Cluster-tilted algebras without clusters

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Abstract

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1 Introduction

Cluster-tilted algebras were introduced in [BMR] and also, independently, in [CCS] for type $\mathbb{A}$, as a by-product of the theory of cluster algebras of

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Fomin and Zelevinsky [FZ]. They are the endomorphism algebras of the so-called tilting objects in the cluster category of [BMRRT]. Since their introduction, they have been the subject of several investigations, see, for instance, [BMR,CCS,ABS1,KR,BFPPT,BOW]. Part of their interest comes from the fact that the cluster category is a 2-Calabi-Yau category. In particular, the representation theory of cluster-tilted algebras has been shown to be very similar to that of the self-injective algebras, see [ABS1,ABS2,ABS3]. One of the essential tools in the study of self-injective algebras is the notion of reflection of a tilted algebra, introduced by Hughes and Waschbüsch in [HW]. This allowed to prove that, if C is a tilted algebra, then its trivial extension $T(C)$ by the minimal injective cogenerator bimodule is representation-finite if and only if C is of Dynkin type and, in this case, $T(C) \cong T(B)$ if and only if B is an iterated reflection of C (or, equivalently, B is iterated tilted of the same type as C), see also [BLR,AHR,Ho]. Moreover, the proofs of these results developed into algorithms allowing to compute explicitly the module category of $T(C)$, starting from that of C, see [HW,BLR].

We recall from [ABS1] that, if C is a tilted algebra, then the trivial extension $\tilde{C}$ of C by the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$ is cluster-tilted, and conversely, every cluster-tilted algebra is of this form. On the other hand, this (surjective) map from tilted algebras to cluster-tilted algebras is certainly not injective and it is an interesting question to find all the tilted algebras B such that $\tilde{B} = \tilde{C}$. This problem has already been considered in [ABS2] and [BOW], see also [BFPPT]. In the present paper, we define notions of reflections (and, dually coreflections) of complete slices and of tilted algebras. Our main result may now be stated as follows.

Theorem 1 *Let C be a tilted algebra having a tree Σ as a complete slice. A tilted algebra B is such that $\tilde{B} = \tilde{C}$ if and only if there exists a sequence of reflections and coreflections $\sigma_1, \ldots, \sigma_t$ such that $B = \sigma_1 \cdots \sigma_t C$ has $\Omega = \sigma_1 \cdots \sigma_t \Sigma$ as a complete slice and $B = \tilde{C}/\mathrm{Ann}\,\Omega$.*

The restriction to tilted algebras of tree type seems to be necessary to ensure the existence of reflections.

As a consequence of this construction and our proof, we obtain, as in [HW], an algorithm allowing to compute explicitly the transjective component of the module category of $\tilde{C}$, having as starting data only the knowledge of the tilted algebra C. In particular, if C is of Dynkin type, this yields the whole module category of $\tilde{C}$. We observe that, since the transjective component of the module category of $\tilde{C}$ is standard, then it is uniquely determined by combinatorial data.

The paper is organised as follows. After a short preliminary section, in which we fix the notation and recall the needed results, we devote our section 3 to general properties of the Auslander-Reiten quiver of a cluster-tilted algebra. In section 4, we define reflections of complete slices and of tilted algebras. Section 5 is devoted to the proof of our main results, and section 6 to the algorithm. We end the paper in section 7 by showing how our algorithm may be applied to construct the tubes of cluster-tilted algebras of euclidean type.

2 Preliminaries

2.1 Notation

Throughout this paper, algebras are basic and connected, locally finite dimensional over an algebraically closed field k. For an algebra C, we denote by $\operatorname{mod} C$ the category of finitely generated right C-modules. All subcategories are full and so are identified with their object classes. Given a category $\mathcal{C}$, we sometimes write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. If $\mathcal{C}$ is a full subcategory of $\operatorname{mod} C$, we denote by $\operatorname{add} \mathcal{C}$ the full subcategory of $\operatorname{mod} C$ having as objects the finite direct sums of summands of modules in $\mathcal{C}$.

Following [BG], we sometimes consider equivalently an algebra C as a locally bounded k-category, in which the object class C_0 is (in bijection with) a complete set $\{e_x\}$ of primitive orthogonal idempotents of C, and the space of morphisms from e_x to e_y is $C(x, y) = e_x C e_y$. A full subcategory B of C is *convex* if, for any path $x = x_0 \to x_1 \to \ldots \to x_t = y$ in the quiver Q_C of C, with $x, y \in B$, we have $x_i \in B$ for all i. For a point x in Q_C, we denote by P_x, I_x, S_x respectively the indecomposable projective, injective and simple C-modules corresponding to x. We denote by $\Gamma(\operatorname{mod} C)$ the Auslander-Reiten quiver of C and by $\tau_C = D\,Tr, \tau_C^{-1} = Tr\,D$ the Auslander-Reiten translations. Given two points M, N in $\Gamma(\operatorname{mod} C)$, we denote by $M \rightsquigarrow N$ or by $M \leq N$ the existence of a path (of non-zero morphisms between indecomposable modules) from M to N in $\operatorname{mod} C$. More generally, if $\mathcal{S}_1, \mathcal{S}_2$ are two sets of indecomposable modules, we write $\mathcal{S}_1 \leq \mathcal{S}_2$ if every module in $\mathcal{S}_1$ has a successor in $\mathcal{S}_2$, no module in $\mathcal{S}_2$ has a successor in $\mathcal{S}_1$, and no module in $\mathcal{S}_1$ has a predecessor in $\mathcal{S}_2$. The notation $\mathcal{S}_1 < \mathcal{S}_2$ stands for $\mathcal{S}_1 \leq \mathcal{S}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.

For further definitions and facts, we refer the reader to [ARS,ASS]. For tilting theory, we refer to [ASS,Ri].

2.2 Cluster-tilted algebras

Let A be a finite dimensional hereditary k-algebra, The *cluster category* $\mathcal{C}_A$ of A is defined as follows. Let F be the automorphism of the bounded derived category $\mathcal{D}^b(\operatorname{mod} A)$ defined as the composition $\tau_{\mathcal{D}}^{-1}[1]$, where $\tau_{\mathcal{D}}^{-1}$ is the Auslander-Reiten translation in $\mathcal{D}^b(\operatorname{mod} A)$ and $[1]$ is the shift (suspension) functor. Then $\mathcal{C}_A$ is the orbit category $\mathcal{D}^b(\operatorname{mod} A)/F$, its objects are the F-orbits $\tilde{X} = (F^i X)_{i\in\mathbb{Z}}$ of the objects $X \in \mathcal{D}^b(\operatorname{mod} A)$ and the space of morphisms from $\tilde{X} = (F^i X)_i$ to $\tilde{Y} = (F^i Y)_i$ is

$$\operatorname{Hom}_{\mathcal{C}_A}(\tilde{X}, \tilde{Y}) = \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(X, F^i Y).$$

$\mathcal{C}_A$ is a triangulated Krull-Schmidt category with almost split triangles. The projection $\pi : \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{C}_A$ is a triangle functor which commutes with the Auslander-Reiten translations [BMRRT,K]. Moreover, for any two objects $\tilde{X}, \tilde{Y}$ in $\mathcal{C}_A$, we have a functorial isomorphism $\operatorname{Ext}^1_{\mathcal{C}_A}(\tilde{X}, \tilde{Y}) \cong D\operatorname{Ext}^1_{\mathcal{C}_A}(\tilde{Y}, \tilde{X})$, in other words, the category $\mathcal{C}_A$ is 2-Calabi-Yau.

An object $\tilde{T} \in \mathcal{C}_A$ is *tilting* if $\operatorname{Ext}^1_{\mathcal{C}_A}(\tilde{T}, \tilde{T}) = 0$, and the number of isomorphism classes of indecomposable summands of $\tilde{T}$ equals the rank of the Grothendieck group $K_0(A)$ of A. The endomorphism algebra $B = \operatorname{End}_{\mathcal{C}_A} \tilde{T}$ is then called *cluster-tilted*. Moreover, we have an equivalence $\operatorname{mod} B \cong \mathcal{C}_A/\operatorname{iadd}(\tau_{\mathcal{C}_A}\tilde{T})$, where $\tau_{\mathcal{C}_A}$ is the Auslander-Reiten translation in $\mathcal{C}_A$ and $\operatorname{iadd}(\tau_{\mathcal{C}_A}\tilde{T})$ is the ideal of $\mathcal{C}_A$ consisting of all morphisms factoring through objects of $\operatorname{add}(\tau_{\mathcal{C}_A}\tilde{T})$. Also, this equivalence commutes with the Auslander-Reiten translations in both categories [BMR].

We now describe the Auslander-Reiten quivers of $\mathcal{C}_A$ and B. If $A = kQ$ is representation-finite, the $\Gamma(\mathcal{C}_A)$ is of the form $\mathbb{Z}Q/\langle\varphi\rangle$, where φ is the automorphism of $\mathbb{Z}Q$ induced by F. If $A = kQ$ is representation infinite, then $\Gamma(\mathcal{C}_A)$ has a unique component of the form $\mathbb{Z}Q$, called *transjective*, because it is the image (under π) of the transjective components of $\Gamma(\mathcal{D}^b(\operatorname{mod} A))$. Moreover, $\Gamma(\mathcal{C}_A)$ also has components called *regular*, because they are the image of the regular components of $\Gamma(\mathcal{C}_A)$. In both cases, we deduce $\Gamma(\operatorname{mod} B)$ from $\Gamma(\mathcal{C}_A)$ by simply deleting the $|Q_0|$ points corresponding to the summands of $\tau_{\mathcal{C}_A}\tilde{T}$.

2.3 Relation-extensions and slices

If B is cluster-tilted, then there exists a hereditary algebra A and a tilting A-module T such that $B = \operatorname{End}_{\mathcal{C}_A}\tilde{T}$, see [BMRRT, 3.3]. Moreover, if $C = \operatorname{End}_A T$ is the corresponding tilted algebra, then the trivial extension $\tilde{C} = C \ltimes \operatorname{Ext}^2_C(DC, C)$ (the *relation-extension* of C) is cluster-tilted and, actually, isomorphic to B, see [ABS1]. Now, tilted algebras are characterised by the presence of so-called complete slices in the connecting components of their Auslander-Reiten quivers [ASS,Ri]. The corresponding notion for cluster-tilted algebras is as follows [ABS2]. A full subquiver Σ of $\Gamma(\text{mod}\,\tilde{C})$ is a *local slice* if :

(LS1) Σ is a presection, that is
  (a) If $X \in \Sigma$ and $X \to Y$ is an arrow, then either $Y \in \Sigma$ or $\tau_{\tilde{C}} Y \in \Sigma$.
  (b) If $Y \in \Sigma$ and $X \to Y$ is an arrow, then either $X \in \Sigma$ or $\tau_{\tilde{C}}^{-1} X \in \Sigma$.

(LS2) Σ is sectionally convex, that is, if $X = X_0 \to X_1 \to \cdots \to X_t = Y$ is a sectional path in $\Gamma(\text{mod}\,\tilde{C})$, with $X, Y \in \Sigma$, then $X_i \in \Sigma$ for all i.

(LS3) $|\Sigma_0| = \text{rk} K_0(C)$.

Let C be tilted, then, under the standard embedding $\text{mod}\,C \to \text{mod}\,\tilde{C}$ any complete slice in $\text{mod}\,C$ embeds as a local slice in $\text{mod}\,\tilde{C}$, and any local slice occurs in this way. If B is cluster-tilted, then a tilted algebra C is such that $B = \tilde{C}$ if and only if there exists a local slice Σ in $\Gamma(\text{mod}\,B)$ such that $C = B/\text{Ann}_B\Sigma$, where $\text{Ann}_B\Sigma = \cap_{X\in\Sigma}\text{Ann}_B X$, see [ABS2].

2.4 Cluster-repetitive algebras

Let C be a tilted algebra. Its *cluster-repetitive algebra* $\check{C}$ is the locally finite dimensional algebra given by

$$\check{C} \quad = \quad \begin{bmatrix} \ddots & & & 0 & \\ & C_{-1} & & & \\ & E_0 & C_0 & & \\ & & E_1 & C_1 & \\ & 0 & & & \ddots \end{bmatrix}$$

where matrices have only finitely many non-zero coefficients, $C_i = C$ and $E_i = \text{Ext}^2_C(DC, C)$ for all $i \in \mathbb{Z}$, all the remaining coefficients are zero, and the multiplication is induced from that of C, the C-C-bimodule structure of $\text{Ext}^2_C(DC, C)$ and the zero map $\text{Ext}^2_C(DC, C) \otimes_C \text{Ext}^2_C(DC, C) \to 0$. The identity maps $C_i \to C_{i-1}$, $E_i \to E_{i-1}$ induce an automorphism φ of $\check{C}$. The orbit category $C/\langle\varphi\rangle$ is isomorphic to $\tilde{C} = C \ltimes \text{Ext}^2_C(DC, C)$. The projection $G : \check{C} \to \tilde{C}$ is thus a Galois covering with infinite cyclic group generated by φ. It is shown in [ABS3] that the corresponding pushdown functor $\text{mod}\,\check{C} \to \text{mod}\,\tilde{C}$ is always dense, so it induces an isomorphism $\Gamma(\text{mod}\,\tilde{C}) \cong \Gamma(\text{mod}\,\check{C})/\mathbb{Z}$. Also, if $C = \text{End}\,_A T$, where T is a tilting module over the hereditary algebra A, then $\operatorname{mod}\check{C} \cong \mathcal{D}^b(\operatorname{mod} A)/\operatorname{iadd}(\tau_{\mathcal{D}}F^iT)_{i\in\mathbb{Z}}$, where $\tau_{\mathcal{D}}$ is the Auslander-Reiten translation in $\mathcal{D}^b(\operatorname{mod} A)$ and $\operatorname{iadd}(\tau_{\mathcal{D}}F^iT)_{i\in\mathbb{Z}}$ is the ideal of $\mathcal{D}^b(\operatorname{mod} A)$ consisting of all morphisms which factor through $\operatorname{add}(\tau_{\mathcal{D}}F^iT)_{i\in\mathbb{Z}}$. Finally, every local slice in $\Gamma(\operatorname{mod}\tilde{C})$ is the image under G_λ of (several) local slices in $\Gamma(\operatorname{mod}\check{C})$ (that is, full subquiver of $\Gamma(\operatorname{mod}\check{C})$ satisfying the axioms (LS1),(LS2),(LS3) of (2.4) above). Throughout this paper, we identify C_0 with C, and thus any complete slice of $\operatorname{mod} C$ can be considered as a local slice in $\operatorname{mod}\check{C}$.

3 Properties of the Auslander-Reiten quiver of a cluster-tilted algebra

3.1

In this section, we let C be a tilted algebra, having Σ as a complete slice, and $\tilde{C} = C \ltimes \operatorname{Ext}^2_C(DC, C)$ be its relation extension. The following lemma is borrowed from [ADLS]; we include the proof for the convenience of the reader.

Lemma 2 *Let C be a tilted algebra, Σ a complete slice in $\operatorname{mod} C$ and $M \in \Sigma$, then we have:*

(a) $M \otimes_C \operatorname{Ext}^2_C(DC, C) = 0$, *and*
(b) $\operatorname{Hom}_C(\operatorname{Ext}^2_C(DC, C), \tau_C M) = 0$.

PROOF. (a) Let $A = \operatorname{End}(\oplus_{X\in\Sigma} X)$ and T_A be a tilting module such that $C = \operatorname{End} T_A$. Since $M \in \Sigma$, there exists an injective A-module I such that $M_C \cong \operatorname{Hom}_A(T, I)$. Using standard functorial isomorphisms, we have:

$$\begin{aligned}
D(M \otimes_C \operatorname{Ext}^2_C(DC, C)) &\cong \operatorname{Hom}_C(M, D\operatorname{Ext}^2_C(DC, C)) \\
&\cong \operatorname{Hom}_C\Big(\operatorname{Hom}_A(T, I), D\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(T, FT)\Big) \\
&\cong \operatorname{Hom}_C\Big(\operatorname{Hom}_A(T, I), D\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(T, \tau^{-1}T[1])\Big) \\
&\cong \operatorname{Hom}_C\Big(\operatorname{Hom}_A(T, I), D\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\tau T, T[1])\Big) \\
&\cong \operatorname{Hom}_C\Big(\operatorname{Hom}_A(T, I), D\operatorname{Ext}^1_{\mathcal{D}^b(\operatorname{mod} A)}(\tau T, T)\Big) \\
&\cong \operatorname{Hom}_C\Big(\operatorname{Hom}_A(T, I), \operatorname{Hom}_A(T, \tau^2 T)\Big) \\
&\cong \operatorname{Hom}_A(I, t(\tau^2 T)),
\end{aligned}$$

where $t(\tau^2 T) \cong \operatorname{Hom}_A(T, \tau^2 T) \otimes_C T$ is the torsion part of the A-module $\tau^2 T$ in the torsion pair induced by T in $\operatorname{mod} A$. Since $\tau^2 T$ is not an injective

A-module, neither is its submodule $t(\tau^2 T)$. Since A is hereditary, and I is injective, we get $\mathrm{Hom}_A(I, \tau^2 T) = 0$.

(b) Since $\tau_C M$ precedes the complete slice Σ in $\mathrm{mod}\, C$, it suffices to prove that $\mathrm{Ext}^2_C(DC, C)$ succedes it. Note first that

$$\begin{aligned}\mathrm{Ext}^2_C(DC, C) &\cong \mathrm{Ext}^1_C(DC, \Omega^{-1}C) \\ &\cong D\underline{\mathrm{Hom}}_C(\tau^{-1}\Omega^{-1}C, DC),\end{aligned}$$

using the first cosyzygy $\Omega^{-1}C$ of C and the Auslander-Reiten formula. Now notice that for every indecomposable summand X of $\Omega^{-1}C$, there exists an injective C-module J such that $\mathrm{Hom}_C(J, X) \neq 0$. But all injectives are successors of Σ, so there exists $L \in \Sigma$ such that we have a path $L \to J \to X \to * \to \tau^{-1}X$. This shows that every indecomposable summand of $\tau^{-1}\Omega^{-1}C$ succedes (properly) the slice Σ. Since no indecomposable projective module is a successor of Σ, we get

$$\underline{\mathrm{Hom}}_C(\tau^{-1}\Omega^{-1}C, DC) = \mathrm{Hom}_C(\tau^{-1}\Omega^{-1}C, DC).$$

Hence

$$\mathrm{Ext}^2_C(DC, C)_C \cong D\mathrm{Hom}_C(\tau^{-1}\Omega^{-1}C, DC) \cong \tau^{-1}\Omega^{-1}C_C.$$

But as we have already shown, every indecomposable summand of $\tau^{-1}\Omega^{-1}C_C$ is a (proper) successor of Σ. The required statement follows at once. □

3.2

Proposition 3 *Let C be a tilted algebra, Σ be a complete slice in* mod C *and $M \in \Sigma$. Then:*

(a) $\tau_C M \cong \tau_{\tilde{C}} M$, *and*
(b) $\tau_C^{-1} M \cong \tau_{\tilde{C}}^{-1} M$.

PROOF. Part (a) follows directly from Lemma 2 and the main result of [AZ]. Part (b) follows by duality. □

3.3

We need to apply Proposition 3 also to the cluster repetitive algebra $\check{C}$ of C.

Corollary 4 *Let C be a tilted algebra, Σ be a complete slice in* mod C *and $M \in \Sigma$. Then:*

(a) $\tau_C M \cong \tau_{\overset{\vee}{C}} M$,
(b) $\tau_C^{-1} M \cong \tau_{\overset{\vee}{C}}^{-1} M$. □

3.4

For the next lemma, we need some notations: let A be a hereditary algebra, T be a tilting A-module such that $\mathrm{End}\,_A T = C$ and $\mathrm{End}\,_{\mathcal{C}_A} T = \tilde{C}$ (where $\mathcal{C}_A$ denotes the cluster category associated to A). Let also $\tilde{P}_x, \tilde{I}_x$ and T_x be the indecomposable projective $\tilde{C}$-module, the indecomposable injective $\tilde{C}$-module and the indecomposable summand of T corresponding to an object x in $\tilde{C}$.

Lemma 5 *With the above notation:*

(a) *For every object x in $\tilde{C}$, we have* $\mathrm{Hom}\,_{\mathcal{C}_A}(T, \tau^2 T_x) \cong \tilde{I}_x$.
(b) *For every pair of objects x, y in $\tilde{C}$, we have an isomorphism of the spaces of irreducible morphisms* $\mathrm{Irr}_{\tilde{C}}(\tilde{P}_x, \tilde{P}_y) \cong \mathrm{Irr}_{\tilde{C}}(\tilde{I}_x, \tilde{I}_y)$.

PROOF. Using standard functorial isomorphisms we have:

(a)
$$\begin{aligned}\tilde{I}_x &\cong D\mathrm{Hom}\,_{\mathcal{C}_A}(T_x, T)\\ &\cong D\mathrm{Hom}\,_{\mathcal{D}^b(\mathrm{mod}\,A)}(T_x, T) \oplus D\mathrm{Hom}\,_{\mathcal{D}^b(\mathrm{mod}\,A)}(T_x, \tau^{-1}T[1])\\ &\cong \mathrm{Ext}^1_{\mathcal{D}^b(\mathrm{mod}\,A)}(T, \tau T_x) \oplus D\mathrm{Ext}^1_{\mathcal{D}^b(\mathrm{mod}\,A)}(T_x, \tau^{-1}T)\\ &\cong \mathrm{Hom}\,_{\mathcal{D}^b(\mathrm{mod}\,A)}(T, \tau T[1]) \oplus \mathrm{Hom}\,_{\mathcal{D}^b(\mathrm{mod}\,A)}(T, \tau^2 T_x)\\ &\cong \mathrm{Hom}\,_{\mathcal{C}_A}(T, \tau^2 T_x).\end{aligned}$$

(b)
$$\begin{aligned}\mathrm{Irr}_{\tilde{C}}(\tilde{P}_x, \tilde{P}_y) &\cong \mathrm{Irr}_{\mathcal{C}_A}(T_x, T_y)\\ &\cong \mathrm{Irr}_{\mathcal{C}_A}(\tau^2 T_x, \tau^2 T_y)\\ &\cong \mathrm{Irr}_{\tilde{C}}\Big(\mathrm{Hom}\,_{\mathcal{C}_A}(T, \tau^2 T_x), \mathrm{Hom}\,_{\mathcal{C}_A}(T, \tau^2 T_y)\Big)\\ &\cong \mathrm{Irr}_{\tilde{C}}(\tilde{I}_x, \tilde{I}_y),\end{aligned}$$

where we have used the category equivalence $\mathrm{Hom}\,_{\mathcal{C}_A}(T, -) : \mathcal{C}_A/\mathrm{iadd}(\tau T) \to \mathrm{mod}\,\tilde{C}$ of [BMR], and part (a) above. □

Remark 6 *Statement (b) above does not hold true in general, even for tilted algebras. Let indeed C be given by the quiver*

$$1 \xleftarrow{\gamma} 2 \xleftarrow{\beta} 3 \xleftarrow{\alpha} 4$$

bound by $\alpha\beta = 0$. Note that $\mathrm{Irr}_C(I_1, I_2) = 0$ *while* $\mathrm{Irr}_C(P_1, P_2) = k$.

4 Reflections

4.1

The objective of this section is to define a notion of reflection on a local slice in a cluster-tilted algebra. This will in turn induce a notion of reflection on a tilted subalgebra of the given cluster-tilted algebra.

Let, as before, C be a tilted algebra, $\tilde{C} = C \ltimes \mathrm{Ext}^2_C(DC, C)$ its relation-extension algebra and $\check{C}$ its cluster repetitive algebra. We still identify C with the full convex subcategory C_0 of $\check{C}$. We assume throughout that C is of tree type.

Let Γ be a connecting component of $\mathrm{mod}\, C$, and Σ be a complete slice in Γ.

Assume first that $M \in \Sigma$ is a source in Σ which is not injective, then $(\Sigma \setminus \{M\}) \cup \{\tau_C^{-1} M\}$ is also a complete slice in Γ. In the language of [BOW], these two slices are *homotopic*. Homotopy is clearly an equivalence relation on slices, and there are either one or two equivalence classes in $\mathrm{mod}\, C$ (two if and only if C is concealed). We need distinguished representatives of these classes. If there exists a complete slice in which all sources are injective C-modules, then such a slice is unique and is called the *rightmost slice* of $\mathrm{mod}\, C$. We denote it as Σ^+. Dually, we define the *leftmost slice* Σ^- of $\mathrm{mod}\, C$. Note that, if C is representation-finite, then rightmost and leftmost slices exist.

We recall from [HW] that a point $x \in C_0$ is a *strong sink* if the injective module I_x has no injective module as a proper predecessor in $\mathrm{mod}\, C$. Clearly, strong sinks are sinks. The following Lemma is obvious.

Lemma 7 *A point $x \in C_0$ is a strong sink if and only if I_x is an injective source of the rightmost slice Σ^+.*

PROOF. Assume first that I_x is an injective source of Σ^+. If x is not a strong sink, then there exists $y \neq x$ in C such that we have a path $I_y \rightsquigarrow I_x$. Since Σ^+ is sincere, there exists $M \in \Sigma^+$ and a morphism $M \to I_y$ yielding a path $M \to I_y \to I_x$. Since Σ^+ is convex in $\mathrm{ind}\, C$, we get $I_y \in \Sigma^+$ which contradicts the hypothesis that I_x is a source in Σ^+.

Conversely, assume x to be a strong sink in C, and suppose that I_x is not an injective source of Σ^+. Because Σ^+ is sincere, then there exist $N \in \Sigma^+$ and a morphism $N \to I_x$. Now there exists a source (necessarily injective) I_z in Σ^+ and a path $I_z \rightsquigarrow N$ in Σ^+. This yields a path $I_z \rightsquigarrow N \to I_x$, contrary to the hypothesis. □

4.2 The completion G_x

Let x be a strong sink in C. We define the *completion* G_x of x in Σ^+ to be a non-empty full connected subquiver of Σ^+ such that

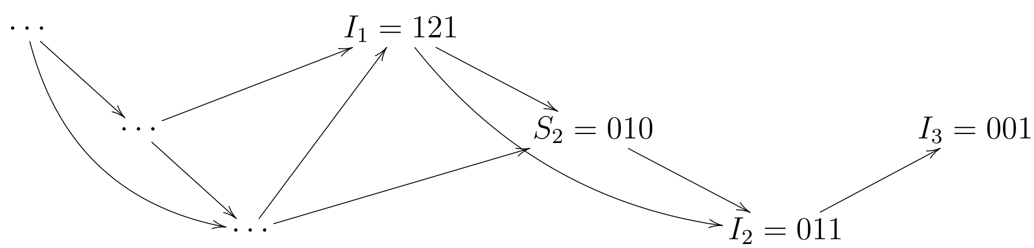
(a) $I_x \in G_x$,
(b) G_x is closed under predecessors in Σ^+,
(c) If $I \to M$ is an arrow in Σ^+, with $I \in G_x$ injective, then $M \in G_x$,
(d) If $N \to I$ is an arrow in Σ^+, with $I \in G_x$ injective, then N is injective (and in G_x).

Completions do not always exist.

Example 8 *The tilted algebra C given by the quiver*

$$1 \underset{\gamma}{\overset{\beta}{\leftleftarrows}} 2 \xleftarrow{\alpha} 3$$

bound by $\alpha\,\beta = 0$ admits the complete rightmost slice consisting of the modules I_1, S_2 and I_2, and I_1 is the only source. A part of the Auslander-Reiten quiver of mod C *containing this slice is shown below, where modules are represented by their dimension vectors.*



In this example G_1 does not exist, because by condition (c) it would contain both S_2 and I_2, and this contradicts condition (d).

The tilted algebra C in the example above is of euclidean type $\tilde{\mathbb{A}}_2$, so it is not of tree type. The following Lemma guarantees the existence of some completion in a rightmost slice, if the tilted algebra is of tree type.

Lemma 9 *Let C be a tilted algebra of tree type having a rightmost slice Σ^+. Then there exists a strong sink x in C such that the completion G_x exists.*

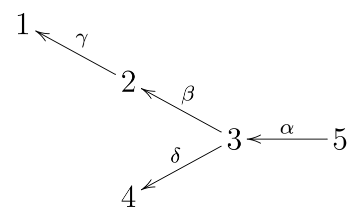
PROOF. Let I_{x_1} be a source in Σ^+ and G_1' its closure under condition (c) above, then let G_1 be the closure of G_1' under condition (b).

If G_1 satisfies condition (d), then we are done. Otherwise there exist an injective $I \in G_1$ and an arrow $N \to I$ in Σ^+ with N not injective. Then there exists a sectional path in Σ^+ ending at N. Let I_{x_2} be the source of such a path.

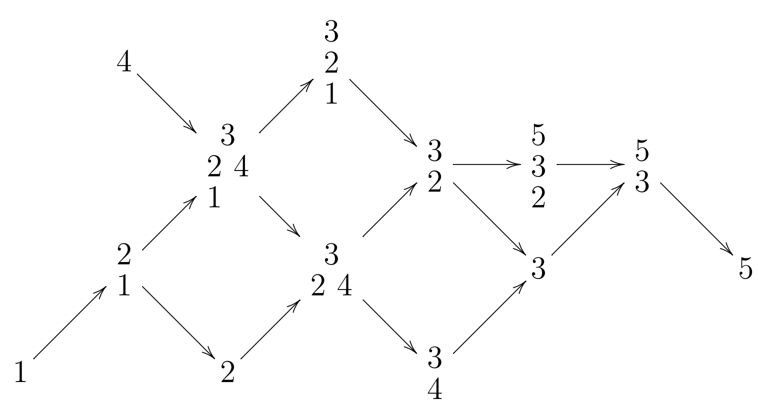
Let G_2' be the closure of I_{x_2} under condition (c), and then let G_2 be the closure of G_2' under condition (b). Clearly, G_2' does not contain the injective I, since there is an arrow $N \to I$ in the sectional path, with N non-injective. Using that Σ^+ is a tree, we see that $I_{x_1} \notin G_2$.

If G_2 satisfies condition (d), then we are done. Otherwise we repeat the procedure. Since Σ^+ is a tree, this procedure must ultimately stop. $\square$

Example 10 *Let C be the tilted algebra of tree type $\mathbb{D}_5$ given by the quiver*



bound by $\alpha\,\beta\,\gamma = 0$ and $\alpha\,\delta = 0$. Its Auslander-Reiten quiver is shown below.



(here, modules are represented by their composition factors). The rightmost slice

$$\left\{ \begin{matrix}3\\2\\1\end{matrix},\ \begin{matrix}3\\2\end{matrix},\ \begin{matrix}3\\4\end{matrix},\ \begin{matrix}5\\3\\2\end{matrix},\ 3 \right\}$$

in this example has the two injective sources: I_1 and I_4. We have

$$G_1 = \left\{ \begin{matrix}3\\2\\1\end{matrix},\ \begin{matrix}3\\2\end{matrix} \right\} \quad \text{and} \quad G_4 = \left\{ \begin{matrix}3\\2\\1\end{matrix},\ \begin{matrix}3\\2\end{matrix},\ \begin{matrix}3\\4\end{matrix},\ 3 \right\}.$$

4.3 The reflection of a slice

Let now x be a sink in C such that the completion G_x exists. We then say that x is an *admissible sink*. We are now able to define the reflection $\Sigma' = \sigma_x^+\Sigma^+$ of the complete slice Σ^+. The set of objects in G_x is of the form $\mathcal{J} \sqcup \mathcal{M}$, where $\mathcal{J}$ and $\mathcal{M}$ consist respectively of the injective, and the non-injectives in G_x. Let $\mathcal{P} = \{P_x \in \operatorname{mod} C_1 \mid I_x \in \mathcal{J}\}$, where we recall that C_1 is the copy of C next to C_0 on the diagonal blocks of $\check{C}$. We then set

$$\sigma_x^+\Sigma^+ = \Big(\Sigma^+ \setminus G_x\Big) \cup \mathcal{P} \cup \tau_{\check{C}}^{-1}\mathcal{M}.$$

Recall that, by Corollary 4, $\tau_{\check{C}}^{-1}M \cong \tau_C^{-1}M$ for every $M \in \Sigma^+$.

Lemma 11 *$\sigma_x^+\Sigma^+$ is a local slice in* $\operatorname{mod}\check{C}$*.*

PROOF. We first consider in the cluster category $\mathcal{C}_A$ the full subquiver defined by:

$$\Sigma'' = \Big(\Sigma^+ \setminus G_x\Big) \cup \tau_C^{-1}\mathcal{M} \cup \tau_{\mathcal{C}_A}^{-1}\mathcal{I}.$$

Thus Σ'' is a local slice in $\mathcal{C}_A$ because G_x is closed under predecessors and we have $\Sigma' = (\Sigma'' \setminus \tau_{\mathcal{C}_A}^{-1}\mathcal{I}) \sqcup \mathcal{P}$.

We claim that Σ' is connected. The objects lying in Σ' and Σ'' are in one-to-one correspondence, since any object of Σ' is either an object of Σ'' or the Auslander-Reiten translate of an object in Σ''. Hence it is enough to show that whenever there is an arrow between M'', N'' in Σ'', then there is an arrow between the two corresponding objects M', N' in Σ'.

Because of Lemma 5 (b), we only need to consider the case where $M'' \in (\Sigma^+ \setminus G_x) \cup \tau_C^{-1}\mathcal{M}$ and $N'' \in \tau_{\mathcal{C}_A}^{-1}\mathcal{I}$. Thus $M' = M''$ and $N' = \tau_{\mathcal{C}_A}^{-1}N'' = \tau_{\mathcal{C}_A}^{-2}I$ for some $I \in \mathcal{I} \subset G_x$.

Either we have $M'' \to N''$ or $N'' \to M''$ in Σ''. In the latter case, there is an arrow from $(M' = M'')$ to $(N' = \tau_{\mathcal{C}_A}^{-1}N'')$ in Σ', and we are done. On the other hand, if $M'' \to N''$, then there is an arrow $\tau_{\mathcal{C}_A}N'' \to M''$ with $\tau_{\mathcal{C}_A}N'' = I \in G_x$ injective, and thus $M' \in G_x$, by condition (c) of the completion G_x. This establishes our claim.

Consequently, Σ' may be identified to a local slice in $\mathcal{D}^b(\operatorname{mod} C)$. Since, furthermore, Σ' consists of $\check{C}$-modules then, by [ABS3], σ' is a local slice in $\operatorname{mod}\check{C}$. $\square$

4.4 A hereditary subcategory

We deduce from our definition of reflection of Σ^+ a definition of reflection of the tilted algebra C, which we denote by $\sigma_x^+ C$.

Define $\mathcal{S}_x$ to be the full subcategory of C consisting of the objects y such that $I_y \in G_x$.

Lemma 12 *With the above notation*

(a) $\mathcal{S}_x$ *is hereditary,*
(b) $\mathcal{S}_x$ *is closed under successors in* C*,*
(c) C *may be written in the form*

$$C = \begin{bmatrix} H & 0 \\ M & C' \end{bmatrix}$$

with H *hereditary,* C' *tilted and* M *a* C'-H*-bimodule.*

PROOF. (a) We let $H = \operatorname{End}(\oplus_{y \in \mathcal{S}_x} I_y)$. Then H is a full subcategory of the hereditary algebra $A = \operatorname{End}(\oplus_{X \in \Sigma^+} X)$. Therefore H is also hereditary, that is, $\mathcal{S}_x$ is hereditary.

(b) Let $y \in \mathcal{S}_x$ and $y \to z$ be an arrow in C. Then there exists a morphism $I_z \to I_y$. Since I_z is an injective C-module and Σ^+ is sincere, there exist $N \in \Sigma^+$ and a morphism $N \to I_z$. Thus we have $N \to I_z \to I_y$. Since $N, I_y \in \Sigma^+$ and Σ^+ is convex in $\operatorname{mod} C$, then $I_z \in \Sigma^+$ and so $z \in \mathcal{S}_x$.

(c) This follows at once from (a) and (b). □

4.5 The structure of the cluster duplicated algebra

We recall from [ABS3] that the cluster duplicated algebra $\overline{C}$ of C is the (finite dimensional) matrix algebra

$$\overline{C} = \begin{bmatrix} C & 0 \\ \operatorname{Ext}^2_C(DC, C) & C \end{bmatrix}$$

with the ordinary matrix addition and the multiplication induced from that of C and from the C-C-bimodule structure of $\operatorname{Ext}^2_C(DC, C)$. Clearly, $\overline{C}$ is useful as a "building block" for the cluster repetitive algebra $\check{C}$.

Corollary 13 *The cluster duplicated algebra of C is of the form*

$$\overline{C} = \begin{bmatrix} H & 0 & 0 & 0 \\ M & C' & 0 & 0 \\ 0 & F_0 & H & 0 \\ 0 & F_1 & M & C' \end{bmatrix},$$

where $F_0 = \mathrm{Ext}^2_C(DC', H)$ *and* $F_1 = \mathrm{Ext}^2_C(DC', C')$.

PROOF. We start by writing C in the matrix form of Lemma 12 (c). Since, by definition, H consists of the objects y in C such that $I_y \in G_x \subset \Sigma^+$, then the projective dimension $\mathrm{pd}_C DH$ is at most 1, hence $\mathrm{Ext}^2_C(DH, -) = 0$. The result follows upon multiplying by idempotents. □

4.6 The reflection of a tilted algebra

We can now define the *reflection* $\sigma_x^+ C$ of C to be the matrix algebra

$$\sigma_x^+ C = \begin{bmatrix} C' & 0 \\ F_0 & H \end{bmatrix},$$

where $F_0 = \mathrm{Ext}^2_C(DC', H)$. Note that $\sigma_x^+ C$ is a quotient algebra of $\check{C}$.

We now prove that this definition is compatible with the definition of reflection of local slices. We recall that the *support* $\mathrm{Supp}\,\mathcal{X}$ of a subclass $\mathcal{X}$ of $\check{C}$ is the full subcategory of $\check{C}$ having as objects the x in $\check{C}$ such that there exists a module $M \in \mathcal{X}$ satisfying $M(x) \neq 0$.

Proposition 14 *The reflection $\sigma_x^+ C$ is a tilted algebra having $\sigma_x^+ \Sigma^+$ as a complete slice. Moreover, the cluster-tilted algebras of C and $\sigma_x^+ C$ and the cluster repetitive algebras of C and $\sigma_x^+ C$ are isomorphic.*

PROOF. It follows directly from the definition of $\sigma_x^+ \Sigma^+$ that $\mathrm{Supp}\,(\sigma_x^+ \Sigma^+) \subset \sigma_x^+ C$. Indeed, in the notation of Lemma 11, we have $\sigma_x^+ \Sigma^+ = (\Sigma^+ \setminus G_x) \cup \mathcal{P} \cup \tau_{\check{C}}^{-1} \mathcal{M}$. Since, as observed before, $\tau_{\check{C}}^{-1} \mathcal{M} \cong \tau_C^{-1} \mathcal{M}$ by Corollary 4, and the injectives in $\mathcal{I}$ are replaced by the projectives in $\mathcal{P}$, then we get the wanted inclusion.

Now, as shown in Lemma 11, $\sigma_x^+\Sigma^+$ is a local slice in $\operatorname{mod} \check{C}$. Denoting by $G_\lambda : \operatorname{mod} \check{C} \to \operatorname{mod} \tilde{C}$ the pushdown functor associated to the Galois covering $G : \check{C} \to \tilde{C}$, we get that $G_\lambda(\sigma_x^+\Sigma^+)$ is a local slice in $\operatorname{mod} \tilde{C}$. By [ABS2], $C^* = \tilde{C}/\operatorname{Ann}(G_\lambda(\sigma_x^+\Sigma^+))$ is a tilted algebra of the same type as C. Moreover we have $\tilde{C} = C \ltimes \operatorname{Ext}^2_C(DC, C) \cong C^* \ltimes \operatorname{Ext}^2_{C^*}(DC^*, C^*)$ so that we also have $\check{C} = \check{C^*}$.

On the other hand, $\sigma_x^+\Sigma^+$ is a complete slice in $\operatorname{mod} C^*$ so, in particular, it is sincere over C^*. Therefore, $\operatorname{Supp} \sigma_x^+\Sigma^+ = C^*$. Using that $\check{C} = \check{C^*}$, we thus have $C^* \subset \sigma_x^+ C$. Finally, since the Grothendieck groups of $C^*, \sigma_x^+ C$ and C are all of the same rank, it follows that the full subcategories C^* and $\sigma_x^+ C$ of $\tilde{C}$ are equal. This completes the proof. □

Dually, one defines coreflections σ_x^- with respect to admissible sources x. We leave the straightforward statements to the reader.

5 Main result

5.1 The distance between two local slices

We introduce the following notation. Let Σ_1, Σ_2 be two local slices in $\operatorname{mod} \check{C}$, considered as embedded in $\mathcal{D}^b(\operatorname{mod} C)$. We define $\check{d}(\Sigma_1, \Sigma_2)$ to be the number of $\tau F^j T_i$ (where $1 \leq i \leq \operatorname{rk} K_0(C)$ and $j \in \mathbb{Z}$) in $\mathcal{D}^b(\operatorname{mod} C)$ such that either $\Sigma_1 < \tau F^j T_i < \Sigma_2$, or $\Sigma_2 < \tau F^j T_i < \Sigma_1$.

Note that $\check{d}(\Sigma_1, \Sigma_2)$ is always a non-negative integer but it can be arbitrarily large. Also, if $\check{C}$ is locally representation-finite (that is, $\tilde{C}$ is representation-finite), then $\check{d}(\Sigma_1, \Sigma_2) = 0$ if and only if the local slices $G_\lambda \Sigma_1$ and $G_\lambda \Sigma_2$ in $\operatorname{mod} \tilde{C}$ are homotopic in the sense of [BOW] (see section (4.1) above).

Lemma 15 *Let $\Sigma_1, \Sigma_2, \Sigma_3$ be local slices in $\operatorname{mod} \tilde{C}$, then:*

(a) $\check{d}(\Sigma_1, \Sigma_2) = \check{d}(\Sigma_2, \Sigma_1)$,
(b) $\check{d}(\Sigma_1, \Sigma_3) \leq \check{d}(\Sigma_1, \Sigma_2) + \check{d}(\Sigma_2, \Sigma_3)$.

PROOF. (a) is obvious and (b) follows from a straightforward counting argument. □

Clearly, $\check{d}$ is not yet a distance function. Our objective is to use it in order to define a distance function. We say that an algebra C' is a *fibre quotient* of $\check{C}$ if C' is tilted and such that $\check{C'} \cong \check{C}$. This terminology is motivated by the observation that such an algebra C' lies in the fibre of $\check{C}$ under the mapping $C \mapsto \check{C}$ from the class of tilted algebras to the class of cluster repetitive algebras.

Let now C_1, C_2 be two fibre quotients of $\check{C}$, and Σ_1, Σ_2 be complete slices in $\operatorname{mod} C_1, \operatorname{mod} C_2$ respectively, considered as local slices in $\operatorname{mod} \check{C}$. Then we set

$$\check{d}(C_1, C_2) = \check{d}(\Sigma_1, \Sigma_2).$$

This does not depend on the choice of the complete slices Σ_1 and Σ_2. Indeed, let Σ_1, Σ_1' be two complete slices in $\operatorname{mod} C_1$, then it is clear that $\check{d}(\Sigma_1, \Sigma_1') = 0$. Hence Lemma 15 (b) yields $\check{d}(\Sigma_1, \Sigma_2) \leq \check{d}(\Sigma_1, \Sigma_1') + \check{d}(\Sigma_1', \Sigma_2) = \check{d}(\Sigma_1', \Sigma_2)$. Similarly, $\check{d}(\Sigma_1', \Sigma_2) \leq \check{d}(\Sigma_1, \Sigma_2)$, so $\check{d}(\Sigma_1, \Sigma_2) = \check{d}(\Sigma_1', \Sigma_2)$, and our notion is well-defined.

Proposition 16 *Let C_1, C_2 be two fibre quotients of $\check{C}$, then $\check{d}(C_1, C_2) = 0$ if and only if $C_1 = C_2$.*

PROOF. Assume indeed that $\check{d}(C_1, C_2) = 0$. Let Σ_1, Σ_2 be complete slices in $\operatorname{mod} C_1, \operatorname{mod} C_2$, respectively, considered as local slices in $\operatorname{mod} \check{C}$. By [ABS2], we have $C_1 = \check{C}/\operatorname{Ann} \Sigma_1$ and $C_2 = \check{C}/\operatorname{Ann} \Sigma_2$.

Let T be a tilting module over the hereditary algebra A such that $\operatorname{End}_A T \cong C$, and $\operatorname{End}_{\mathcal{C}_A} T \cong \tilde{C}$, (so that $\operatorname{End}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i \in \mathbb{Z}} F^i T) = \check{C}$). By [ABS2], the annihilator $\operatorname{Ann} \Sigma_1$ is generated by the arrows $\alpha : (x_0, i) \to (y_0, j)$ of $\check{C}$ (here x_0, y_0 are points of C_1, while $i, j \in \mathbb{Z}$) such that the corresponding morphism $f_\alpha : F^j T_{y_0} \to F^i T_{x_0}$ in the derived category lies in $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(F^j T, F^{j+1} T)$ and $\Sigma_1 = F^j DA$. Now, this is the case if and only if

$$F^j T_{y_0} \leq \Sigma_1 \leq \tau^2 F^{j+1} T_{x_0}$$

in $\mathcal{D}^b(\operatorname{mod} A)$. Indeed, notice first that the existence of the arrow α means that $i \in \{j, j+1\}$. Moreover $\tau^2 F T_{x_0} = \tau T_{x_0}[1] \geq DA$ implies $\tau^2 F^{j+1} T_{x_0} \geq F^j DA = \Sigma_1$. On the other hand, $T_{y_0} \leq DA$ gives clearly $F^j T_{y_0} \leq F^j DA = \Sigma_1$.

We next claim that $\overset{\vee}{d}(\Sigma_1, \Sigma_2) = 0$ implies

$$F^j T_{y_0} \leq \Sigma_2 \leq \tau^2 F^{j+1} T_{x_0}.$$

Indeed, if $F^j T_{y_0} \not\leq \Sigma_2$, then $\Sigma_2 < F^j T_{y_0}$, so that $\Sigma_2 < \tau F^j T_{y_0}$ because $\tau F^j T_{y_0} \notin \Sigma_2$. This implies that $\Sigma_2 < \tau F^j T_{y_0} < \Sigma_1$ and we have a contradiction to $\overset{\vee}{d}(\Sigma_1, \Sigma_2) = \overset{\vee}{d}(C_1, C_2) = 0$. On the other hand, if $\Sigma_2 \not\leq \tau^2 F^{j+1} T_{x_0}$, then $\tau^2 F^{j+1} T_{x_0} < \Sigma_2$ and so $\tau F^{j+1} T_{x_0} < \Sigma_2$ because $\tau F^{j+1} T_{x_0} \notin \Sigma_2$. This implies that $\Sigma_1 < \tau F^{j+1} T_{x_0} < \Sigma_2$, another contradiction to $\overset{\vee}{d}(\Sigma_1, \Sigma_2) = \overset{\vee}{d}(C_1, C_2) = 0$. This establishes our claim.

Now, that claim implies that the annihilators of Σ_1 and Σ_2 have the same generators. Therefore $C_1 = C_2$. Since the converse is obvious, the proof of the proposition is complete. □

Corollary 17 *The set $\overset{\vee}{\mathcal{F}}$ of all fibre quotients of $\overset{\vee}{C}$ is a discrete metric space with the distance $\overset{\vee}{d}$.*

PROOF. It follows from Lemma 15 and Proposition 16 that $\overset{\vee}{d}$ is a distance in $\overset{\vee}{\mathcal{F}}$. It is clear that the resulting metric space is discrete. □

5.3 The metric space of fibre quotients of a cluster-tilted algebra

We now bring down this information to $\tilde{C}$. We say that an algebra C' is a *fibre quotient* of $\tilde{C}$ if C' is tilted and such that $\tilde{C'} \cong \tilde{C}$. Let C_1, C_2 be two fibre quotients of $\tilde{C}$, then we set

$$d(C_1, C_2) = \min_{C_1^*, C_2^* \in \overset{\vee}{\mathcal{F}}} \{\overset{\vee}{d}(C_1^*, C_2^*) \mid GC_1^* = C_1, GC_2^* = C_2\}.$$

Corollary 18 *Let C_1, C_2 be two fibre quotients of $\tilde{C}$, then $d(C_1, C_2) = 0$ if and only if $C_1 = C_2$.*

PROOF. This follows immediately from Proposition 16. □

Remark 19 *This gives another interpretation and proof of [BOW, Theorem 4.13].*

Notice that while our definition implies that the set $\overset{\vee}{\mathcal{F}}$ of fibre quotients of $\overset{\vee}{C}$ is infinite, clearly the set $\tilde{\mathcal{F}}$ of fibre quotients of $\tilde{C}$ is finite. Moreover, it is easily seen that $\overset{\vee}{\mathcal{F}}$ is (trivially) a topological covering of $\tilde{\mathcal{F}}$.

Corollary 20 *The set $\tilde{\mathcal{F}}$ of all fibre quotients of $\tilde{C}$ is a discrete metric space with the distance d.*

PROOF. This follows from Corollary 17. $\square$

5.4

The following lemma and its proof, which relate fibre quotients of $\tilde{C}$ and $\check{C}$, are valid without assuming that C is of tree type.

Lemma 21 *Let C be a tilted algebra. If C' is a fibre quotient of $\tilde{C}$, then $G^{-1}(C)$ is the φ-orbit of a fibre quotient of $\check{C}$. Conversely, if C^* is a fibre quotient of $\check{C}$, then $G(C^*)$ is a fibre quotient of $\tilde{C}$.*

Remark 22 *By abuse of language, we quote from now on this lemma by saying that C' is a fibre quotient of $\tilde{C}$ if and only if C' is a fibre quotient of $\check{C}$.*

PROOF. Suppose $\check{C} = \check{C^*}$. Let Σ be a complete slice in $\operatorname{mod} C$ considered as a local slice in $\check{C} = \check{C^*}$. By [ABS2], Σ lifts isomorphically as a section both in $\mathcal{D}^b(\operatorname{mod} C)$ and in $\mathcal{D}^b(\operatorname{mod} C^*)$. This implies that we have equivalences of triangulated categories $\phi : \mathcal{D}^b(\operatorname{mod} C) \xrightarrow{\cong} \mathcal{D}^b(\operatorname{mod} k\Sigma)$ and $\phi^* : \mathcal{D}^b(\operatorname{mod} C^*) \xrightarrow{\cong} \mathcal{D}^b(\operatorname{mod} k\Sigma)$. Let $T = \phi\, C$ and $T^* = \phi^* C^*$. Then:

$$\begin{aligned}
\operatorname{End}_{\mathcal{D}^b(\operatorname{mod} k\Sigma)}(\oplus_{j\in\mathbb{Z}} F^j T) &\cong \operatorname{End}_{\mathcal{D}^b(\operatorname{mod} C)}(\oplus_{j\in\mathbb{Z}} F^j C) \\
&\cong \check{C} \\
&\cong \check{C^*} \\
&\cong \operatorname{End}_{\mathcal{D}^b(\operatorname{mod} C')}(\oplus_{j\in\mathbb{Z}} F^j C^*) \\
&\cong \operatorname{End}_{\mathcal{D}^b(\operatorname{mod} k\Sigma)}(\oplus_{j\in\mathbb{Z}} F^j T^*).
\end{aligned}$$

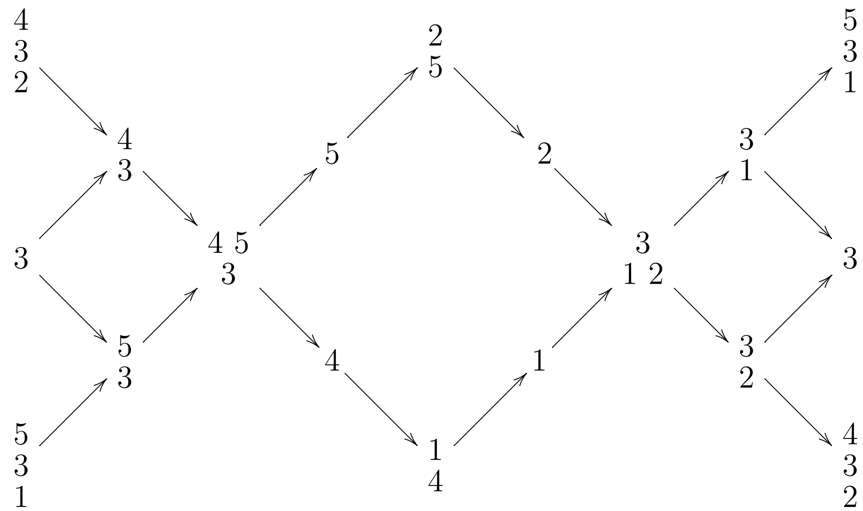


Fig. 1. Auslander-Reiten quiver of Example 5.5

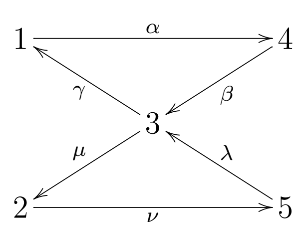
Define $C' = G(C^*)$, then, passing to the cluster category, we have $\mathcal{C}_C \cong \mathcal{C}_{k\Sigma} \cong \mathcal{C}_{C'}$ and

$$\begin{aligned}\tilde{C} &\cong \operatorname{End}_{\mathcal{C}_C} C \\ &\cong \operatorname{End}_{\mathcal{C}_{k\Sigma}} T \\ &\cong \operatorname{End}_{\mathcal{C}_{k\Sigma}} T^* \\ &\cong \operatorname{End}_{\mathcal{C}_{C'}} C' \\ &\cong \tilde{C}'.\end{aligned}$$

This proves the sufficiency. The necessity is obvious. □

5.5 Example

Let $\tilde{C}$ be the cluster-tilted algebra of type $\mathbb{A}_5$ given by the quiver



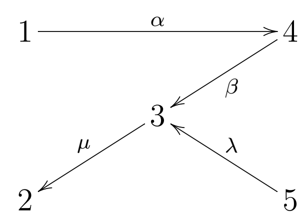
bound by $\alpha\beta = 0$, $\beta\gamma = 0$, $\gamma\alpha = 0$ $\lambda\mu = 0$, $\mu\nu = 0$ and $\nu\lambda = 0$. Its Auslander-Reiten quiver is shown in Figure 1, where modules are represented by their Loewy series and we identify the vertices that have the same label, thus creating a Moebius strip. Let $\Sigma_1, \Sigma_2, \Sigma_3$ be respectively given by

$$\Sigma_1 = \left\{ \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix} , \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} , \begin{smallmatrix} 4\ 5 \\ 3 \end{smallmatrix} , 4 , \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right\}$$

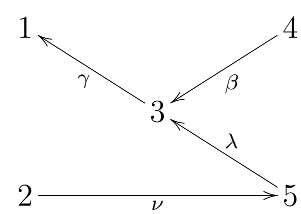
$$\Sigma_2 = \left\{ \begin{smallmatrix} 5 \\ 3 \\ 1 \end{smallmatrix} , \begin{smallmatrix} 5 \\ 3 \end{smallmatrix} , \begin{smallmatrix} 4\ 5 \\ 3 \end{smallmatrix} , 5 , \begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \right\}$$

$$\Sigma_3 = \left\{ \begin{smallmatrix} 2 \\ 5 \end{smallmatrix} , 2 , \begin{smallmatrix} 3 \\ 1\ 2 \end{smallmatrix} , \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} , \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix} \right\}.$$

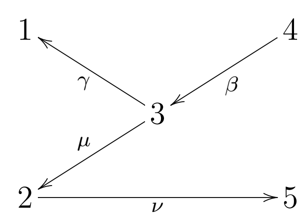
Then $C_1 = \tilde{C}/\mathrm{Ann}\,\Sigma_1$ is given by the quiver



while $C_2 = \tilde{C}/\mathrm{Ann}\,\Sigma_2$ is given by the quiver



and $C_3 = \tilde{C}/\mathrm{Ann}\,\Sigma_3$ is given by the quiver



with the inherited relations in each case. Then we have $d(C_1, C_2) = d(C_1, C_3) = d(C_2, C_3) = 2$. Notice that if $\tilde{C}$ has n points, then clearly, for any two fibre quotients C_1, C_2 of $\tilde{C}$, we have $d(C_1, C_2) \leq \lfloor \frac{n}{2} \rfloor$.

5.6

We are now able to state and prove the key lemma.

Lemma 23 *Let Σ_1, Σ_2 be two local slices in the same transjective component of* $\operatorname{mod} \overset{\vee}{C}$ *such that* $\overset{\vee}{d}(\Sigma_1, \Sigma_2) \neq 0$*. Then either:*

(a) *there exists a rightmost slice Σ_1^+ such that $\overset{\vee}{d}(\Sigma_1, \Sigma_1^+) = 0$ and a reflection σ_x^+ such that $\overset{\vee}{d}(\sigma_x^+ \Sigma_1^+, \Sigma_2) < \overset{\vee}{d}(\Sigma_1, \Sigma_2)$, or*
(b) *there exists a leftmost slice Σ_1^- such that $\overset{\vee}{d}(\Sigma_1, \Sigma_1^-) = 0$ and a coreflection σ_y^- such that $\overset{\vee}{d}(\sigma_y^- \Sigma_1^-, \Sigma_2) < \overset{\vee}{d}(\Sigma_1, \Sigma_2)$.*

PROOF. (1) Assume first that $\Sigma_1 \cap \Sigma_2 = \emptyset$, then we can assume without loss of generality that $\Sigma_1 < \Sigma_2$. Let Σ_1^+ be the rightmost slice such that $\overset{\vee}{d}(\Sigma_1, \Sigma_1^+) = 0$. Such a rightmost slice exists since $\overset{\vee}{d}(\Sigma_1, \Sigma_2) \neq 0$ and the two slices lie in the same transjective component. Let $x = (x_0, j)$ be an admissible sink in Σ_1^+. We claim that $\sigma_x^+ \Sigma_1^+$ gives the result. Indeed, T_{x_0} is such that

$$\Sigma_1 < \tau F^j T_{x_0} < \Sigma_2$$

in $\mathcal{D}^b(\operatorname{mod} C)$, but $\tau F^j T_{x_0} < \sigma_x^+ \Sigma_1^+$. Also, if T_{y_0} is such that $\sigma_x^+ \Sigma_1^+ < \tau F^i T_{y_0} < \Sigma_2$ in $\mathcal{D}^b(\operatorname{mod} C)$, then $\Sigma_1 \leq \Sigma_1^+ < \tau F^i T_{y_0} < \Sigma_2$. Moreover, $\Sigma_2 < \tau F^i T_{y_0} < \sigma_x^+ \Sigma_1^+$ is impossible, because $\sigma_x^+ \Sigma_1^+ \leq \Sigma_2$. We deduce that $d(\sigma_x^+ \Sigma_1^+, \Sigma_2) < d(\Sigma_1, \Sigma_2)$. This proves (a). Similarly, assuming $\Sigma_2 < \Sigma_1$ yields (b).

(2) Suppose now that $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. Since $\overset{\vee}{d}(\Sigma_1, \Sigma_2) \neq 0$, there exists $z = (z_0, j)$ such that either $\Sigma_1 < \tau F^j T_{z_0} < \Sigma_2$ or $\Sigma_2 < \tau F^j T_{z_0} < \Sigma_1$. Assume $\Sigma_1 < \tau F^j T_{z_0} < \Sigma_2$ and let $x = (x_0, i)$ be an admissible sink in Σ_1^+ such that

$$\Sigma_1^+ < \tau F^i T_{x_0} < \Sigma_2.$$

We claim that $\overset{\vee}{d}(\sigma_x^+ \Sigma_1^+, \Sigma_2) < \overset{\vee}{d}(\Sigma_1, \Sigma_2)$.

We first prove that $G_x < \Sigma_2$ (see section 4.2 for the notation G_x). By definition, G_x is constructed by taking closures under socle factors of injectives (lying on the slice) and predecessors. Taking predecessors (of predecessors) of Σ_2 cannot create elements of Σ_2 or successors of Σ_2. Therefore, it suffices to show that, if I is an injective predecessor of Σ_2 and $I \to M$, then $M < \Sigma_2$. Suppose that this is not the case, then $M \in \Sigma_2$ and, since Σ_2 is a local slice and I is injective, then I must belong to Σ_2, a contradiction.

Now the same argument as in case (1) above completes the proof of (a). Similarly, in case $\Sigma_2 < \tau F^j T_{z_0} < \Sigma_1$, we get (b). $\square$

5.7 The main result

We may now state and prove our main theorem.

Theorem 24 *Let C be a tilted algebra having a tree Σ as complete slice. The following conditions are equivalent:*

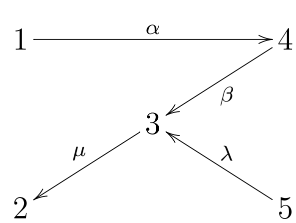
(a) *C' is a fibre quotient of $\tilde{C}$.*
(b) *C' is a fibre quotient of $\check{C}$.*
(c) *There exists a sequence of reflections and coreflections $\sigma_1, \ldots, \sigma_t$ such that $C' = \sigma_1 \cdots \sigma_t C$ has $\Sigma' = \sigma_1 \cdots \sigma_t \Sigma$ as complete slice and $C' = \tilde{C}/\mathrm{Ann}\,\Sigma'$.*

PROOF. Since the equivalence of (a) and (b) follows from Lemma 21, and since Proposition 14 yields easily that (c) implies (a), it suffices to prove that (a) implies (c).

Let C' be a fibre quotient of $\tilde{C}$. Then there exist two local slices Σ and Σ'' in $\mathrm{mod}\,\tilde{C}$ such that $C = \tilde{C}/\mathrm{Ann}\,\Sigma$ and $C' = \tilde{C}/\mathrm{Ann}\,\Sigma''$ (because of [ABS2]). Lifting this information to $\check{C}$, there exist two local slices $\check{\Sigma}$ and $\check{\Sigma}''$ lying in the same transjective component of $\Gamma(\mathrm{mod}\,\check{C})$ such that $G_\lambda\,\check{\Sigma} = \Sigma$ and $G_\lambda\,\check{\Sigma}'' = \Sigma''$. Applying Lemma 23 and an obvious induction, the finiteness of the distance function yields a sequence of reflections and coreflections $\sigma_1, \ldots, \sigma_t$ such that $\check{d}(\sigma_1 \cdots \sigma_t\,\check{\Sigma}, \check{\Sigma}'') = 0$. This implies that $d(\sigma_1 \cdots \sigma_t \Sigma, \Sigma'') = 0$. Let $\Sigma' = \sigma_1 \cdots \sigma_t \Sigma$. By Proposition 14, $C' = \sigma_1 \cdots \sigma_t C$ is tilted and has Σ' as a complete slice. Let $C^* = \tilde{C}/\mathrm{Ann}\,\Sigma'$, then $d(\Sigma', \Sigma'') = 0$ implies $d(C^*, C') = 0$. Because of Corollary 18, we get indeed $C' = C^*$. This completes the proof. □

5.8 Example

Let again $\tilde{C}$ be the cluster-tilted algebra of Example 5.5. We assume that C is the tilted algebra given by the quiver



bound by $\alpha\,\beta = 0, \lambda\,\mu = 0$. A rightmost complete slice Σ of $\mathrm{mod}\,C$ is given by

$$\Sigma = \left\{ \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix} ,\ \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} ,\ \begin{smallmatrix} 4\ 5 \\ 3 \end{smallmatrix} ,\ 4 ,\ \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right\}$$

Reflecting successively at all admissible sinks yields successively the local slices

$$\sigma_2\Sigma = \left\{ \begin{smallmatrix} 4\,5 \\ 3 \end{smallmatrix}, 5, 4, \begin{smallmatrix} 2 \\ 5 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right\},$$

$$\sigma_3\sigma_2\Sigma = \left\{ \begin{smallmatrix} 2 \\ 5 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 4 \end{smallmatrix}, 2, 1, \begin{smallmatrix} 3 \\ 1\,2 \end{smallmatrix} \right\},$$

$$\sigma_4\sigma_3\sigma_2\Sigma = \left\{ \begin{smallmatrix} 2 \\ 5 \end{smallmatrix}, 2, \begin{smallmatrix} 3 \\ 1\,2 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix} \right\},$$

$$\sigma_5\sigma_3\sigma_2\Sigma = \left\{ \begin{smallmatrix} 1 \\ 4 \end{smallmatrix}, 1, \begin{smallmatrix} 3 \\ 1\,2 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 5 \\ 3 \\ 1 \end{smallmatrix} \right\}.$$

$$\sigma_5\sigma_4\sigma_3\sigma_2\Sigma = \left\{ \begin{smallmatrix} 3 \\ 1\,2 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 5 \\ 3 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix} \right\}.$$

Then we have $\Sigma' = \sigma_5\sigma_4\sigma_3\sigma_2\Sigma = \sigma_4\sigma_5\sigma_3\sigma_2\Sigma$. The rightmost slice corresponding to Σ' is

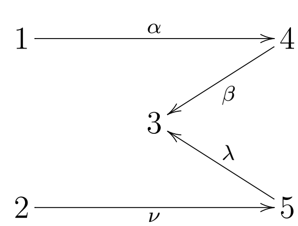
$$\Sigma'^{+} = \left\{ \begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 5 \\ 3 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 4\,5 \\ 3 \end{smallmatrix} \right\},$$

therefore

$$\sigma_2\Sigma'^{+} = \left\{ \begin{smallmatrix} 5 \\ 3 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 4\,5 \\ 3 \end{smallmatrix}, 5, \begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \right\},$$

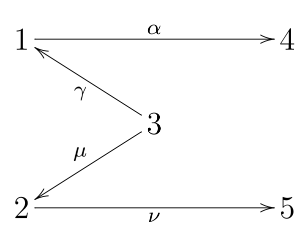
while $\sigma_1\Sigma'^{+} = \Sigma$. Therefore the fibre quotients of $\tilde{C}$ are the algebras

(1) $\sigma_2 C$ given by the quiver



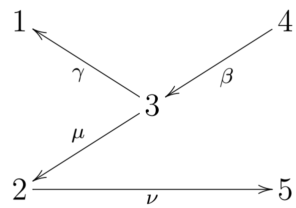
bound by $\alpha\,\beta = 0$ and $\nu\,\lambda = 0$.

(2) $\sigma_3\sigma_2 C$ given by the quiver



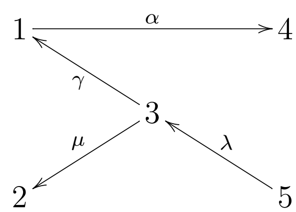
bound by $\gamma\,\alpha = 0$ and $\mu\,\nu = 0$.

(3) $\sigma_4\sigma_3\sigma_2C$ given by the quiver



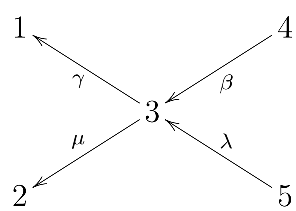
bound by $\beta\,\gamma = 0$ and $\mu\,\nu = 0$.

(4) $\sigma_5\sigma_3\sigma_2C$ given by the quiver



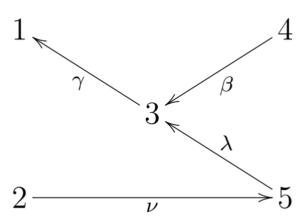
bound by $\gamma\,\alpha = 0$ and $\lambda\,\mu = 0$.

(5) $\sigma_5\sigma_4\sigma_3\sigma_2C = \sigma_4\sigma_5\sigma_3\sigma_2C$ given by the quiver



bound by $\beta\,\gamma = 0$ and $\lambda\,\mu = 0$.

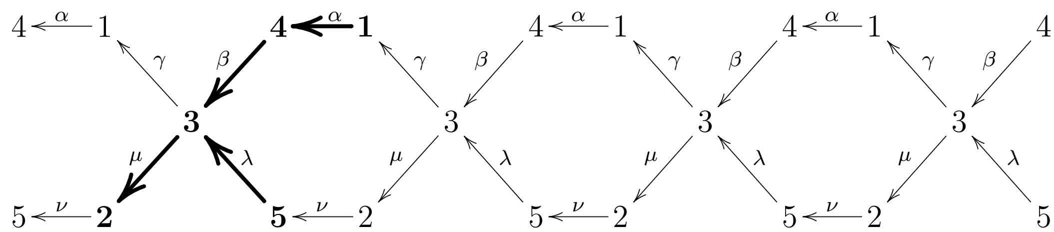
(6) $\sigma_2\sigma_5\sigma_4\sigma_3\sigma_2C$ given by the quiver



bound by $\beta\,\gamma = 0$ and $\nu\,\lambda = 0$.

Finally $\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2C = C$. It is easily seen that we so obtain all fibre quotients of $\tilde{C}$.

The reader can easily locate these reflections (fibre quotients) of C in the quiver of $\check{C}$:



bound by the lifted relations $\alpha\,\beta = 0$, $\beta\,\gamma = 0$, $\gamma\,\alpha = 0$ $\lambda\,\mu = 0$, $\mu\,\nu = 0$ and $\nu\,\lambda = 0$. We have illustrated one copy of C in bold face.

6 Algorithm

6.1

Let C be a tilted algebra of tree type, and Γ a connecting component of $\operatorname{mod} C$. We recall that a tilted algebra has a unique connecting component, except if it is concealed, in which case it has two. We denote by Σ^+ and Σ^-, respectively, the rightmost and leftmost slice in Γ. We assume both Σ^+ and Σ^- exist. Let Γ_1 be the full subquiver of Γ having as points

$$\Gamma_1 = \{M \in \operatorname{ind} C \mid \tau\Sigma^- \leq M \leq \tau^{-1}\Sigma^+\}.$$

Lemma 25 *With the above notation,*

(a) Γ_1 *embeds as a full subquiver of* $\Gamma(\operatorname{mod} \check{C})$.
(b) *Let M be a $\check{C}$-module such that $\tau\Sigma^- \leq M \leq \tau^{-1}\Sigma^+$ then M is a C-module lying in Γ_1.*

PROOF. (a) follows from Proposition 3.

(b) Let M be such a $\check{C}$-module. It follows from the structure of $\Gamma(\operatorname{mod} \check{C})$ that M lies in a transjective component and furthermore there exists $t \geq 0$ such that $\tau_{\check{C}}^{-t} M \in \Sigma^+$, that is, there exists a C-module $N \in \Sigma^+$ such that $\tau_{\check{C}}^{-t} M = N$. Applying Proposition 3, we get $M = \tau_{\check{C}}^{t}\tau_{\check{C}}^{-t} M = \tau_{\check{C}}^{t} N \cong \tau_{C}^{t} N$, hence the statement. $\square$

Remark 26 *Note that if, for instance, Σ^- does not exist, but Σ^+ does, then the statement of the Lemma applies to the full subquiver of Γ with points*

$\{M \in \operatorname{ind} C \mid M \leq \tau^{-1}\Sigma^+\}$.

6.2

Let now x be an admissible sink in C such that G_x is contained in the rightmost slice Σ^+ of $\operatorname{mod} C$. Let I_y be a source in G_x and define a $\check{C}$-module $\overline{P}_y$ by

$$\begin{aligned}\operatorname{top} \overline{P}_y &= S_y \\ \operatorname{rad} \overline{P}_y &= \tau_C^{-1}(I_y/S_y) = \bigoplus_{I_y \to M} (\tau_C^{-1} M).\end{aligned}$$

Note that, since I_y is a source, then all indecomposable modules M such that there exists an arrow $I_y \to M$ in $\Gamma(\operatorname{mod} C)$ lie in G_x (see Section 4.2). Also, as morphisms from $\operatorname{top} \overline{P}_y$ to $\operatorname{rad} \overline{P}_y$, we take, for every arrow $\alpha : y \to z$, the linear map $f_\alpha : \overline{P}_y(y) \to \overline{P}_y(z)$ defined by the right multiplication by the residual class of the arrow α in $\check{C} = k\check{Q} / \check{I}$.

Recursively, for every I_z in G_x with the property that for each predecessor I_w of I_z in G_x, we have already introduced a corresponding projective module $\overline{P}_w$, we define $\overline{P}_z$ by

$$\begin{aligned}\operatorname{top} \overline{P}_z &= S_z \\ \operatorname{rad} \overline{P}_z &= \tau_C^{-1}(I_z/S_z) \oplus \left(\bigoplus_{I_w \to I_z} \overline{P}_w \right),\end{aligned}$$

where the second direct sum is taken over all arrows $I_w \to I_z$ in G_x.

Again, for morphisms from $\operatorname{top} \overline{P}_z$ to $\operatorname{rad} \overline{P}_z$, we take, for every arrow $\alpha : z \to v$, the linear map $f_\alpha : \overline{P}_z(z) \to \overline{P}_z(v)$ defined by the right multiplication by the residual class of the arrow α in $\check{C} = k\check{Q} / \check{I}$. The module $\overline{P}_z$ is thus located at the position $\tau^{-2} I_z$ in $\Gamma(\operatorname{mod} \check{C})$.

Lemma 27 *For each injective module I_y in G_x, the $\check{C}$-module $\overline{P}_y$ thus constructed is isomorphic to the indecomposable projective $\check{C}$-module $\check{P}_y$ at y.*

PROOF. Clearly, it suffices to show that $\operatorname{rad} \check{P}_y = \operatorname{rad} \overline{P}_y$. We have that $\operatorname{rad} \check{P}_y$ is the direct sum of all $N \in \operatorname{ind} \check{C}$ such that there exists an arrow $N \to \check{P}_y$ in $\Gamma(\operatorname{mod} \check{C})$. There are two possibilities for such a radical summand N:

Either N is not projective, and then there exists an arrow $I_y \to M$ with $M \cong \tau_{\check{C}} N$ because $\check{P}_y$ is also situated at the position $\tau^{-2} I_y$ in $\Gamma(\operatorname{mod} \check{C})$ (see Lemma 5 (a)), or $N = \check{P}_w$ is projective, and then there exists an arrow $\check{P}_w \to \check{P}_z$ in $\Gamma(\operatorname{mod} \check{C})$.

Thus

$$\operatorname{rad} \check{P}_y = \left(\bigoplus_{I_y \to M} \tau_{\check{C}}^{-1} M \right) \bigoplus \left(\bigoplus_{\check{P}_w \to \check{P}_z} \check{P}_w \right),$$

where the two direct sums are taken over arrows in $\Gamma(\operatorname{mod} \check{C})$.

Now, if $I_y = \check{I}_y$ is a source in G_x, then there is no arrow $I_z \to I_y$ in $\Gamma(\operatorname{mod} C)$ and, because of Lemma 25, there is no arrow $\check{I}_z \to \check{I}_y$ in $\Gamma(\operatorname{mod} \check{C})$. By Lemma 5 (b), there is no arrow $\check{P}_z \to \check{P}_x$ in $\Gamma(\operatorname{mod} \check{C})$. Therefore, using Proposition 3,

$$\operatorname{rad}_{\check{C}} \check{P}_y = \bigoplus_{\check{I}_y \to M} \tau_{\check{C}}^{-1} M = \bigoplus_{I_y \to M} \tau_C^{-1} M = \operatorname{rad}_{\check{C}} \overline{P}_y,$$

where the first direct sum is taken over arrows in $\Gamma(\operatorname{mod} \check{C})$ and the second over arrows in $\Gamma(\operatorname{mod} C)$.

Now assume that I_z is not a source in G_z, By induction, we may suppose that $\check{P}_w = \overline{P}_w$ for all w such that I_w precedes I_z in G_x. Thus

$$\bigoplus_{\check{P}_w \to \check{P}_z} \check{P}_w \cong \bigoplus_{\check{I}_w \to \check{I}_z} \check{P}_w \cong \bigoplus_{I_w \to I_z} \check{P}_w \cong \bigoplus_{I_w \to I_z} \overline{P}_w,$$

where the last equality holds by induction. Since we have, as before,

$$\bigoplus_{\check{I}_z \to M} \tau_{\check{C}}^{-1} M = \bigoplus_{I_z \to M} \tau_C^{-1} M,$$

the proof is complete. □

6.3

Corollary 28 *With the above notation, we have*

$\sigma_x^+ \Sigma^+ = \{\Sigma \backslash G_x\} \cup \{\overline{P}_y \mid I_y \in G_x \text{ injective}\} \cup \{\tau_C^{-1} M \mid M \in G_x \text{ not injective}\}.$

PROOF. This follows directly from Lemma 27 and the construction in Section 4.3.

Remark 29 *Clearly, the dual construction, starting from an admissible source y in C and constructing the local slice $\sigma_y \Sigma^-$ in $\Gamma(\mathrm{mod}\, \check{C})$ holds as well. We leave its statement to the reader.*

6.4

We now describe an algorithm allowing to construct the transjective component of a cluster-tilted algebra $\tilde{C}$ knowing only a complete slice of a tilted algebra C. Since the pushdown functor $G_\lambda : \mathrm{mod}\, \check{C} \to \mathrm{mod}\, \tilde{C}$ is dense and thus induces an isomorphism of quivers $\Gamma(\mathrm{mod}\, \tilde{C}) \cong \Gamma(\mathrm{mod}\, \check{C})/\mathbb{Z}$ (see [ABS3]), it suffices to construct a transjective component of $\check{C}$.

Let Σ be a complete slice in $\mathrm{mod}\, C$, then Σ embeds as a local slice in a transjective component Γ of the cluster repetitive algebra $\check{C}$. For clarity, we treat separately the representation-finite and the representation-infinite case.

(a) Assume $\tilde{C}$ is representation-finite, that is, $\check{C}$ is locally representation-finite. In this case, Σ is a Dynkin quiver. We carry out the following steps.

(1) If there exists a source M of Σ which is not injective, then we replace Σ by

$$\Sigma' = \{\Sigma \setminus \{M\}\} \cup \{\tau^{-1} M\}$$

(here, the Auslander-Reiten translation τ is computed with respect to the support of Σ which, at the start, is equal to C). If not go to 2. Repeat until every source is injective.

(2) If all sources of Σ are injective then there exists a source I_x in Σ such that G_x exists (because of Lemma 9). Then we replace Σ by

$$\Sigma' = \sigma_x^+ \Sigma.$$

Go to 1.

Since $\check{C}$ is locally representation-finite, we eventually construct a slice Σ such that for every module M in Σ, the module $\varphi^{-1} M$ has already been constructed before, where φ is the automorphism of $\check{C}$ inducing the covering $\check{C} \to \tilde{C}$ (see Section 3.3). At this point the algorithm stops. After identification under φ, we have thus obtained the Auslander-Reiten quiver of the cluster-tilted algebra $\tilde{C}$.

(b) Assume $\tilde{C}$ is representation-infinite, that is, $\check{C}$ is locally representation-infinite. We carry out the following steps.

(1) If there exists a source M of Σ which is not injective, then we replace Σ by

$$\Sigma' = \{\Sigma \setminus \{M\}\} \cup \{\tau^{-1} M\}$$

(where, again, τ^{-1} is computed with respect to the support of Σ). Repeat. If this procedure produces a Σ in which every source is injective, then go to 2. If not, then this procedure produces the right stable part of Γ. Then go to 3.

(2) If all sources of Σ are injective then there exists a source I_x in Σ such that G_x exists. Then we replace Σ by

$$\Sigma' = \sigma_x \Sigma.$$

Go to 1. Since there are finitely many injectives in Γ then, at some point, we get to 3.

(3) Return to the initial slice Σ.

(4) If there exists a sink N of Σ which is not projective, then we replace Σ by

$$\Sigma' = \{\Sigma \setminus \{N\}\} \cup \{\tau N\}$$

(where, again, τ is computed with respect to the support of Σ). Repeat. If this procedure produces a Σ in which every sink is projective, then go to 5. If not, then this procedure produces the left stable part of Γ. Then the algorithm stops.

(5) If all sinks of Σ are projective then there exists a sink P_y in Σ such that G_y exists. Then we replace Σ by

$$\Sigma' = \sigma_y \Sigma.$$

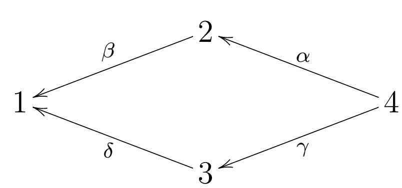
Go to 4. Since there are finitely many projectives in Γ then, at some point, the algorithm stops.

Theorem 30 *Let C be a tilted algebra of tree type. Then the transjective component of $\Gamma(\mathrm{mod}\,\tilde{C})$ is constructed by the preceding algorithm. Moreover, if C is of Dynkin type, then the algorithm yields $\Gamma(\mathrm{mod}\,\tilde{C})$.*

PROOF. This follows from Corollary 28 and the density of the pushdown functor $G_\lambda : \mathrm{mod}\,\check{C} \to \mathrm{mod}\,\tilde{C}$. $\square$

6.5 A representation-finite example

Let C be the tilted algebra of type $\mathbb{D}_4$ given by the quiver



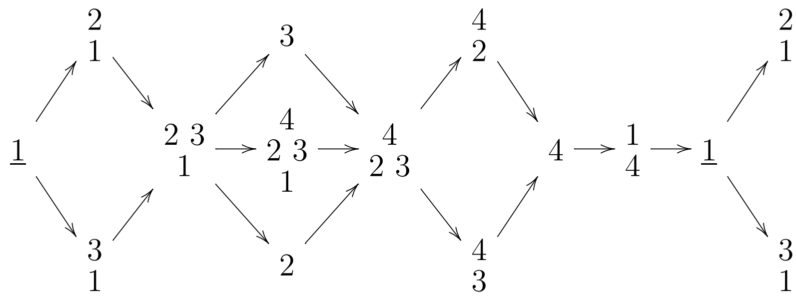


Fig. 2. Auslander-Reiten quiver of Example 6.5

bound by $\alpha\beta = \gamma\delta$. We construct its Auslander-Reiten quiver until we reach its rightmost slice

$$\Sigma^+ = \left\{ \begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 4 & \\ 2 & & 3 \end{smallmatrix}, \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \right\}.$$

Since Σ^+ has a unique source $\begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix}$, the corresponding sink 1 is admissible and so we get

$$\sigma_1^+\Sigma^+ = \left\{ \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}, 4, , \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right\}.$$

In the next step we must move the points $\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}$ and $\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}$ simultaneously (because $G_2 = G_3$), hence we get

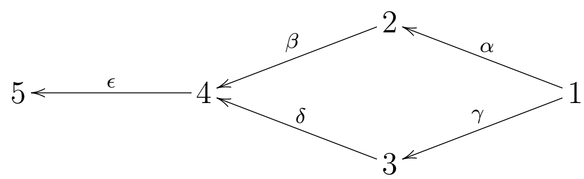
$$\sigma_2^+\sigma_1^+\Sigma^+ = \sigma_3^+\sigma_1^+\Sigma^+ = \left\{ \begin{smallmatrix} 1 \\ 4 \end{smallmatrix}, 1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right\}.$$

A further reflection yields

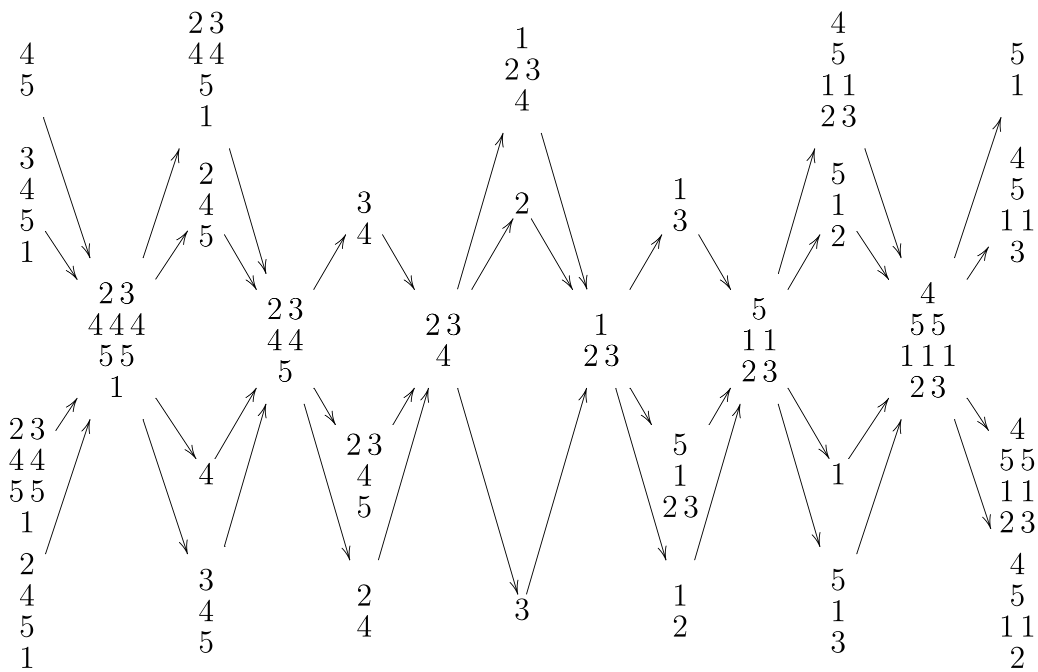
$$\sigma_4^+\sigma_2^+\sigma_1^+\Sigma^+ = \left\{ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 2 & & 3 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix} \right\},$$

which is the leftmost slice Σ^- in $\Gamma(\operatorname{mod} C)$. The Auslander-Reiten quiver of $\tilde{C}$ is of the form shown in Figure 2.

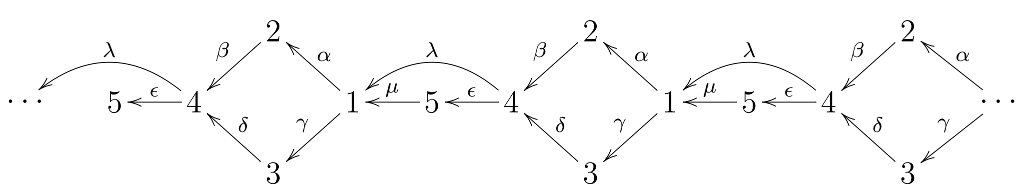
Let C be the tilted algebra of type $\tilde{\mathbb{D}}_4$ given by the quiver



bound by $\alpha\,\beta = \gamma\delta$ and $\alpha\,\beta\,\epsilon = 0$. Here, $\tilde{C}$ is representation-infinite. We show part of its transjective component.



The rest of the transjective component is constructed by the "knitting" procedure, constructing successively the Auslander-Reiten translates of the modules thus obtained. The remaining projectives lie in the tubes. The cluster repetitive algebra $\check{C}$ is given by the quiver



bound by $\alpha\,\beta = \gamma\,\delta,\ \alpha\,\beta\,\epsilon = 0,\ \beta\,\lambda = \beta\,\epsilon\,\mu,\ \lambda\,\alpha = \epsilon\,\mu\,\alpha,\ \delta\,\lambda = 0$ and $\lambda\,\gamma = 0$.

7 Tubes

The same algorithm seems to work for the tubes of the cluster-tilted algebras of Euclidean type. We have no proof of this fact but we give partial results and an example here.

Let A be a hereditary algebra of Euclidean type and T be a tilting A-module without preinjective summands. Assume that T_i is a summand of T that lies in a tube and such that i is a source of $C = \mathrm{End}_A T$. Denote by r the quasi length of T_i and let M be the quasi simple module that lies on the same ray as T_i on the mouth of the tube.

Lemma 31 *The immediate predecessor of T_i on the semi-ray ending at T_i is a summand of T.*

PROOF. If $r = 1$, then $M = T_i$ and the result holds since there is no such predecessor. If $r > 1$, it follows from the assumption that i is a source in C. $\square$

We denote this predecessor by T_j. Thus there is a sectional path $M \to \cdot \to \cdots \to T_j \to T_i$ of length $r-1$, and M lies on the mouth of the tube.

Lemma 32 *In the above situation, we have*

$$\mathrm{Hom}_A(T, \tau^2 T_i) \cong \mathrm{Hom}_A(T, \tau^2 M).$$

PROOF. Applying the functor $\mathrm{Hom}_A(T, -)$ to the short exact sequence

$$0 \to \tau^2 M \to \tau^2 T_i \to \tau T_j \to 0,$$

the result follows from $\mathrm{Hom}_A(T, \tau T_j) = D\mathrm{Ext}_A(T_j, T) = 0.$ $\square$

Lemma 33 *In the above situation, let $\tilde{I}_i$ denote the indecomposable injective and $\tilde{S}_i$ the indecomposable simple module of the cluster-tilted algebra $C \ltimes \mathrm{Ext}^2_C(DC, C)$ corresponding to the point i. Then*

$$\tilde{I}_i / \tilde{S}_i = \mathrm{Hom}_A(T, \tau^2 T_i).$$

PROOF. A straightforward computation shows that

$$\begin{aligned}\tilde{I}_i &= \mathrm{Hom}_{\mathcal{C}}(T, \tau^2 T_i)\\ &= \mathrm{Hom}_A(T, \tau^2 T_i) \oplus \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\tau T[-1], \tau^2 T_i)\\ &= \mathrm{Hom}_A(T, \tau^2 T_i) \oplus D\mathrm{Hom}_A(\tau^2 T_i, \tau^2 T).\end{aligned}$$

The simple socle of $\tilde{I}_i$ corresponds in this description to the direct summand $D\mathrm{Hom}_A(\tau^2 T_i, \tau^2 T_i)$ of the second term. Thus

$$\tilde{I}_i/\tilde{S}_i = \mathrm{Hom}_A(T, \tau^2 T_i) \oplus D\mathrm{Hom}_A(\tau^2 T_i, \tau^2 \overline{T}),$$

where $\overline{T} \oplus T_i = T$. The statement now follows, because $\mathrm{Hom}_A(\tau^2 T_i, \tau^2 \overline{T}) = \mathrm{Hom}_A(T_i, \overline{T}) = 0$, because i is a source in C. □

Now consider the image of the tube in the module category of the tilted algebra $C = \mathrm{End}_A T$. The A-modules T_j and T_i correspond to the indecomposable projective C-modules P_j and P_i respectively. Moreover P_j is a direct summand of the radical of P_i. Since P_i lies in a tube its radical $\mathrm{rad}\, P_i = P_j \oplus N$, for some indecomposable C-module N. Since i is a source, it follows from the construction of the tube in $\mathrm{mod}\, C$ from the tube in $\mathrm{mod}\, A$ that $\tau_C N = \mathrm{Hom}_A(T, \tau^2 M)$.

Lemma 34 *With the notation above,*

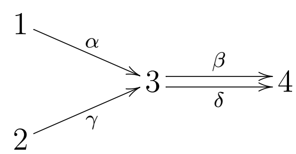
$$\tilde{I}_i/\tilde{S}_i = \tau_C N.$$

PROOF. $\tau_C N = \mathrm{Hom}_A(T.\tau^2 M) = \mathrm{Hom}_A(T, \tau^2 T_i) = \tilde{I}_i/\tilde{S}_i$, where the second equality follows from Lemma 32 and the last from Lemma 33. □

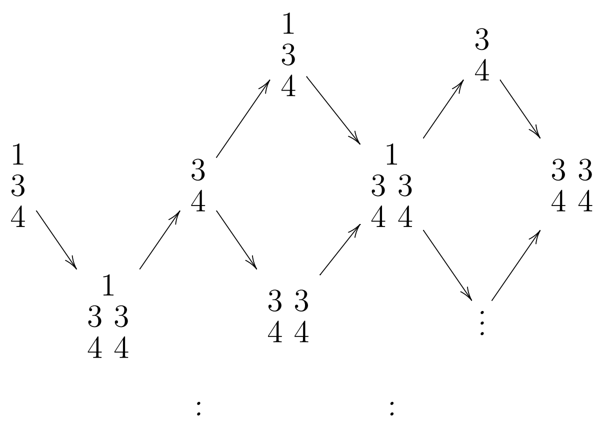
This shows that at least in certain cases, a similar algorithm as for the transjective component can be used to construct the tubes of the cluster-tilted algebra. Starting from the tube of the tilted algebra, we use knitting to the left until we reach an indecomposable projective C-module P_i. We insert a new injective at the position $\tau^2 P_i$ by requiring that its socle quotient is equal to τ_C of the unique non-projective indecomposable summand of the radical of P_i in mod C. Lemma 34 shows that this module is actually the indecomposable injective module $\tilde{I}_i$ of the cluster-tilted algebra.

The arguments above will stop functioning if we come to another projective P_ℓ inside the same tube for which there is no sectional path from P_ℓ to P_i. The algorithm still seems to work in all the examples we have computed, but we do not know how to prove it.

Example 35 *We conclude with an example of a tube. Let C be given by the quiver*

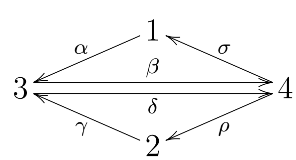


bound by the relations $\alpha\beta = 0$ and $\gamma\delta = 0$. One of the two exceptional tubes in $\operatorname{mod} C$ *is given as*

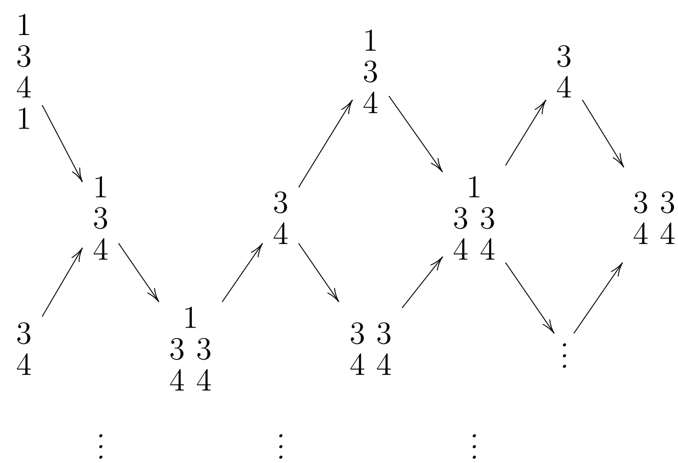


where modules with identical labels must be identified. The module $P_1 = \begin{smallmatrix}1\\3\\4\end{smallmatrix}$ is projective and each module in the tube lies in the τ-orbit of P_1.

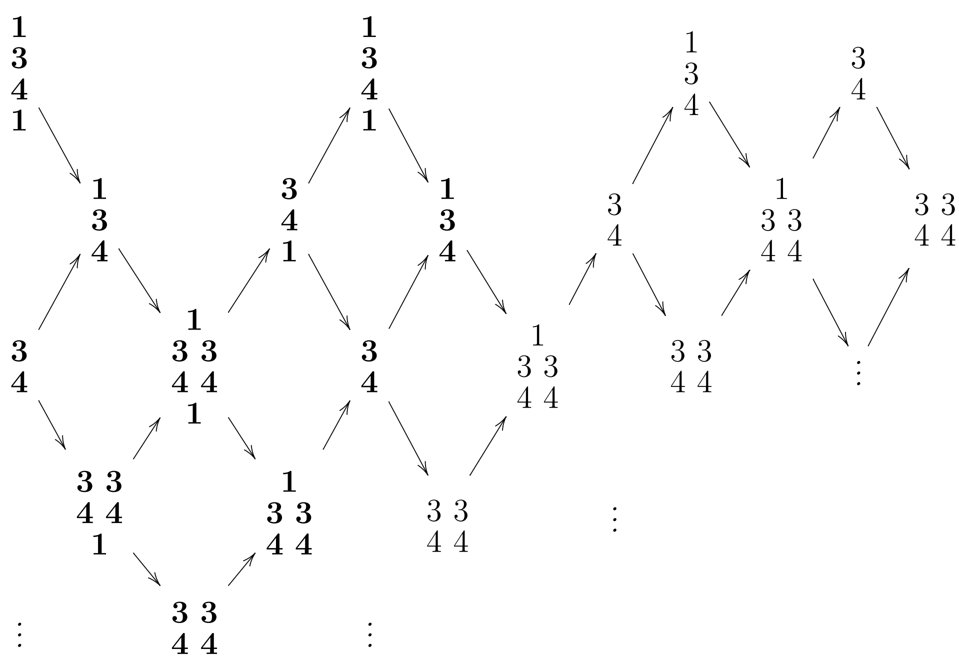
We use our algorithm to construct the tube of the corresponding cluster-tilted algebra $\tilde{C} = C \ltimes \operatorname{Ext}^2_C(DC, C)$ which is given by the quiver



bound by the relations $\alpha\beta = \beta\sigma = \sigma\alpha = \gamma\delta = \delta\rho = \rho\gamma = 0$. First we construct *the new injective module*



and then we continue knitting to the left until the modules start repeating.



The tube in the cluster-tilted algebra consists of the modules in bold face. Modules (in bold face) with identical labels must be identified. Note that the tube of the cluster-tilted algebra in this example is obtained by inserting a coray into the tilted tube.

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