable* from T if T and T' are in the same connected component of the cluster-tilting graph of $\mathcal{C}$.

A fundamental example of triangulated 2-Calabi-Yau category is the so-called *cluster category* $\mathcal{C}_Q$ of an acyclic quiver Q introduced in [BMR$^+$06] (see also [CCS06] for Dynkin type $\mathbb{A}$). The path algebra $\mathbf{k}Q$ of Q can be identified with a cluster-tilting object in $\mathcal{C}_Q$, called *canonical cluster-tilting object in* $\mathcal{C}_Q$ and in this case, every rigid object in $\mathcal{C}$ is reachable from $\mathbf{k}Q$.

Given a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ with M, N in $\mathcal{C}$, the *cone* of f is the unique (up to isomorphism) object $\mathrm{Cone}\,(f)$ in $\mathcal{C}$ such that there exists a triangle

$$M \xrightarrow{f} N \longrightarrow \mathrm{Cone}\,(f) \longrightarrow M[1].$$

Throughout the paper, we will assume that the category $\mathcal{C}$ has constructible cones in the sense of [Pal09, §1.3]. This situation includes in particular cluster categories, stable categories of Frobenius categories and Amiot generalized cluster categories [Ami10] as shown in [Pal09].

Definition 2.1. A $\mathcal{T}$*-morphism* in $\mathcal{C}$ is a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ for some objects T_0, T_1 in $\mathcal{T}$.

We denote by F_T the functor

$$F_T = \mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \longrightarrow B_T\text{-mod}\,.$$

inducing an equivalence of categories

$$\mathcal{C}/(T[1]) \xrightarrow{\sim} B_T\text{-mod}$$

where B_T-mod denotes the category of finitely generated left B_T-modules. Note that projective B_T-modules are given by the $F_T M$ where M runs over $\mathcal{T}$.

As shown in [KR08], for any object M in $\mathcal{C}$, there exist (non-unique) triangles

$$T_1^M \longrightarrow T_0^M \longrightarrow M \longrightarrow T_1^M[1],$$
$$M \longrightarrow T_M^0[2] \longrightarrow T_M^1[2] \longrightarrow M[1]$$

with T_i^M, T_M^i in $\mathcal{T}$ for any $i \in \{1, 2\}$.

Following [Pal08], the *index* of M (with respect to $\mathcal{T}$)

$$\mathrm{ind}_{\mathcal{T}}(M) = [T_0^M] - [T_1^M]$$

and the *coindex* of M (with respect to $\mathcal{T}$)

$$\mathrm{coind}_{\mathcal{T}}(M) = [T_M^0] - [T_M^1]$$

are well-defined elements in the Grothendieck group $K_0(\mathcal{T})$.

2.2. **Cluster characters.** Given a B_T-module M and $\mathbf{e} \in K_0(B_T\text{-mod})$, we denote by $\mathrm{Gr}_{\mathbf{e}}(M)$ the variety of sub-B_T-modules of M whose class in $K_0(B_T\text{-mod})$ is $\mathbf{e}$. This is a projective variety and we denote by $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$ its Euler characteristic with respect to the simplicial cohomology.

Let $\langle -, - \rangle$ be the bilinear form on the split Grothendieck group $K_0(B_T\text{-mod})^{\mathrm{split}}$ given by

$$\langle M, N \rangle = \dim \mathrm{Hom}_{B_T}(M, N) - \dim \mathrm{Ext}^1_{B_T}(M, N).$$

It is well-defined on the Grothendieck group $K_0(B_T\text{-mod})$ if B_T is hereditary but not in general.

For any $i \in \{1, \ldots, n\}$, let S_i be the simple B_T-module associated to i. Then the linear form

$$\langle S_i, - \rangle_a : M \mapsto \langle S_i, M \rangle - \langle M, S_i \rangle$$

is well-defined on $K_0(B_T\text{-mod})$ [Pal08, Lemma 1.3].

Definition 2.2 ([Pal08])**.** The *cluster character associated to* T is the map

$$X_?^T : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathbb{Z}[\mathbf{x}^{\pm 1}]$$

given by

$$X_M^T = \begin{cases} x_i \text{ if } M \simeq T_i[1]\ ; \\ \displaystyle\sum_{\mathbf{e} \in K_0(B_T\text{-mod})} \chi(\mathrm{Gr}_{\mathbf{e}}(F_T M)) \prod_{i=1}^n x_i^{\langle S_i, \mathbf{e} \rangle_a - \langle S_i, F_T M \rangle} \text{ otherwise.} \end{cases}$$

Then, it follows from [Pal08, Corollary 5.4] that

$$\left\{X_M^T | M \text{ is indecomposable rigid in } \mathcal{C} \text{ and reachable from } T\right\} = \mathrm{Cl}(Q_T, \mathbf{x})$$

and

$$\left\{X_M^T | M \text{ is rigid in } \mathcal{C} \text{ and reachable from } T\right\} = \mathcal{M}(Q_T, \mathbf{x}).$$

When $\mathcal{C} = \mathcal{C}_Q$ is the cluster category of an acyclic quiver and $T = \mathbf{k}Q$, the cluster character $X_?^T$ coincides with the so-called *Caldero-Chapoton map* introduced in [CC06, CK06].

3. Generic cluster characters for 2-Calabi-Yau triangulated categories

3.1. **Generic characters.** In [Dup08], we observed that the Caldero-Chapoton map takes generic values on the irreducible components of the set of objects in the corresponding cluster category. For an arbitrary 2-Calabi-Yau category $\mathcal{C}$, there is in general no obvious geometry on $\mathrm{Ob}(\mathcal{C})$. However, since $\mathcal{C}$ is $\mathbf{k}$-linear, Hom-spaces in $\mathcal{C}$ are $\mathbf{k}$-vector spaces and in particular, they are irreducible affine varieties. Thus, for geometric statements, it is more convenient to consider morphisms instead of objects. This philosophy is also clear from the point of view of representation theory of algebras, as shown in [IOTW09, DF09].

Definition 3.1. For any objects T_0, T_1 in $\mathcal{T}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$, we set

$$X : \left\{ \begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(T_1, T_0) & \longrightarrow & \mathbb{Z}[\mathbf{x}^{\pm 1}] \\ f & \mapsto & X^T_{\mathrm{Cone}\,(f)} \end{array} \right.$$

The group $\mathrm{Aut}_{\mathcal{C}}(T_0) \times \mathrm{Aut}_{\mathcal{C}}(T_1)^{\mathrm{op}}$ acts on $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ by

$$(g_0, g_1).f = g_0 f g_1$$

and the map X is invariant under this action.

If M is a rigid object in $\mathcal{C}$, consider a triangle

$$T_1^M \xrightarrow{f} T_0^M \longrightarrow M \longrightarrow T_1^M[1]$$

with T_0^M, T_1^M in $\mathcal{T}$. Then, it follows from [DK08, §2.1] that the orbit of f under this action is a dense open subset in $\mathrm{Hom}_{\mathcal{C}}(T_1^M, T_0^M)$ so that X is constant over a dense open subset of $\mathrm{Hom}_{\mathcal{C}}(T_1^M, T_0^M)$. The following lemma proves that such a dense open subset actually exists for any two objects T_0, T_1 in $\mathcal{T}$.

Lemma 3.2. *Let T_0, T_1 be objects in $\mathcal{T}$. Then there exists a Zariski dense open subset $U_{(T_1,T_0)} \subset \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ such that X is constant on $U_{(T_1,T_0)}$.*

Moreover, if $U'_{(T_1,T_0)}$ is another such open dense subset, then the values of X on $U_{(T_1,T_0)}$ and $U'_{(T_1,T_0)}$ coincide. We denote by $X(T_1, T_0)$ this common value.

Proof. With the assumptions on the category $\mathcal{C}$, it follows from [Pal09] that the map $X : f \mapsto X^T_{\mathrm{Cone}\,(f)}$ is constructible on $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$. Thus, there exists a non-empty open (for the Zariski topology) subset $U(T_0, T_1) \subset \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ such that X is constant over $U(T_0, T_1)$. Since $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ is an irreducible affine variety, $U(T_0, T_1)$ is dense and the lemma follows. $\square$

3.2. **Stabilization maps and cluster characters.** Let T_0, T_1, T_0', T_1' be objects in $\mathcal{T}$. Then $[T_0]-[T_1] = [T_0']-[T_1']$ in $K_0(\mathcal{T})$ if and only if there exists $T, T' \in \mathcal{T}$ such that $T_0 \oplus T = T_0' \oplus T'$ and $T_1 \oplus T = T_1' \oplus T'$. We define on $\mathcal{T}$ a structure of right-filter by setting $T' \leq T''$ iff there exists $T^{(3)} \in \mathcal{T}$ such that $T'' = T' \oplus T^{(3)}$. For any T_0, T_1 in $\mathcal{T}$, if $T' \leq T''$ in $\mathcal{T}$ as above, the *stabilization map* from $\mathrm{Hom}_{\mathcal{C}}(T_1 \oplus T', T_0 \oplus T')$ to $\mathrm{Hom}_{\mathcal{C}}(T_1 \oplus T'', T_0 \oplus T'')$ is

$$\mathrm{St}_{T',T''} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(T_1 \oplus T', T_0 \oplus T') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(T_1 \oplus T'', T_0 \oplus T'') \\ f & \mapsto & f \oplus \mathrm{id}_{T^{(3)}}. \end{array} \right.$$

We set $\mathrm{St}_{T'} = \mathrm{St}_{0,T'}$ the stabilization map $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(T_1 \oplus T', T_0 \oplus T')$ sending f to $f \oplus \mathrm{id}_{T'}$.

Given an element $\gamma \in K_0(\mathcal{T})$, there exists a unique pair $(T_0^{\min}(\gamma), T_1^{\min}(\gamma))$ of objects in $\mathcal{T}$ such that

$$\gamma = [T_0^{\min}(\gamma)] - [T_1^{\min}(\gamma)]$$

and such that $T_0^{\min}(\gamma)$ and $T_1^{\min}(\gamma)$ have no common direct factor.

Remark 3.3. Let T', T_0, T_1 be objects in $\mathcal{T}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$. The triangles

$$T_1 \xrightarrow{f} T_0 \longrightarrow \mathrm{Cone}\,(f) \longrightarrow T_1[1] \text{ and } T' \xrightarrow{\mathrm{id}_{T'}} T' \longrightarrow 0 \longrightarrow T[1]$$

give rise to

$$T_1 \oplus T' \xrightarrow{f \oplus \mathrm{id}_{T'}} T_0 \oplus T' \longrightarrow \mathrm{Cone}\,(f) \longrightarrow T_1[1] \oplus T'[1]$$

so that $\mathrm{Cone}\,(f \oplus \mathrm{id}_{T'}) \simeq \mathrm{Cone}\,(f)$ and thus $X(f \oplus \mathrm{id}_{T'}) = X(f)$. Thus, X is invariant under stabilization.

Note that if f is generic in $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$, the image of its $\mathrm{Aut}(P_0) \times \mathrm{Aut}(P_1)^{\mathrm{op}}$-orbit under a stabilization map is a Zariski dense open subset in the image of $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ under this stabilization map. Theorem 3.4 proves that the image of f under stabilization is actually generic in $\mathrm{Hom}_{\mathcal{C}}(T_1 \oplus T, T_0 \oplus T)$.

Theorem 3.4 (Stability Theorem)**.** *Let T_0, T_1 be objects in $\mathcal{T}$ and $\gamma = [T_0] - [T_1] \in K_0(\mathcal{T})$. Then, the generic element in $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ is isomorphic to an element in the image of the stabilization map $\mathrm{Hom}_{\mathcal{C}}(T_1^{\min}(\gamma), T_0^{\min}(\gamma)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$.*

Proof. The proof is the same as [IOTW09, Theorem 5.2.2]. We recall it for completeness. Let T_0, T_1 be objects in $\mathcal{T}$ and $\gamma = [T_0] - [T_1] \in K_0(\mathcal{T})$. Then, there exists some $T \in \mathcal{T}$ such that $T_0 = T_0^{\min}(\gamma) \oplus T$ and $T_1 = T_1^{\min}(\gamma) \oplus T$. Let $\phi \in \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ be a generic element. We now prove that there exists $(g_0, g_1) \in \mathrm{Aut}(T_0 \oplus T) \times \mathrm{Aut}(T_1 \oplus T)^{\mathrm{op}}$ such that $\phi = (g_0, g_1).\mathrm{St}_T(\phi^{\min})$ where $\phi^{\min} \in \mathrm{Hom}_{\mathcal{C}}(T_1^{\min}(\gamma), T_0^{\min}(\gamma))$.

The element $\phi \in \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ can be viewed as a 2×2 matrix

$$\phi = \begin{bmatrix} f & h \\ g & r \end{bmatrix}$$

with $f \in \mathrm{Hom}_{\mathcal{C}}(T_1^{\min}(\gamma), T_0^{\min}(\gamma))$, $g \in \mathrm{Hom}_{\mathcal{C}}(T_1^{\min}(\gamma), T)$, $h \in \mathrm{Hom}_{\mathcal{C}}(T, T_0^{\min}(\gamma))$ and $r \in \mathrm{End}_{\mathcal{C}}(T)$. Since ϕ is generic, r is generic in $\mathrm{End}_{\mathcal{C}}(T)$ so that it is invertible. Thus, we get

$$\phi = \begin{bmatrix} f & h \\ g & r \end{bmatrix} = \begin{bmatrix} 1_{T_0} & hr^{-1} \\ 0 & 1_T \end{bmatrix} \begin{bmatrix} f - hr^{-1}g & 0 \\ 0 & 1_T \end{bmatrix} \begin{bmatrix} 1_{T_1} & 0 \\ g & r \end{bmatrix}$$

so that

$$\phi = (g_0, g_1)\begin{bmatrix} f - hr^{-1}g & 0 \\ 0 & 1_T \end{bmatrix} = (g_0, g_1)\mathrm{St}_T(\phi^{\min})$$

where

$$g_0 = \begin{bmatrix} 1_{T_0} & hr^{-1} \\ 0 & 1_T \end{bmatrix} \in \mathrm{Aut}_{\mathcal{C}}(T_0 \oplus T),$$

$$g_1 = \begin{bmatrix} 1_{T_1} & 0 \\ g & r \end{bmatrix} \in \mathrm{Aut}_{\mathcal{C}}(T_1 \oplus T)^{\mathrm{op}}$$

and $\phi^{\min} = f - hr^{-1}g \in \mathrm{Hom}_{\mathcal{C}}(T_1^{\min}(\gamma), T_0^{\min}(\gamma))$. This proves the theorem. □

Corollary 3.5. *For any $T_0, T_1 \in \mathcal{T}$, $X(T_1, T_0)$ only depends on $[T_0] - [T_1]$.*

Proof. Let T_0, T_1 be objects in $\mathcal{T}$ and set $\gamma = [T_0] - [T_1]$. It follows from the stability theorem that $X(T_1, T_0) = X(\mathrm{St}_T(\phi^{\min}))$ where $\phi^{\min}$ is a generic element in $\mathrm{Hom}_{\mathcal{C}}(T_1^{\min}(\gamma), T_0^{\min}(\gamma))$. In particular, $X(T_1, T_0) = X(T_1^{\min}(\gamma), T_0^{\min}(\gamma))$ only depends on γ. □

Definition 3.6. For any $\gamma \in K_0(\mathcal{T})$, a *generic morphism of index* γ is a morphism in the dense open subset $U_{(T_1^{\min}(\gamma), T_0^{\min}(\gamma))} \subset \mathrm{Hom}_{\mathcal{C}}(T_1^{\min}(\gamma), T_0^{\min}(\gamma))$.

If T_0, T_1 are objects in $\mathcal{T}$ such that $\gamma = [T_0] - [T_1] \in K_0(\mathcal{T})$, we will sometimes abuse terminology and view the generic morphism of index γ as an element of $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ by considering its image under the stabilization map.

We now define the generic characters as the images under X of the generic morphisms.

Definition 3.7. For any $\gamma \in K_0(\mathcal{T})$, the *generic character of index* γ *is*

$$X(\gamma) = X(T_0^{\min}(\gamma), T_1^{\min}(\gamma)).$$

The set

$$\mathcal{G}^T(\mathcal{C}) = \{X(\gamma) | \gamma \in K_0(\mathcal{T})\}$$

is called the set of *generic characters* associated to T in $\mathcal{C}$.

3.3. **Generic morphism and general presentations in B_T-mod.** The classical generic variables introduced in [Dup08] for an acyclic quiver Q were closely related to the generic representation theory of the finite dimensional hereditary algebras $\mathbf{k}Q$ developed in [Kac80, Kac82, Sch92]. In general, a 2-Calabi-Yau tilted algebra B_T is a basic finite dimensional $\mathbf{k}$-algebra but is not hereditary [KR07]. In [DF09], the authors develop a generic representation theory for any finite dimensional basic $\mathbf{k}$-algebra. This is done by replacing the usual notion of generic representation by the notion of generic presentation. We recall briefly this notion.

Given a finite dimensional basic $\mathbf{k}$-algebra B, we denote by $\epsilon_1, \ldots, \epsilon_n$ the idempotents of B. The indecomposable projective B-modules are ${}_BP_i = B\epsilon_i$ for $i \in \{1, \ldots, n\}$. We denote by $K_0(B\text{-proj})$ the Grothendieck group of the additive category B-proj $= \mathrm{add}({}_BP_1 \oplus \cdots \oplus {}_BP_n)$ and for any B-module M, we denote by $[M]$ its class in $K_0(B\text{-proj})$.

For any projective B-modules M_1, M_0, and any morphism $f \in \mathrm{Hom}_B(M_1, M_0)$, the *$\delta$-vector* of f is $[M_0] - [M_1]$. Note that, identifying $K_0(B\text{-proj})$ with $\mathbb{Z}^n$ by sending each $[{}_BP_i]$ to the i-th vector α_i of the canonical basis of $\mathbb{Z}^n$, this coincides with the definition provided in [DF09].

The group $\mathrm{Aut}_B(M_0) \times \mathrm{Aut}_B(M_1)^{\mathrm{op}}$ acts on $\mathrm{Hom}_B(M_1, M_0)$ by $(g_0, g_1).f = g_0 f g_1$. A morphism $f \in \mathrm{Hom}_B(M_1, M_0)$ is called a *generic* if its $\mathrm{Aut}_B(M_0) \times \mathrm{Aut}_B(M_1)^{\mathrm{op}}$-orbit is a Zariski dense open subset in $\mathrm{Hom}_B(M_1, M_0)$. If M is a B_T-module, a *general presentation of* M is a generic morphism in $\mathrm{Hom}_{B_T}(M_1, M_0)$ where M_0, M_1 are projective B_T-modules such that there is a projective resolution $M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$ of B_T-modules.

We now prove that, under the functor $F_T : \mathcal{C} \longrightarrow B_T\text{-mod}$, generic $\mathcal{T}$-morphisms in the category $\mathcal{C}$ correspond to generic morphisms in B_T-mod.

Lemma 3.8. *For any T_0, T_1 in $\mathcal{T}$ and any $f \in \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$, the following are equivalent :*

1. *f is a generic $\mathcal{T}$-morphism in $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$;*
2. *$F_T f$ is a general presentation in $\mathrm{Hom}_{B_T}(F_T T_1, F_T T_0)$.*

Proof. The functor $F_T = \mathrm{Hom}_{\mathcal{C}}(T, -)$ induces a $\mathbf{k}$-linear equivalence of categories

$$F_T : \mathcal{C}/\mathcal{T}[1] \xrightarrow{\sim} B_T\text{-mod}\,.$$

Let T_0, T_1 be objects in $\mathcal{T}$. Since T is a cluster-tilting object, for any X in $\mathcal{T}[1]$, $\mathrm{Hom}_{\mathcal{C}}(T_1, X) = 0$ so that there are no morphisms from T_1 to T_0 in $\mathcal{C}$ factorizing through $\mathcal{T}[1]$. In particular, F_T induces an isomorphism of $\mathbf{k}$-vector spaces

$$\mathrm{Hom}_{\mathcal{C}}(T_1, T_0) \simeq \mathrm{Hom}_{B_T}(F_T T_1, F_T T_0).$$

Thus, f is generic in $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ if and only if $F_T f$ is generic $\mathrm{Hom}_{B_T}(F_T T_1, F_T T_0)$ □

Corollary 3.9. *A presentation of B_T-modules is generic if and only if it is the image under F_T of a generic $\mathcal{T}$-morphism.*

Proof. The projective B_T-modules are the $F_T(M)$ where M is an object in $\mathcal{T}$. Fix thus two projective B_T-modules $F_T T_0, F_T T_1$ and fix $g \in \mathrm{Hom}_{B_T}(F_T T_1, F_T T_0)$. The functor F_T induces an isomorphism of $\mathbf{k}$-vector spaces

$$\mathrm{Hom}_{\mathcal{C}}(T_1, T_0) \simeq \mathrm{Hom}_{B_T}(F_T T_1, F_T T_0)$$

so that a $g = F_T f$ for some $f \in \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$. The corollary thus follows from Lemma 3.8. □

Remark 3.10. Note that for any $f \in \mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$, the δ-vector of $F_T f$ is $[F_T T_0] - [F_T T_1]$. Thus, using the above identifications with $\mathbb{Z}^n$,

$$\delta(F_T f) = [F_T T_0] - [F_T T_1] = [T_0] - [T_1] = \mathrm{ind}_{\mathcal{T}}(\mathrm{Cone}\,(f)).$$

Thus F_T induces a 1-1 correspondence between generic $\mathcal{T}$-morphism and general presentations in B_T-mod. Under this correspondence, the generic $\mathcal{T}$-morphism of index γ corresponds to the generic presentation of δ-vector γ.

4. Cluster monomials and generic characters

We now prove that cluster monomials in $\mathcal{A}(B_T, \mathbf{x})$ are generic cluster characters. We will see in Corollary 7.5 that the converse is not true in general.

Theorem 4.1. *Let $\mathcal{C}$ be a triangulated 2-Calabi-Yau category with constructible cones and T be a cluster tilting object in $\mathcal{C}$. Then, the following hold :*

1. *If M is a rigid object in $\mathcal{C}$, then*

$$X_M^T = X(\mathrm{ind}_{\mathcal{T}}(M)) \;;$$

(2) *If M is rigid object in $\mathcal{C}$, reachable from T, then*
$$X(\mathrm{ind}_{\mathcal{T}}(M)) \in \mathcal{M}(Q_T, \boldsymbol{x})\ ;$$
(3) *If M is an indecomposable rigid in $\mathcal{C}$, reachable from T, then*
$$X(\mathrm{ind}_{\mathcal{T}}(M)) \in \mathrm{Cl}(Q_T, \boldsymbol{x})\ ;$$
(4)
$$\mathcal{M}(Q_T, \boldsymbol{x}) \subset \mathcal{G}^T(\mathcal{C}).$$

Proof. Let M be an object in $\mathcal{C}$ and
$$T_1^M \xrightarrow{f_M} T_0^M \longrightarrow M \longrightarrow T_1^M[1]$$
be a triangle in $\mathcal{C}_Q$ with $T_0^M, T_1^M \in \mathrm{add}\, T$. Set $\gamma = \mathrm{ind}_{\mathcal{T}}(M) = [T_0^M] - [T_1^M]$. If M is rigid in $\mathcal{C}$, then T_0^M and T_1^M have no common direct summands [DK08, Proposition 2.2] so that $T_0^M = T_0^{\min}(\gamma)$ and $T_1^M = T_1^{\min}(\gamma)$. Moreover, the $\mathrm{Aut}_{\mathcal{C}}(T_0) \times \mathrm{Aut}_{\mathcal{C}}(T_1)^{\mathrm{op}}$-orbit of f_M is open and dense in $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ [DK08, §2.1]. Since X is invariant under the action of $\mathrm{Aut}_{\mathcal{C}}(T_1) \times \mathrm{Aut}_{\mathcal{C}}(T_0)^{\mathrm{op}}$ and $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ is an irreducible variety, it follows that $X(f_M)$ is the generic value of X on $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$. Thus,
$$X(f_M) = X([T_0^M] - [T_1^M]) = X(\mathrm{ind}_{\mathcal{T}}(M)).$$
Since $\mathrm{Cone}\,(f_M) \simeq M$, it follows from the definition of X that $X(f_M) = X_M^T$. This proves the first point.

By [Pal08], $X_?^T$ induces a 1-1 correspondence from the set of reachable indecomposable rigid (resp. rigid) objects to cluster variables (resp. cluster monomials) in $\mathcal{A}(Q_T, \mathbf{x})$. Together with the first point, this proves the second, third and fourth points. □

5. Indices and dimension vectors in cluster categories

From now on, we assume that $\mathcal{C} = \mathcal{C}_Q$ is the cluster category of an acyclic quiver Q and that $T = \mathbf{k}Q$ is the canonical cluster-tilting object in $\mathcal{C}_Q$. We recall that the cluster category is the orbit category in $D^b(\mathbf{k}Q\text{-mod})$ of the functor $\tau^{-1}[1]$ where $[1]$ is the suspension functor in the bounded derived category $D^b(\mathbf{k}Q\text{-mod})$ and τ is the Auslander-Reiten translation. It is a canonically triangulated 2-Calabi-Yau category [Kel05] with constructible cones [Pal09].

For any $i \in Q_0$, we denote by S_i the simple $\mathbf{k}Q$-module at vertex i and by P_i its projective cover. Indecomposable objects in $\mathcal{C}_Q$ can be identified with indecomposable $\mathbf{k}Q$-modules and shifts of indecomposable projective modules. The canonical cluster-tilting object T can thus be written as $T = \mathbf{k}Q = \bigoplus_{i \in Q_0} P_i$.

5.1. **Dimension vectors and index.** We shall now compare the notion of index in the cluster category $\mathcal{C}_Q$ to the notion of dimension vector in the underlying module category $\mathbf{k}Q$-mod.

For any $\mathbf{k}Q$-module M, the *dimension vector* of M is the element $\mathbf{dim}\, M = (\dim_k \mathrm{Hom}_{\mathbf{k}Q}(P_i, M))_{i \in Q_0} \in \mathbb{N}^{Q_0}$. Let $K_0(\mathbf{k}Q\text{-mod})$ denote the Grothendieck group of $\mathbf{k}Q$-mod. It is known that $\mathbf{dim}$ induces an isomorphism of abelian groups $K_0(kQ) \xrightarrow{\sim} \mathbb{Z}^{Q_0}$ sending the isoclass of S_i to the i-th vector α_i of the canonical basis of $\mathbb{Z}^{Q_0}$ for any $i \in Q_0$.

As usual, we identify $\mathbf{k}Q$-mod with the category $\mathrm{rep}(Q)$ of finite dimensional representations of Q over $\mathbf{k}$. We recall that a representation M of Q is a pair

$M = ((M(i))_{i\in Q_0}, (M(\alpha))_{\alpha\in Q_1})$ such that each $M(i)$ is a finite dimensional $\mathbf{k}$-vector space and each $M(\alpha)$ is a $\mathbf{k}$-linear map $M(i) \longrightarrow M(j)$ where $\alpha : i \longrightarrow j \in Q_1$. Note that $\mathbf{dim}\, M = (\dim_{\mathbf{k}} M(i))_{i\in Q_0}$ for any representation M of Q.

For any $\mathbf{d} \in \mathbb{N}^{Q_0}$, we denote by $\mathrm{rep}(Q, \mathbf{d})$ the set of representations M of Q such that $\mathbf{dim}\, M = \mathbf{d}$ which can is identified with the irreducible affine variety

$$\mathrm{rep}(Q, \mathbf{d}) = \prod_{\alpha:i\longrightarrow j} \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}^{d_i}, \mathbf{k}^{d_j}),$$

called *representation space of dimension* $\boldsymbol{d}$.

We define the *dimension vector* $\mathbf{dim}_{\mathcal{C}}\, M$ of an object M in the cluster category by

$$\mathbf{dim}_{\mathcal{C}}\, M = \begin{cases} \mathbf{dim}\, M & \text{if } M \text{ is an indecomposable } \mathbf{k}Q\text{-module ;} \\ -(E^t)^{-1}\mathbf{dim}\, S_i & \text{if } M \simeq P_i[1] \text{ ;} \\ \mathbf{dim}_{\mathcal{C}}\, M_1 + \mathbf{dim}_{\mathcal{C}}\, M_2 & \text{if } M = M_1 \oplus M_2. \end{cases}$$

Since $\mathbf{dim}_{\mathcal{C}}\, M = \mathbf{dim}\, M$ for any $\mathbf{k}Q$-module M, we will simply write $\mathbf{dim}\, M$ for the dimension vector of an arbitrary object M in $\mathcal{C}$.

Remark 5.1. Note that our convention for dimension vectors of objects in $\mathcal{C}_Q$ agrees with the one considered in [IOTW09] but differs from the one taken in [CK06, Dup08]. Indeed, in [CK06, Dup08], the dimension vector of $P_i[1]$ was set to $-\mathbf{dim}\, S_i$. This was more accurate from the point of view of denominator vectors of the corresponding cluster characters (see for instance [CK06, Theorem 3] and [Dup08, Lemma 3.9]). Nevertheless, as we shall see, it appears that the convention of [IOTW09] we use here is more natural from the point of view of indices, $\mathbf{g}$-vectors, virtual generic decompositions and generic characters.

Since Q is acyclic, $\mathbf{k}Q$ is hereditary and thus, the bilinear form $\langle -, -\rangle$ is well defined on $\mathbb{Z}^{Q_0} \simeq K_0(\mathbf{k}Q\text{-mod})$. The *Euler matrix* of Q is thus the matrix $E \in M_{Q_0}(\mathbb{Z})$ of the (non-symmetric) bilinear form $\langle -, -\rangle$ on $K_0(\mathbf{k}Q\text{-mod})$. We refer the reader to [ASS05, §III.3] for classical properties of E.

Lemma 5.2. *Let $\mathcal{C} = \mathcal{C}_Q$ be the cluster category of an acyclic quiver Q. Let $T = \boldsymbol{k}Q$ be the canonical cluster tilting object in $\mathcal{C}$ and $\mathcal{T} = \mathrm{add}\, T$. Then, for any object M be an object in $\mathcal{C}$, we have*

$$\mathrm{ind}_{\mathcal{T}}(M) = E^t \boldsymbol{dim}\, M,$$
$$\mathrm{coind}_{\mathcal{T}}(M) = E\, \boldsymbol{dim}\, M.$$

Proof. Without loss of generality, we can assume that X is indecomposable. If M is a $\mathbf{k}Q$-module, then it follows from [Pal08, Lemma 2.3] that

$$\begin{aligned} \mathrm{coind}_{\mathcal{T}}(M) &= (\langle S_i, M\rangle)_{i\in Q_0} \\ &= (\alpha_i E \mathbf{dim}\, M)_{i\in Q_0} \\ &= E \mathbf{dim}\, M \end{aligned}$$

and

$$\begin{aligned} \mathrm{ind}_{\mathcal{T}}(M) &= (\langle M, S_i\rangle)_{i\in Q_0} \\ &= (\mathbf{dim}\, M^t E \alpha_i)_{i\in Q_0} \\ &= E^t \mathbf{dim}\, M. \end{aligned}$$

□

5.2. **Presentation spaces and index.** As we mentioned, classical approaches in geometric representation theory considered objects rather than morphisms. In particular, the geometric representation theory of acyclic quivers focuses on representation spaces. Nevertheless, generalizations can be carried out to arbitrary $\mathbf{k}$-algebras B if one focuses on morphisms in B-mod instead of objects in B-mod. In this settings, Kac's concept of generic representations of quivers [Kac80, Kac82] is replaced by Derksen-Fei's concept of general presentations of $\mathbf{k}Q$-modules [DF09]. Following the approach of [IOTW09], we will thus consider presentation spaces in $\mathbf{k}Q$-mod instead of representations spaces in $\mathrm{rep}(Q)$.

Given an element $\alpha \in \mathbb{Z}^{Q_0}$, a *projective decomposition* of α is a pair $(\gamma_0, \gamma_1) \in \mathbb{N}^{Q_0} \times \mathbb{N}^{Q_0}$ such that $E^t\alpha = \gamma_0 - \gamma_1$. It is called *minimal* if γ_0 and γ_1 have disjoint support. A minimal decomposition of a given element $\gamma \in \mathbb{N}^{Q_0}$ is unique.

For any $\gamma \in \mathbb{N}^{Q_0}$, we set $P(\gamma) = \bigoplus_{i \in Q_0} P_i^{\oplus \gamma_i}$. For any $\alpha \in \mathbb{Z}^{Q_0}$, the *presentation space* associated to a projective decomposition (γ_0, γ_1) of α is

$$R(\gamma_0, \gamma_1) = \mathrm{Hom}_{\mathbf{k}Q}(P(\gamma_1), P(\gamma_0)).$$

The *minimal presentation space* $R^{\min}(\alpha)$ is the representation space associated to the minimal projective decomposition of α.

Let α be an element $\mathbb{Z}^{Q_0}$, which shall be thought as a dimension vector of an object in $\mathcal{C}_Q$. Then, by Lemma 5.2, $\gamma = E^t\alpha \in \mathbb{Z}^{Q_0}$ may be thought as an index vector of an object in $\mathcal{C}_Q$. Let (γ_0, γ_1) be a projective decomposition of α then $\gamma = \gamma_0 - \gamma_1 = [P(\gamma_0)] - [P(\gamma_1)]$. The objects $T_0 = P(\gamma_0)$ and $T_1 = P(\gamma_1)$ belong to $\mathcal{T} = \mathrm{add}\,(\mathbf{k}Q) = \mathbf{k}Q\text{-proj}$ and the representation space $R(\gamma_0, \gamma_1)$ is thus isomorphic to $\mathrm{Hom}_{\mathbf{k}Q}(T_1, T_0)$ which is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(T_1, T_0)$ since T_1 is a projective $\mathbf{k}Q$-module [BMR+06].

If $(\gamma_0^{\min}, \gamma_1^{\min})$ is the minimal projective decomposition of α, we have $T_0^{\min}(\gamma) = P(\gamma_0^{\min})$ and $T_1^{\min}(\gamma) = P(\gamma_1^{\min})$. Thus, the minimal presentation space $R^{\min}(\alpha)$ is isomorphic to the space of $\mathcal{T}$-morphisms $\mathrm{Hom}_{\mathcal{C}}(T_1^{\min}(\gamma), T_0^{\min}(\gamma))$. In particular, the generic morphism of index γ can be viewed as a generic element of the minimal presentation space $R^{\min}((E^t)^{-1}\gamma)$.

5.3. **g-vectors.** We now prove that generic cluster characters are naturally parameterized by their $\mathbf{g}$-vectors, up to a Coxeter transformation. For details concerning $\mathbf{g}$-vectors, we refer the reader to [FZ07] in general and [FK10] for $\mathbf{g}$-vectors in the context of cluster characters.

We denote by $C = -E^tE^{-1}$ the *Coxeter matrix* of $\mathbf{k}Q$. If a Laurent polynomial $L \in \mathbb{Z}[\mathbf{x}^{\pm 1}]$ has a $\mathbf{g}$-vector, we denote it by $\mathbf{g}_L$. For any object M in $\mathcal{C}$, X_M^T has a $\mathbf{g}$-vector ([FK10]) and we denote this $\mathbf{g}$-vector by $\mathbf{g}_M$.

Proposition 5.3. *Let $\mathcal{C}$ be the cluster category of an acyclic quiver Q and $T = \mathbf{k}Q$ be the canonical cluster-tilting object. Then, for any $\gamma \in \mathbb{Z}^n$,*

$$\mathbf{g}_{X(\gamma)} = C^{-1}\gamma.$$

Proof. It follows from [FK10], that $\mathbf{g}_M = -\mathrm{coind}_T(M)$ but Lemma 5.2 implies that $-\mathrm{coind}_T(M) = -E\,\mathbf{dim}\,M$. Fix $\gamma \in \mathbb{Z}^n$. Then $X(\gamma) = X^T_{\mathrm{Cone}\,(f)}$ for some $\mathcal{T}$-morphism f such that $\gamma = \mathrm{ind}_T(\mathrm{Cone}\,(f))$. Let $M = \mathrm{Cone}\,(f)$. By Lemma 5.2,

$$\gamma = \mathrm{ind}_T(M) = E^t\mathbf{dim}\,M.$$

It follows that

$$\begin{aligned}\gamma &= E^t \mathbf{dim}\, M \\ &= E^t(-E^{-1})\mathbf{g}_M \\ &= C\mathbf{g}_M\end{aligned}$$

but the Coxeter transformation is invertible so that

$$\mathbf{g}_{X(\gamma)} = \mathbf{g}_M = C^{-1}\gamma.$$

□

Corollary 5.4. *Let M be a $\boldsymbol{k}Q$-module, and $P_0^M \xrightarrow{f_M} P_1^M \longrightarrow M \longrightarrow 0$ be a projective presentation of $\boldsymbol{k}Q$-modules. Then,*

$$\delta(f_M) = C\boldsymbol{g}_M.$$

Proof. It follows from Remark 3.10 that $\delta(f_M) = \mathrm{ind}_T(\mathrm{Cone}\,(f_M)) = \mathrm{ind}_T(M) = E^t \mathbf{dim}\, M = C\mathbf{g}_M$. □

6. Generic cluster characters and virtual generic decomposition

In this section, we still assume that $\mathcal{C} = \mathcal{C}_Q$ is the cluster category of an acyclic quiver Q and that $T = \mathbf{k}Q$ is the canonical cluster-tilting object in $\mathcal{C}$.

Following [Sch92], given two elements $\beta, \gamma \in \mathbb{N}^{Q_0}$, we say that $\mathrm{Ext}^1_{\mathbf{k}Q}(\beta, \gamma)$ *vanishes generally* if there exists $M_\beta \in \mathrm{rep}(Q, \beta)$, $M_\gamma \in \mathrm{rep}(Q, \gamma)$ such that $\mathrm{Ext}^1_{\mathbf{k}Q}(M_\beta, M_\gamma) = 0$. This is equivalent to the fact that there exists dense open subsets $O_\beta \subset \mathrm{rep}(Q, \beta)$, $O_\gamma \subset \mathrm{rep}(Q, \gamma)$ such that $\mathrm{Ext}^1_{\mathbf{k}Q}(M_\beta, M_\gamma) = 0$ for any $M_\beta \in O_\beta$ and $M_\gamma \in O_\gamma$.

Given an element $\alpha \in \mathbb{N}^{Q_0}$, Kac proved that there exists a unique decomposition

$$\alpha = \beta_1 + \cdots + \beta_k$$

such that :

(1) $\mathrm{Ext}^1_{\mathbf{k}Q}(\beta_i, \beta_j)$ vanishes generally if $i \neq j$;
(2) each β_i is a Schur root of Q.

This decomposition is called the *generic (or canonical) decomposition of* α.

This was generalized in [IOTW09] to arbitrary elements $\mathbb{Z}^{Q_0}$. Namely, for any $\alpha \in \mathbb{Z}^{Q_0}$ there exists a unique decomposition

$$\alpha = \beta_1 + \cdots + \beta_k - (E^t)^{-1}\gamma$$

such that :

(1) $\beta_1, \ldots, \beta_k, \gamma \in \mathbb{N}^{Q_0}$;
(2) β_i, γ have disjoint support for all i ;
(3) $\mathrm{Ext}^1_{\mathbf{k}Q}(\beta_i, \beta_j)$ vanishes generally if $i \neq j$;
(4) each β_i is a Schur root of Q.

This decomposition is called the *virtual generic decomposition of* α. Note that if $\alpha \in \mathbb{N}^n$, generic and virtual generic decomposition coincide [IOTW09].

Theorem 6.1. *Let $\mathcal{C}$ be the cluster category of an acyclic quiver Q and $T = \boldsymbol{k}Q$ be the canonical cluster-tilting object in $\mathcal{C}$. Let $\alpha \in \mathbb{Z}^n$ with virtual generic decomposition*

$$\alpha = \beta_1 + \cdots + \beta_k - (E^t)^{-1}\gamma.$$

Then,
$$X(E^t\alpha) = X(E^t\beta_1)\cdots X(E^t\beta_k)X(-\gamma).$$

Proof. Let $\alpha \in \mathbb{Z}^n$ with virtual canonical decomposition $\alpha = \beta_1+\cdots+\beta_k-(E^t)^{-1}\gamma$. For any $i = 1,\dots,k$, let M_i be a generic representation in $\mathrm{rep}(Q,\beta_i)$. Let p_{M_i} be the canonical projective presentation of M_i and 0_γ be the zero map $0_\gamma : P(\gamma)\longrightarrow 0$. Then, it follows from [IOTW09, Theorem 6.3.1] that $X(E^t\alpha) = X(f)$ where $f = \bigoplus_{i=1}^k p_{M_i} \oplus 0_\gamma$. Thus,

$$\begin{aligned} X(E^t\alpha) &= X(f) \\ &= X(\bigoplus_{i=1}^k p_{M_i} \oplus 0_\gamma) \\ &= \prod_{i=1}^k X(p_{M_i})X(0_\gamma) \end{aligned}$$

but it follows from [IOTW09] that p_{M_i} is generic for every $i = 1,\dots,k$ so that $X(p_{M_i}) = X(\mathrm{ind}_T(M_i)) = X(E^t\mathbf{dim}\, M_i) = X(E^t\beta_i)$. Also, 0_γ is generic so that $X(0_\gamma) = X(\mathrm{ind}_T(P_\gamma[1])) = X(-\gamma)$. Thus,
$$X(E^t\alpha) = X(E^t\beta_1)\cdots X(E^t\beta_k)X(-\gamma).$$
□

7. Generic cluster characters and classical generic variables

In this section, $\mathcal{C}$ still denotes the cluster category of an acyclic quiver Q and $T = \mathbf{k}Q$ is the canonical cluster-tilting object in $\mathcal{C}$. In this case, the cluster character X^T coincides with the Caldero-Chapoton map $CC : \mathrm{Ob}(\mathcal{C})\longrightarrow \mathbb{Z}[\mathbf{x}^{\pm 1}]$ introduced in [CC06]. In [Dup08], we introduced a family of Laurent polynomials in $\mathbb{Z}[\mathbf{x}^{\pm 1}]$, called *generic variables*, by considering generic values of the Caldero-Chapoton map. We shall now see that generic characters coincide with these generic variables when $\mathcal{C} = \mathcal{C}_Q$ and $T = \mathbf{k}Q$. First, we briefly review the construction of [Dup08].

For any $\alpha \in \mathbb{N}^{Q_0}$, there exists a dense open subset $U_\alpha \subset \mathrm{rep}(Q,\alpha)$ such that CC is constant over U_α. The value of CC on U_α does not depend on the chosen open set U_α and is denoted by $CC(\alpha)$ [Dup08, Lemma 3.1].

More generally, if $\alpha \in \mathbb{Z}^n$, set $\alpha = \alpha_+ - \alpha_-$ with $\alpha_\pm \in \mathbb{N}^{Q_0}$ having disjoint supports and set
$$CC(\alpha) = CC(\alpha_+)\prod_{i\in Q_0} x_i^{(\alpha_-)_i} = CC(\alpha_+)CC(P(\alpha_-)[1]).$$

The set
$$\mathcal{G}(Q) = \left\{CC(\alpha)|\alpha \in \mathbb{Z}^{Q_0}\right\}$$
is called the set of *generic variables* in $\mathcal{A}(Q)$ and for any $\alpha \in \mathbb{Z}^{Q_0}$, $CC(\alpha)$ is called the generic variable of dimension α.

Generic variables satisfy multiplicative properties with respect to Kac's generic decomposition [Dup08, Proposition 3.7]. Namely, for $\alpha \in \mathbb{N}^n$, if
$$\alpha = \beta_1 + \cdots + \beta_k$$

is the generic decomposition of α, then

$$CC(\alpha) = CC(\beta_1)\cdots CC(\beta_k).$$

Lemma 7.1. *For any $\alpha \in \mathbb{N}^n$,*

$$X(E^t\alpha) = CC(\alpha).$$

Proof. Let $\alpha \in \mathbb{N}^n$. For any projective decomposition $E^t\alpha = \gamma_0 - \gamma_1$, $X(E^t\alpha) = X(f)$ for a generic element $f \in \mathrm{Hom}_{kQ}(P(\gamma_1), P(\gamma_0))$. We can thus assume that $E^t\alpha = \gamma_0 - \gamma_1$ is the projective canonical decomposition of [IOTW09]. Consider the triangle

$$P(\gamma_1) \xrightarrow{f} P(\gamma_0) \longrightarrow \mathrm{Cone}\,(f) \longrightarrow P(\gamma_1)[1].$$

Applying F_T, we get

$$P(\gamma_1) \xrightarrow{F_T f} P(\gamma_0) \longrightarrow F_T(\mathrm{Cone}\,(f)) \longrightarrow 0$$

so that $X(f) = X^T_{\mathrm{Cone}\,(F_T(f))} = X^T_{\mathrm{Coker} F_T(f)}$. According to Corollary 3.9, $F_T f$ is generic in $\mathrm{Hom}_{kQ}(P(\gamma_1), P(\gamma_0))$. Thus, we get

$$X(f) = X^T(\mathrm{Cone}\, f) = CC(\mathrm{Coker} F_T f).$$

Consider the map $\kappa : g \mapsto \mathrm{Coker} g$ on $\mathrm{Hom}_{kQ}(P(\gamma_1), P(\gamma_0))$. For any morphism $g \in \mathrm{Hom}_{kQ}(P(\gamma_1), P(\gamma_0))$, we have $\mathbf{dim}\,\mathrm{Coker}(g) = \mathbf{dim}\, P(\gamma_0) - \mathbf{dim}\, P(\gamma_1) = (E^t)^{-1}(\gamma_0 - \gamma_1) = \alpha$ so that κ is a map

$$\kappa : \mathrm{Hom}_{kQ}(P(\gamma_1), P(\gamma_0)) \longrightarrow \mathrm{rep}(Q, \alpha).$$

Now, it follows from [IOTW09, Proposition 4.1.7] that there exists a Zariski dense open subset $\mathcal{U} \subset \mathrm{Hom}_{kQ}(P(\gamma_1), P(\gamma_0))$ such that $\kappa_{|\mathcal{U}} : \mathcal{U} \longrightarrow \mathrm{rep}(Q, \alpha)$ is algebraic and injective. Thus, $\dim \kappa(\mathcal{U}) \geq \dim \mathcal{U} > 0$ and thus $\kappa(\mathcal{U}) \cap U_\alpha \neq \emptyset$ where U_α denotes, as before, the Zariski dense open subset for the Caldero-Chapoton map. It follows that $\kappa^{-1}(U_\alpha)$ is a Zariski dense open subset in $\mathrm{Hom}_{kQ}(P(\gamma_1), P(\gamma_0))$ and we can thus assume that the generic element f belongs to $\kappa^{-1}(U_\alpha)$. It follows that

$$X(E^t\alpha) = X(f) = CC(\mathrm{Coker} F_T f) = CC(\alpha)$$

which proves the lemma. $\square$

Lemma 7.2. *For any $\alpha \in (-\mathbb{N})^n$,*

$$X(\alpha) = CC(\alpha).$$

Proof. Let $\gamma \in \mathbb{N}^n$ such that $\alpha = -\gamma$. Consider the morphism $0_\gamma : \mathrm{Hom}_{kQ}(P(\gamma), 0)$. We have the triangle

$$P(\gamma) \xrightarrow{0_\gamma} 0 \longrightarrow P(\gamma)[1] \xrightarrow{\sim} P(\gamma)[1]$$

so that

$$X(0_\gamma) = X^T_{P(\gamma)[1]} = CC(P(\gamma)[1]) = CC(-\gamma) = CC(\alpha).$$

Now, by definition, $X(f)$ is the generic character of index α, that is,

$$X(\alpha) = X(0_\gamma) = CC(\alpha).$$

$\square$

Let $\alpha, \beta \in \mathbb{Z}^n$ such that $\alpha = \alpha_+ - \alpha_-$ (resp. $\beta = \beta_+ - \beta_-$) where α_-, α_+ (resp. β_-, β_+) have disjoint support. Following [Dup08], we say that $\mathrm{Ext}^1_{\mathcal{C}_Q}(\alpha, \beta)$ *vanishes generally* if there exists $M_\alpha \in \mathrm{rep}(Q, \alpha_+), M_\beta \in \mathrm{rep}(Q, \beta_+)$ such that

$$\mathrm{Ext}^1_{\mathcal{C}_Q}(M_\alpha \oplus P(\alpha_-)[1], M_\beta \oplus P(\beta_-)[1]) = 0.$$

It follows from [Dup08, Lemma 3.9] that if $\mathrm{Ext}^1_{\mathcal{C}_Q}(\alpha, \beta)$ vanishes generally, then

$$CC(\alpha + \beta) = CC(\alpha)CC(\beta).$$

Theorem 7.3. *Let $\mathcal{C} = \mathcal{C}_Q$ be the cluster category of an acyclic quiver Q and $T = \boldsymbol{k}Q$ be the canonical cluster-tilting object in $\mathcal{C}$. Let $\alpha \in \mathbb{Z}^n$ with virtual generic decomposition*

$$\alpha = \beta_1 + \cdots + \beta_k - (E^t)^{-1}\gamma.$$

Then,

$$X(E^t\alpha) = CC(\beta_1 + \cdots + \beta_k - \gamma).$$

Proof. According to Theorem 6.1, we have

$$X(E^t\alpha) = X(E^t\beta_1)\cdots X(E^t\beta_k)X(-\gamma).$$

For any $i = 1, \ldots, k$, it follows from 7.2 that $X(E^t\beta_i) = CC(\beta_i)$. Also, it follows from 7.2 that $X(-\gamma) = CC(-\gamma)$. Now, $\beta_1 + \cdots + \beta_k$ is the (classical) generic decomposition of $\sum_{i=1}^k \beta_i$ so that

$$CC(\beta_1)\cdots CC(\beta_k) = CC(\beta_1 + \cdots + \beta_k).$$

Moreover, β_i and γ have disjoint support for any $i \in \{1, \ldots, k\}$ so that γ and $\sum_{i=1}^k \beta_i$ have disjoint support and $\mathrm{Ext}^1_{\mathcal{C}}(\sum_{i=1}^k \beta_i, -\gamma)$ vanishes generally. Thus,

$$CC(\sum_{i=1}^k \beta_i)CC(-\gamma) = CC(\sum_{i=1}^k \beta_i - \gamma)$$

which proves the theorem. $\square$

Theorem 7.4. *Let $\mathcal{C} = \mathcal{C}_Q$ be the cluster category of an acyclic quiver Q and $T = \boldsymbol{k}Q$ be the canonical cluster-tilting object in $\mathcal{C}$. Then,*

$$\mathcal{G}(Q) = \mathcal{G}^T(\mathcal{C}).$$

Proof. E^t is an invertible matrix so $E^t\alpha$ runs over $\mathbb{Z}^n$ when α runs over $\mathbb{Z}^n$. It thus follows from Theorem 7.3 that $\mathcal{G}(Q) \supset \mathcal{G}^T(\mathcal{C})$.

Conversely, if $\alpha \in \mathbb{Z}^n$, we can write $\alpha = \alpha_+ - \alpha_-$ where $\alpha_\pm \in \mathbb{N}^n$ have disjoint supports. Considering the classical canonical decomposition

$$\alpha_+ = \beta_1 + \cdots + \beta_k$$

we get $\alpha = \beta_1 + \cdots + \beta_k - \alpha_-$. For any $i = 1, \ldots, k$, consider a generic representation $M_i \in \mathrm{rep}(Q, \beta_i)$. It follows from [Dup08] that

$$CC(\alpha) = \prod_{i=1}^k CC(M_i)CC(P(\alpha_-)[1]).$$

Considering the minimal projective resolutions, we obtain triangles in $\mathcal{C}$

$$P_1^{M_i} \xrightarrow{p_{M_i}} P_0^{M_i} \longrightarrow M_i \longrightarrow P_1^{M_i}[1]$$

for any $i = 1, \ldots, k$. Moreover we have the triangle in $\mathcal{C}$

$$P(\alpha_-) \xrightarrow{0} 0 \longrightarrow P(\alpha_-)[1] \simeq P(\alpha_-)[1].$$

So, we get a triangle

$$\bigoplus_{i=1}^{k} P_1^{M_i} \oplus P(\alpha_-) \xrightarrow{f} \bigoplus_{i=1}^{k} P_0^{M_i} \longrightarrow \bigoplus_{i=1}^{k} M_i \oplus P(\alpha_-)[1] \longrightarrow \bigoplus_{i=1}^{k} P_1^{M_i}[1] \oplus P(\alpha_-)[1]$$

where $f = \bigoplus_{i=1}^{k} p_{M_i} \oplus 0$. By definition of X, we have

$$X(f) = X^T_{\mathrm{Cone}\,(f)} = \prod_{i=1}^{k} CC(M_i) CC(P(\alpha_-)[1]) = CC(\alpha).$$

But, it follows from [IOTW09, Theorem 6.3.1] that f is generic, thus

$$\begin{aligned} X(f) &= X(\mathrm{ind}_T(\bigoplus_{i=1}^{k} M_i) + \mathrm{ind}_T(P(\alpha_-)[1])) \\ &= X(\sum_{i=1}^{k} (E^t)\mathbf{dim}\, M_i + (E^t)\mathbf{dim}\, P(\alpha_-)[1]) \\ &= X(\sum_{i=1}^{k} (E^t)\beta_i - \alpha_-) \end{aligned}$$

and thus $\mathcal{G}^T(\mathcal{C}) \subset \mathcal{G}(Q)$. □

Combining with known results on classical generic variables, we get :

Corollary 7.5. *Let $\mathcal{C} = \mathcal{C}_Q$ be the cluster category of an acyclic quiver and $T = \boldsymbol{k}Q$ be the canonical cluster-tilting object in $\mathcal{C}$. Then the following hold :*

1. *$\mathcal{G}^T(\mathcal{C})$ is a $\mathbb{C}$-linear basis in $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}(Q, \boldsymbol{x})$;*
2. *if moreover Q_T is a Dynkin or an affine quiver, then $\mathcal{G}^T(\mathcal{C})$ is a $\mathbb{Z}$-linear basis in $\mathcal{A}(Q, \boldsymbol{x})$;*
3. *$\mathcal{G}^T(\mathcal{C}) = \mathcal{M}(Q, \boldsymbol{x}_T)$ if and only if Q_T is a Dynkin quiver.*

Proof. Let Q be an acyclic quiver. In [GLS10] the authors construct a $\mathbb{C}$-basis $\mathcal{S}^*(Q)$ of $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}(Q, \mathbf{x})$. According to [Sch09], $\mathcal{S}^*(Q)$ coincides with the set $\mathcal{G}(Q)$ of generic variables in $\mathcal{A}(Q, \mathbf{x})$ which coincides with $\mathcal{G}^T(\mathcal{C})$ according to Theorem 7.4. This proves the first point. The second point is a consequence of [Dup08, Theorem 4.20, Theorem 4.21] and [DXX09, Theorem 8.2].The third point is a consequence of [Dup08, Lemma 3.10]. □

More generally, we conjecture :

Conjecture 7.6. *Let $\mathcal{C}$ be a Hom-finite triangulated 2-Calabi-Yau category with constructible cones and let T be a cluster-tilting object in $\mathcal{C}$. We conjecture :*

1. *$\mathcal{G}^T(\mathcal{C})$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q_T, \boldsymbol{x})$;*
2. *if T' is another cluster-tilting object reachable from T, then $\mathcal{G}^T(\mathcal{C})$ coincides with $\mathcal{G}^{T'}(\mathcal{C})$ under the canonical isomorphism of cluster algebras $\mathcal{A}(Q_T, \boldsymbol{x}) \simeq \mathcal{A}(Q_{T'}, \boldsymbol{x}')$.*

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