GENERALIZED CHEBYSHEV POLYNOMIALS AND POSITIVITY FOR REGULAR CLUSTER CHARACTERS

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Abstract. We introduce a positive character on the category of finite dimensional representations of the $\mathbb{A}$-double-infinite quiver. We prove several interactions between this character and generalized Chebyshev polynomials finding applications to acyclic cluster algebras.

If Q is any representation-infinite acyclic quiver, we prove that the positivity of the cluster character of an indecomposable regular $\mathbf{k}Q$-module can be deduced from the positivity of the cluster characters of its quasi-composition factors.

When Q is affine, we can deduce positivity properties of various known $\mathbb{Z}$-bases in the coefficient-free acyclic cluster algebra $\mathcal{A}(Q)$.

Contents

1. Introduction

The aim of this article is triple. Firstly, it intends to provide results concerning the positivity of certain bases in acyclic cluster algebras constructed in [SZ04, CZ06, Cer09, Dup08]. Secondly, it introduces a positive character, giving an analogue of a cluster algebra associated to the $\mathbb{A}$-double-infinite quiver which may be compared to [HJ09]. Finally, it investigates further properties of generalized Chebyshev polynomials continuing the investigation initiated in [Dup09a, Dup09d].

Cluster algebras were introduced by Fomin and Zelevinsky in order to design a combinatorial framework for studying positivity in algebraic groups and canonical bases in quantum groups [FZ02, FZ03, BFZ05, FZ07]. Since then, cluster algebras found applications in various areas of mathematics like Lie theory, combinatorics, Teichmüller theory, Poisson geometry or quiver representations.

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A (coefficient-free) cluster algebra $\mathcal{A}$ is a commutative $\mathbb{Z}$-algebra equipped with a distinguished set of generators, called *cluster variables*, gathered into possibly overlapping sets of fixed cardinality, called *clusters*. Monomials in variables belonging to a same cluster are called *cluster monomials*. According to the so-called *Laurent phenomenon* [FZ02], it is known that $\mathcal{A} \subset \bigcap_{\mathbf{c}} \mathbb{Z}[\mathbf{c}^{\pm 1}]$ where $\mathbf{c}$ runs over the clusters in $\mathcal{A}$. An element $y \in \mathcal{A}$ is called *positive* if $y \in \bigcap_{\mathbf{c}} \mathbb{N}[\mathbf{c}^{\pm 1}] \setminus \{0\}$ where $\mathbf{c}$ runs over the clusters in $\mathcal{A}$. It is conjectured that cluster variables are always positive elements of the cluster algebra. This conjecture, known as the *positivity conjecture*, was established for cluster algebras coming from surfaces [Sch08, MSW09] (including acyclic cluster algebras of type $\mathbb{A}, \widetilde{\mathbb{A}}, \mathbb{D}, \widetilde{\mathbb{D}}$) and for rank two cluster algebras [SZ04, MP06, Dup09b]. It is also known that, in any acyclic cluster algebra, cluster variables can be expressed as substraction-free Laurent polynomials in the initial cluster [CR08].

Beyond the positivity conjecture, a problem of first interest is to construct $\mathbb{Z}$-bases constituted of positive elements in cluster algebras. Some bases were constructed in particular cases (see e.g. [SZ04, CK08, CZ06, Cer09, GLS08, Dup08, DXX09]) but it is in general not known whether these bases are constituted of positive elements or not. It is in general not even known if the elements in theses bases can be expressed as substraction-free Laurent polynomials in the initial cluster.

If $\mathcal{A}(Q)$ is the acyclic cluster algebra associated to an acyclic quiver Q, it is well-known that the *cluster category* $\mathcal{C}_Q$ (over an algebraically closed field $\mathbf{k}$) introduced in [BMR$^+$06] provides a nice categorical framework for studying the cluster algebra $\mathcal{A}(Q)$. The *Caldero-Chapoton map* was introduced in order to have an explicit realization of this categorification [CC06, CK06]. It is an explicit map $X_? : M \mapsto X_M$ from the set of objects in the cluster category $\mathcal{C}_Q$ to the ring of Laurent polynomials in the initial cluster of $\mathcal{A}(Q)$. The element X_M is called the *cluster character of* M (with respect to the canonical cluster-tilting object in $\mathcal{C}_Q$). Roughly speaking, X_M is a generating series for Euler characteristics of quiver grassmannians of M. The Caldero-Chapoton map induces a 1-1 correspondence between the set of indecomposable rigid (that is, without self-extensions) objects in $\mathcal{C}_Q$ and the set of cluster variables in $\mathcal{A}(Q)$. In also induces is a 1-1 correspondence $T \mapsto \mathbf{c}_T$ from the set of cluster-tilting objects in $\mathcal{C}_Q$ to the set of clusters in $\mathcal{A}(Q)$ [CK06]. In these terms, a family of cluster of particular interest is constituted by clusters $\mathbf{c}_T$ such that T does not contain any regular $\mathbf{k}Q$-module as a direct summand. These clusters are called *concealed* clusters. Given a regular component $\mathcal{R}$ in the Auslander-Reiten quiver $\Gamma(\mathbf{k}Q)$ of $\mathbf{k}Q$-mod and a cluster-tilting object T in $\mathcal{C}_Q$, we say that $\mathcal{R}$ and T are *compatible* if $\mathcal{R}$ does not contain any indecomposable direct summand of T as a quasi-simple module. If Q is an affine quiver, we will say that a cluster $\mathbf{c}_T$ is *unmixed* if T is concealed or if there exists an exceptional tube $\mathcal{T}$ in $\Gamma(\mathbf{k}Q)$ compatible with T.

Given an indecomposable rigid object M, the Caldero-Chapoton map provides a closed expansion formula of the cluster variable X_M in the initial cluster. Based on a conjecture of Caldero and Keller, Palu introduced in [Pal08] a generalization of the Caldero-Chapoton that allows to express X_M in any cluster $\mathbf{c}_T$. In these terms, the positivity conjecture amounts to say that $X_M \in \mathbb{N}[\mathbf{c}_T^{\pm 1}]$ for any indecomposable rigid object M in $\mathcal{C}_Q$ and any cluster-tilting object T. In this context, the positivity conjecture can be restated as a problem of positivity of Euler characteristics of quiver grassmannians. This positivity was in particular established for rigid modules over hereditary algebras in [CR08]. Nevertheless, it is known that such positivity results may not be expected for non-rigid modules, even on hereditary algebras [DWZ09, Example 3.5].

When Q is an affine quiver, various $\mathbb{Z}$-bases were constructed in the cluster algebra $\mathcal{A}(Q)$ [SZ04, CZ06, Cer09, GLS08, Dup08]. These bases can always be expressed as a disjoint union of the set of cluster monomials in $\mathcal{A}(Q)$ and of a set of "extra elements" in $\mathcal{A}(Q)$. It was observed that these "extra elements" can be expressed using values of the Caldero-Chapoton map associated to certain non-rigid regular $\mathbf{k}Q$-modules. Since positive elements form a semi-ring, the positivity conjecture would imply positivity of cluster monomials. Nevertheless, the positivity of the "extra elements" is not a direct consequence of the positivity conjecture and positivity of characters associated to non-rigid regular modules must be directly investigated.

In this article, we prove that for any acyclic quiver, if M is an indecomposable regular $\mathbf{k}Q$-module, the positivity of X_M in a cluster $\mathbf{c}_T$ can be deduced from the positivity conjecture as soon as the quasi-composition factors of M are rigid and do not belong to $\mathrm{add}(T[1])$ where $[1]$ denotes the suspension functor in $\mathcal{C}_Q$. As a consequence, if Q is an affine quiver, this implies that the "extra elements" can be expressed as substraction-free Laurent polynomials in any unmixed cluster.

Note that we will prove positivity results using cluster characters but without investigating directly Euler characteristics of quiver grassmannians. Instead, we will rely on the fact that cluster characters of regular $\mathbf{k}Q$-modules are governed by the combinatorics of the so-called *generalized Chebyshev polynomials* [Dup09a, Dup09d, Dup09c]. In general, generalized Chebyshev polynomials allow to express combinatorially the character X_M of a regular $\mathbf{k}Q$-module M in terms of the characters associated to its quasi-composition factors. If Q is affine and M belongs to an exceptional tube, its quasi-composition factors are rigid and thus X_M is a polynomial in cluster variables. If M belongs to a homogeneous tube, its (unique) quasi-composition factor is not unique rigid so that generalized Chebyshev polynomials are not enough to express X_M in terms of cluster variables. The *difference properties* established in [Dup08, DXX09, Dup09c], allow nevertheless to express X_M in terms of the characters associated to the quasi-simple modules in an exceptional tube. In this article, we introduce a new family of polynomials, called Δ-*polynomials*, in order to express these difference properties combinatorially. Using these Δ-polynomials, we can establish positivity results for characters of modules contained in homogeneous tubes. This allows in particular to prove that the elements in the generic basis (resp. Sherman-Zelevinsky basis, Caldero-Zelevinsky basis) of the cluster algebra $\mathcal{A}(Q)$ introduced in [Dup08] (resp. [SZ04, Cer09], [CZ06]) belong to $\mathbb{N}[\mathbf{c}^{\pm 1}]$ for any unmixed cluster $\mathbf{c}$ in $\mathcal{A}(Q)$.

We will establish several properties of generalized Chebyshev polynomials generalizing the classical properties of normalized Chebyshev polynomials of the second kind. In particular we will provide an algebraic characterizations of generalized Chebyshev polynomial analogue to the well-known identity $S_n(t+t^{-1}) = \sum_{k=0}^{n} t^{n-2k}$ for normalized Chebyshev polynomials of the second kind. We will also establish some differential properties of generalized Chebyshev polynomials and Δ-polynomials. This will be useful in order to prove surprising connections between combinatorics of generalized Chebyshev (and Δ-polynomials) and positivity in acyclic cluster algebras. In this article, we will fit to the context of coefficient-free cluster algebras. For this reason, in our applications to cluster characters, we will only use the generalized Chebyshev polynomials as they were introduced in [Dup09a]. Nevertheless all the results on generalized Chebyshev polynomials will be established in the broader context of quantized Chebyshev polynomials, introduced in [Dup09d] in order to study acyclic cluster algebras with principal coefficients.

In order to prove these connections, we introduce a map $M \mapsto f_M$, called *f-map*, from the category of finite dimensional representations of the quiver $\mathbb{A}_\infty^\infty$ to the semi-ring of Laurent polynomials $\mathbb{N}[t_i^{\pm 1} | i \in \mathbb{Z}]$. The characters associated to indecomposable representations of $\mathbb{A}_\infty^\infty$ provide an analogue of a cluster algebra associated to the infinite quiver $\mathbb{A}_\infty^\infty$. Namely, for every indecomposable representation M of $\mathbb{A}_\infty^\infty$, f_M is a substraction-free Laurent polynomials with denominator vector $\mathbf{dim}\, M$. This f-map, together with the differential properties of the generalized Chebyshev polynomials (resp. Δ-polynomials) will allow to establish positivity properties for characters of modules in regular components containing rigid modules (resp. homogeneous tubes).

The paper is organized as follows. In Section 2, we recall the necessary background concerning cluster categories and cluster characters and we state the main results of the article. In Section 3, we introduce the notion of *f-map* on the category rep($\mathbb{A}_\infty^\infty$). Section 4 is devoted to the study of positivity properties of Chebyshev polynomials and Δ-polynomials. We apply these results to cluster characters associated to arbitrary regular $\mathbf{k}Q$-modules in Section 5. Finally in Section 6, we apply our methods to bases in cluster algebras of affine types.

2. Background and main results

Throughout the paper, $\mathbf{k}$ denotes an algebraically closed field. Let Q be a quiver. We always denote by Q_0 its set of vertices and Q_1 its set of arrows. We always assume that Q_0 and Q_1 are finite sets and that the underlying unoriented graph of Q is connected. If Q contains no oriented cycles, it is called *acyclic*.

For any finite dimensional $\mathbf{k}$-algebra B, we denote by mod$-B$ the category of finitely generated right B-modules, by $K_0(\text{mod}-B)$ the Grothendieck group of mod$-B$ and for any B-module M, we denote by $[M]$ its class in $K_0(\text{mod}-B)$.

Let Q be an acyclic quiver, we denote by $\mathbf{k}Q$ the path algebra of Q and by $\mathbf{k}Q$-mod the category of finitely generated left-modules over $\mathbf{k}Q$ which we identify with the category rep(Q) of finite dimensional representations of the quiver Q over $\mathbf{k}$.

We fix a Q_0-tuple $\mathbf{u} = (u_i | i \in Q_0)$ of indeterminates over $\mathbb{Z}$ and we denote by $\mathcal{A}(Q)$ the (coefficient-free) cluster algebra with initial seed $(Q, \mathbf{u})$. We denote by $\mathcal{M}(Q)$ the set of cluster monomials in $\mathcal{A}(Q)$.

2.1. Regular modules over hereditary algebras.

Since we will mainly be interested in cluster characters associated to regular $\mathbf{k}Q$-modules, we shall first briefly recall some terminology and classical results concerning the structure of $\mathbf{k}Q$-mod for any acyclic quiver Q.

A connected component in the Auslander-Reiten quiver $\Gamma(\mathbf{k}Q)$ of $\mathbf{k}Q$-mod is called *regular* if it does not contain any projective nor injective $\mathbf{k}Q$-module. A $\mathbf{k}Q$-module is called *regular* if all its indecomposable direct summands belong to regular components. It is well-known that every regular component $\mathcal{R}$ is of the form $\mathbb{Z}\mathbb{A}_\infty/(p)$ for some $p \geq 0$ (see e.g [ARS97, Section VIII.4, Theorem 4.15]). If

$p \geq 1$, $\mathcal{R}$ is called a *tube of rank p*. If $p = 1$, the tube is called *homogeneous*, if $p > 1$, the tube is called *exceptional*. If $p = 0$, $\mathcal{R}$ is called a *sheet*.

Let $\mathcal{R}$ be a regular component in $\Gamma(\mathbf{k}Q)$ and M be an indecomposable object in $\mathcal{R}$. Then there exists a unique family $\{R^{(0)}, R^{(1)} \ldots, R^{(n)}\}$ of indecomposable $\mathbf{k}Q$-modules in $\mathcal{R}$ such that $R^{(0)} = 0$ and such that there is a sequence of irreducible monomorphisms

$$R^{(1)} \longrightarrow R^{(2)} \longrightarrow \cdots \longrightarrow R^{(n)} = M.$$

$R^{(1)}$ is called the *quasi-socle* of M and is denoted by q.soc$M$, $R^{(n-1)}$ is called the *quasi-radical* of M and is denoted by q.rad$M$. The integer n is called the *quasi-length* of M. The module M is called *quasi-simple* if its quasi-length is equal to 1. The quotients $M_i = R^{(i)}/R^{(i-1)}, i = 1, \ldots, n$ are called the *quasi-composition factors* of M. Quasi-composition factors are always quasi-simple and moreover $\tau M_i \simeq M_{i-1}$ for every $i = 2, \ldots, n$.

If Q is an affine quiver, that is of type $\widetilde{\mathbb{A}}, \widetilde{\mathbb{D}}$ or $\widetilde{\mathbb{E}}$, the regular components form a $\mathbb{P}^1(\mathbf{k})$-family of tubes and at most three of these tubes are exceptional [Rin84, Section 3.6]. If Q is wild, every regular component is a sheet [Rin78]. It is well-known that quasi-simple $\mathbf{k}Q$-modules in exceptional (resp. homogeneous) tubes are always (resp. never) rigid and that quasi-simple modules in sheets can be rigid or not.

2.2. **Cluster categories and cluster characters.** Let $D^b(\mathbf{k}Q)$ be the bounded derived category of $\mathbf{k}Q$-mod. It is a triangulated category with shift functor [1] and Auslander-Reiten translation τ. As introduced in [BMR$^+$06], the *cluster category* is the orbit category $\mathcal{C}_Q$ of the auto-equivalence $\tau^{-1}[1]$ in $D^b(\mathbf{k}Q)$. It is a triangulated category and the canonical functor $D^b(\mathbf{k}Q) \longrightarrow \mathcal{C}_Q$ is triangulated [Kel05].

For any finite-dimensional $\mathbf{k}$-algebra B, any right-B-module M and any $\mathbf{e} \in K_0(\text{mod}-B)$, the *grassmannian of submodules of M of dimension $\mathbf{e}$* is

$$\text{Gr}_{\mathbf{e}}(M) = \{N \subset M | [N] = \mathbf{e}\}.$$

It is a projective variety and we denote by $\chi(\text{Gr}_{\mathbf{e}}(M))$ its Euler characteristic with respect to the simplicial (resp. étale) cohomology if $\mathbf{k} = \mathbb{C}$ (resp. if $\mathbf{k}$ is arbitrary).

Extending an idea of Caldero and Chapoton [CC06] and following an idea of Caldero and Keller [CK08], for any cluster tilting object T in $\mathcal{C}_Q$, Palu defined a map

$$X_?^T : \text{Ob}(\mathcal{C}_Q) \longrightarrow \mathbb{Z}[\mathbf{c}_T^{\pm 1}]$$

called *cluster character associated to T* [Pal08]. We shall now review this definition. Let $T = \bigoplus_i T_{i \in Q_0}$ be a cluster-tilting object in $\mathcal{C}_Q$, we set

$$F_T = \text{Hom}_{\mathcal{C}_Q}(T, -) : \mathcal{C}_Q \longrightarrow \text{mod}-B$$

where $B = \text{End}_{\mathcal{C}_Q}(T)$. We denote by $\langle -, - \rangle$ the truncated Euler form on mod$-B$ and by $\langle -, - \rangle_a$ the skew-symmetrized Euler form (see [Pal08] for details).

The *cluster character (associated to T) of an object M* in $\mathcal{C}_Q$ is :

$$X_M^T = \begin{cases} c_i & \text{if } M \simeq T_i[1] \,; \\ \sum_{\mathbf{e} \in K_0(\text{mod}-B)} \chi(\text{Gr}_{\mathbf{e}}(F_T M)) \prod_{i \in Q_0} c_i^{\langle S_i, \mathbf{e} \rangle_a - \langle S_i, F_T M \rangle} & \text{otherwise,} \end{cases}$$

where $\mathbf{c}_T = \{c_i | i \in Q_0\}$ and S_i denotes is the simple right-B-module associated to the vertex i. It is known that when $T = \mathbf{k}Q$, the cluster character $X_?^T$ coincides with the Caldero-Chapoton map introduced in [CC06, CK06] (see [Pal08, Section

5]). In order to simplify notations, we will simply write X_M instead of $X_M^{\mathbf{k}Q}$ the cluster character of an object M associated to $T = \mathbf{k}Q$.

Definition 2.1. Let T be any cluster-tilting object in $\mathcal{C}_Q$, $B = \mathrm{End}_{\mathcal{C}_Q}(T)$ and $\mathbf{c}_T = \{c_i | i \in Q_0\}$. For any object M in $\mathcal{C}_Q$ which is not in $\mathrm{add}(T[1])$ and for any $\mathbf{e} \in K_0(\mathrm{mod}{-}B)$,the ***e**-component of* X_M^T is

$$X_M^T(\mathbf{e}) = \chi(\mathrm{Gr}_{\mathbf{e}}(F_T M)) \prod_{i \in Q_0} c_i^{\langle S_i, \mathbf{e} \rangle_a}.$$

The *interior of* X_M^T as

$$\mathrm{int}(X_M^T) = X_M^T - \left(X_M^T(0) + X_M^T([F_T M])\right).$$

We may now state our first main result :

Theorem 5.3. *Let Q be an acyclic quiver and T be a cluster-tilting object in $\mathcal{C}_Q$. Let M be an indecomposable regular $\mathbf{k}Q$-module. Assume that, for any quasi-composition factor N of M, N is rigid, not in* $\mathrm{add}(T[1])$ *and* $\mathrm{int}(X_N^T) \in \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$. *Then,*

$$X_M \in \mathbb{N}[\boldsymbol{c}_T^{\pm 1}] \cap \mathcal{A}(Q).$$

Using the positivity conjecture established in particular for affine quivers of types $\widetilde{\mathbb{A}}$ and $\widetilde{\mathbb{D}}$ in [MSW09], we can deduce the following corollary :

Corollary 6.3. *Let Q be a quiver of affine type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$. Then,*

$$X_M \in \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}^{\pm 1}]$$

for any cluster $\mathbf{c}$ in $\mathcal{A}(Q)$ and any object M in $\mathcal{C}_Q$.

2.3. **Generalized Chebyshev Polynomials.** Generalized Chebyshev polynomials were introduced in [Dup09a] in order to understand the behavior of the Caldero-Chapoton map on the regular components of $\Gamma(\mathbf{k}Q)$ where Q is any representation-infinite quiver. A quantized version was introduced in [Dup09d] in order to fit to the context of acyclic cluster algebras with principal coefficients at the initial seed.

For every $n \geq 1$, the *n-th quantized generalized Chebyshev polynomial* if the polynomial P_n given by

$$P_n(q_1, \ldots, q_n, t_1, \ldots, t_n) = \det \begin{bmatrix} t_n & 1 & & & (0) \\ q_n & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ (0) & & & q_2 & t_1 \end{bmatrix}$$

where $\mathbf{q} = \{q_i | i \in \mathbb{Z}\}$ is a family of indeterminates over $\mathbb{Z}$ and $\mathbf{t} = \{t_i | i \in \mathbb{Z}\}$ is a family of indeterminates over $\mathbb{Z}[\mathbf{q}]$. Note that P_n does not actually depend on q_1 but it is convenient to consider P_n as a polynomial in $2n$ variables.

We denote by $\underline{P}_n(t_1, \ldots, t_n) = P_n(1, \ldots, 1, t_1, \ldots, t_n)$ the specialization at $\mathbf{q} = \{1\}$ of the n-th quantized generalized Chebyshev polynomial. The polynomial $\underline{P}_n$ is called the *n-th generalized Chebyshev polynomial* and coincides with the n-th generalized Chebyshev polynomial of infinite rank introduced in [Dup09a]. When there is no possible confusion, we will abuse the terminology and say "generalized Chebyshev polynomials" for quantized generalized Chebyshev polynomials.

Let Q be any representation-infinite quiver. Let $\mathcal{R}$ be a regular component in $\Gamma(\mathbf{k}Q)$ and M be an indecomposable regular $\mathbf{k}Q$-module in $\mathcal{R}$. Then, it is proved in [Dup09a] that

$$X_M = \underline{P}_n(X_{\mathrm{q.soc}M}, \ldots, X_{\tau^{-n+1}\mathrm{q.soc}M})$$

where n denotes the quasi-length of M.

We recall that the n-th *normalized Chebyshev polynomial of the second kind* is the polynomial S_n given by :

$$S_0(x) = 1,\ S_1(x) = x,$$

$$S_n(x) = xS_{n-1} - S_{n-2}(x) \text{ for any } n \geq 2.$$

Generalized Chebyshev polynomials generalize these polynomials in the sense that

$$S_n(x) = \underline{P}_n(x, \ldots, x).$$

It is well-known that S_n is characterized by $S_n(t+t^{-1}) = \sum_{k=0}^{n} t^{n-2k}$. In this article, we obtain a generalization of this characterization for quantized generalized Chebyshev polynomials. Namely, we prove that

$$P_n(q_1, \ldots, q_n, \frac{q_1}{t_1} + t_2, \ldots, \frac{q_n}{t_n} + t_{n+1}) = \frac{1}{t_1 \ldots t_n} \sum_{k=1}^{n+1} \left(\prod_{i=1}^{k-1} q_i \prod_{j=k}^{n} t_j t_{j+1} \right)$$

where by convention, the empty products are equal to 1.

2.4. **Bases in affine cluster algebras.** In this section, we assume that $\mathbf{k} = \mathbb{C}$ is the field of complex numbers. As mentioned in the introduction, the initial aim for getting interested in the positivity of characters of regular modules is that we want to be able to construct positive bases in acyclic cluster algebras and we know that if Q is representation-infinite, we have to consider regular modules to do so.

More precisely, if Q is an affine quiver, there are three known $\mathbb{Z}$-bases of particular interest in $\mathcal{A}(Q)$ which may be constructed using the cluster character $X_?$ and normalized Chebyshev polynomials of first and second kinds.

We recall that for any $n \geq 0$, we denote by $F_n \in \mathbb{Z}[x]$ the *normalized Chebyshev polynomial of the first kind* defined by

$$F_0(x) = 2,\ F_1(x) = x,$$

$$F_n(x) = xF_{n-1} - F_{n-2}(x) \text{ for any } n \geq 2.$$

We denote by δ the positive minimal imaginary root of Q. It is known that given two quasi-simple representations M and N in homogeneous tubes of $\Gamma(\mathbf{k}Q)$, the characters X_M and X_N coincide (see e.g. [Dup08]). This common value is denoted by X_δ and is called the *generic value of dimension* δ, following the terminology of [Dup08].

The first known $\mathbb{Z}$-basis, denoted by $\mathcal{B}(Q)$, originally appeared in [SZ04] for constructing the canonically positive basis in the cluster algebra associated to the Kronecker quiver (see also [Cer09] for type $\widetilde{\mathbb{A}}_{2,1}$ and [Dup09c] for a general representation-theoretic interpretation). It is given by

$$\mathcal{B}(Q) = \mathcal{M}(Q) \sqcup \{F_l(X_\delta)X_R | l \geq 1,\ R \text{ is a rigid regular } \mathbf{k}Q\text{-module}\}.$$

We call $\mathcal{B}(Q)$ the *Sherman-Zelevinsky basis* of the cluster algebra $\mathcal{A}(Q)$.

The second known $\mathbb{Z}$-basis, denoted by $\mathcal{C}(Q)$ was initially introduced in [CZ06] in order to construct another basis in the cluster algebra associated to the Kronecker quiver. It is given by :

$$\mathcal{C}(Q) = \mathcal{M}(Q) \sqcup \{S_l(X_\delta)X_R | l \geq 1, \, R \text{ is a rigid regular } \mathbf{k}Q\text{-module}\}.$$

We call $\mathcal{C}(Q)$ the *Caldero-Zelevinsky basis* of the cluster algebra $\mathcal{A}(Q)$.

The last one is denoted by $\mathcal{G}(Q)$. It was introduced in [Dup08]. For a given affine quiver Q, the set $\mathcal{G}(Q)$ is given by

$$\mathcal{G}(Q) = \mathcal{M}(Q) \sqcup \left\{X_\delta^l X_R | l \geq 1, \, R \text{ is a rigid regular } \mathbf{k}Q\text{-module}\right\}.$$

We call $\mathcal{G}(Q)$ the *generic basis* of the cluster algebra $\mathcal{A}(Q)$.

Initially, the notion of canonically positive bases was introduced in order to provide a combinatorial description of Luzstig's dual canonical basis (see e.g. [FZ02]). It was observed that, if Q is not of finite representation type, it is in general not the case. For example, when Q is the Kronecker quiver, Lusztig's dual canonical basis is known to coincide with the basis $\mathcal{C}(Q)$ constructed by Caldero and Zelevinsky in [CZ06] (see also [Nak09] for a more general construction of $\mathcal{C}(Q)$ using quiver varieties). Also, it follows from works of Geiss-Leclerc-Schröer, that the generic basis $\mathcal{G}(Q)$ constructed in [Dup08] coincides with Lusztig's dual semicanonical basis [GLS08, Sch09].

Our main result concerning these bases is :

Theorem 6.1. *Let Q be an affine quiver with at least three vertices. Let T be a cluster-tilting object in $\mathcal{C}_Q$ such that there exists an exceptional tube $\mathcal{T}$ not containing any T_i as a quasi-simple module. Assume that* $\mathrm{int}(X_M) \in \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$ *for any quasi-simple module M in $\mathcal{T}$. Then, for any $l \geq 1$, the following hold :*

1. $F_l(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$;
2. $S_l(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$;
3. $X_\delta^l \in \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$.

As a corollary, we obtain :

Corollary 6.7. *Let Q be a quiver of affine type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$. Let $\mathbf{c}$ be any unmixed cluster in $\mathcal{A}(Q)$. Then, the following hold :*

1. $\mathcal{B}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}^{\pm 1}]$;
2. $\mathcal{C}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}^{\pm 1}]$;
3. $\mathcal{G}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}^{\pm 1}]$.

3. A positive character on $\mathrm{rep}(\mathbb{A}_\infty^\infty)$

This section is devoted to the introduction and the investigation of a particular map $f_?$ defined over the objects in the category of finite dimensional representations of the $\mathbb{A}$-double-infinite quiver and taking its value in a semi-ring of substraction-free Laurent polynomials. The first aim of this map is to prove positivity properties for generalized Chebyshev polynomials. Also, this map may be thought as an analogue of a cluster character corresponding to a cluster algebra associated to the infinite quiver $\mathbb{A}_\infty^\infty$.

3.1. **The f-map on** $\text{rep}(\mathbb{A}_\infty^\infty)$**.** We denote by $\mathbb{A}_\infty^\infty$ the quiver

$$\mathbb{A}_\infty^\infty : \qquad \cdots \longrightarrow 1 \longrightarrow 0 \longrightarrow -1 \longrightarrow \cdots$$

Let $\text{rep}(\mathbb{A}_\infty^\infty)$ be the category of finite-dimensional representations of $\mathbb{A}_\infty^\infty$ over $\mathbf{k}$. Then $\text{rep}(\mathbb{A}_\infty^\infty)$ is an hereditary abelian length category. For every $i \in \mathbb{Z}$, we denote by Σ_i the simple representation associated to the vertex i and for every $n \geq 1$, we denote by $\Sigma_i^{(n)}$ the unique indecomposable representation with socle Σ_i and length n. By "representation", we will always mean finite dimensional representation over $\mathbf{k}$. If M is a representation of $\mathbb{A}_\infty^\infty$, we denote by $M(i)$ the $\mathbf{k}$-vector space at vertex i.

The AR-quiver of $\text{rep}(\mathbb{A}_\infty^\infty)$ is isomorphic to $\mathbb{Z}\mathbb{A}_\infty$ and can be (locally) depicted as follows :

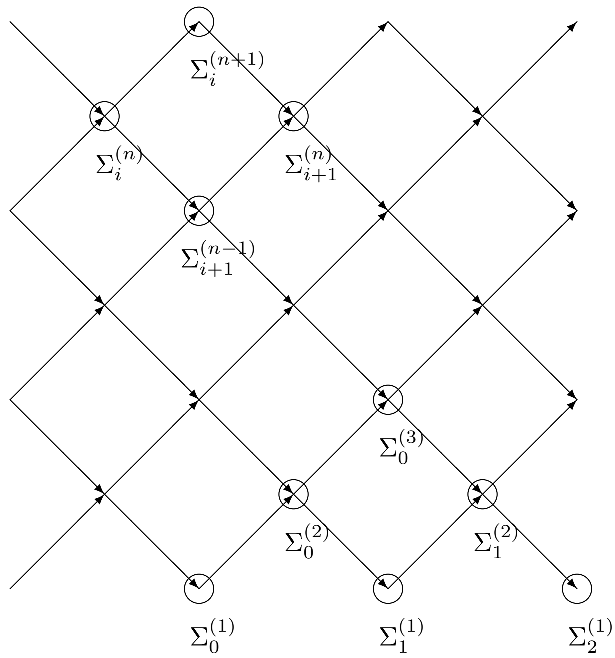

FIGURE 1. The AR-quiver of $\text{rep}(\mathbb{A}_\infty^\infty)$

We denote by $\mathbb{N}_c^{\mathbb{Z}}$ the set of sequence of non-negative integers with finite support parametrized by $\mathbb{Z}$. For every representation M of $\mathbb{A}_\infty^\infty$, we denote by $\mathbf{dim}\, M$ its dimension vector, that is, the element of $\mathbb{N}_c^{\mathbb{Z}}$ given by $(\mathbf{dim}\, M)_i = \dim_{\mathbf{k}} M(i)$ for every $i \in \mathbb{Z}$. The *support* of an element $\mathbf{e} \in \mathbb{N}_c^{\mathbb{Z}}$ is the set

$$\text{supp}(\mathbf{e}) = \{\mathrm{i} \in \mathbb{Z} | \mathrm{e_i} \neq 0\} .$$

The *support* of an object M in $\text{rep}(\mathbb{A}_\infty^\infty)$ is by definition the support of its dimension vector.

Given a representation M of $\mathbb{A}_\infty^\infty$ and an element $\mathbf{e} \in \mathbb{N}_c^{\mathbb{Z}}$, the *grassmannian of subrepresentations of M of dimension $\mathbf{e}$* is the set of subrepresentations N of M such that $\mathbf{dim}\, N = \mathbf{e}$. This is a closed variety in the standard vector spaces grassmannian and it is thus a projective variety. Inspired by the theory cluster characters, we construct a certain normalized generating series for Euler characteristics of this variety.

We fix a family $\mathbf{q} = \{q_i | i \in \mathbb{Z}\}$ of indeterminates over $\mathbb{Z}$ and a family $\mathbf{t} = \{t_i | i \in \mathbb{Z}\}$ of indeterminates over $\mathbb{Z}[\mathbf{q}]$.

Definition 3.1. The *f-map* on $\mathrm{rep}(\mathbb{A}_\infty^\infty)$ is the map

$$f_? : \mathrm{Ob}(\mathrm{rep}(\mathbb{A}_\infty^\infty)) \longrightarrow \mathbb{Z}[\mathbf{q}][\mathbf{t}^{\pm 1}]$$

defined by

$$f_M = \frac{1}{\mathbf{t}^{\mathbf{dim}\, M}} \sum_{\mathbf{e} \in \mathbb{N}_c^{\mathbb{Z}}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) \prod_{i \in \mathbb{Z}} q_i^{e_i} (t_i t_{i+1})^{m_i - e_i}.$$

where $m_i = \dim M(i)$ for every $i \in \mathbb{Z}$.

Note that $f_?$ is invariant under isomorphisms.

We recall that the *denominator vector* of a Laurent polynomial $L \in \mathbb{Z}[\mathbf{q}][\mathbf{t}^{\pm 1}]$ is the element $\mathrm{den}(L) \in \mathbb{Z}^{\mathbb{Z}}$ such that there exists a polynomial $P(\mathbf{t}) \in \mathbb{Z}[\mathbf{q}][\mathbf{t}]$ not divisible by any t_i such that

$$L = \frac{P(\mathbf{t})}{\mathbf{t}^{\mathrm{den}(L)}}.$$

It follows from the definition that $\mathrm{den}(f_M) = \mathbf{dim}\, M$ for any M in $\mathrm{rep}(\mathbb{A}_\infty^\infty)$. This is an analogue of Caldero-Keller's denominator theorem for cluster characters [CK06, Theorem 3].

Example 3.2. For example, for every $i \in \mathbb{Z}$, the f-map of the simple representation Σ_i is given by

$$f_{\Sigma_i} = \frac{q_i}{t_i} + t_{i+1}.$$

The f-map satisfies the following multiplicative property:

Lemma 3.3. *Let M, N be objects in $\mathrm{rep}(\mathbb{A}_\infty^\infty)$, then*

$$f_M f_N = f_{M \oplus N}.$$

Proof. For every dimension vector $\mathbf{e}$, there is an isomorphism of varieties

$$\mathrm{Gr}_{\mathbf{e}}(M \oplus N) \simeq \bigsqcup_{\mathbf{f}+\mathbf{g}=\mathbf{e}} \mathrm{Gr}_{\mathbf{f}}(M) \times \mathrm{Gr}_{\mathbf{g}}(M).$$

Thus, if we set $\mathbf{m} = \mathbf{dim}\, M$ and $\mathbf{n} = \mathbf{dim}\, N$, we get

$$\begin{aligned} f_{M \oplus N} &= \frac{1}{\mathbf{t}^{\mathbf{dim}\, M + \mathbf{dim}\, N}} \sum_{\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{e}}(M \oplus N)) \prod_{i \in \mathbb{Z}} q_i^{e_i} (t_i t_{i+1})^{m_i + n_i - e_i} \\ &= \frac{1}{\mathbf{t}^{\mathbf{dim}\, M + \mathbf{dim}\, N}} \sum_{\mathbf{f}+\mathbf{g}=\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{f}}(M)) \chi(\mathrm{Gr}_{\mathbf{f}}(N)) \prod_{i \in \mathbb{Z}} q_i^{f_i + g_i} (t_i t_{i+1})^{(m_i - f_i) + (n_i - g_i)} \\ &= f_M f_N \end{aligned}$$

□

As a corollary, we obtain the positivity of the f-map.

Corollary 3.4. *For every object M in* $\mathrm{rep}(\mathbb{A}_\infty^\infty)$, $f_M \in \mathbb{N}[\boldsymbol{q}][\boldsymbol{t}^{\pm 1}]$.

Proof. Since $\mathbb{N}[\mathbf{q}][\mathbf{t}^{\pm 1}]$ is a semi-ring, it follows from lemma 3.3 that it is enough to prove the result for M indecomposable. For every indecomposable object M in $\mathrm{rep}(\mathbb{A}_\infty^\infty)$ and every $\mathbf{e} \in \mathbb{N}_c^{\mathbb{Z}}$, $\mathrm{Gr}_{\mathbf{e}}(M)$ is either empty or reduced to a point. Thus, $\chi(\mathrm{Gr}_{\mathbf{e}}(M)) \in \{0, 1\}$ and therefore $f_M \in \mathbb{N}[\mathbf{q}][\mathbf{t}^{\pm 1}]$. □

Remark 3.5. Note that we can obtain a combinatorial expression for the f-map of an indecomposable object in $\mathrm{rep}(\mathbb{A}_\infty^\infty)$. Namely, an element $\mathbf{e} \in \mathbb{N}_c^{\mathbb{Z}}$ will be called an *interval* if its support is an interval in $\mathbb{Z}$. Note that elements $\mathbf{e} \in \mathbb{N}_c^{\mathbb{Z}}$ such that $e_i \leq 1$ for every $i \in \mathbb{Z}$ are completely determined by their support. Let $I \subset \mathbb{Z}$ be an interval, a subset $J \subset I$ is called *terminal in* I if J is empty or if J is an interval such that $\min(J) = \min(I)$. We write $J \lhd I$ if J is terminal in I. Let M be an indecomposable representation of $\mathbb{A}_\infty^\infty$, since $\dim M(i) \leq 1$ for every $i \in \mathbb{Z}$, it is easy to check that

$$\chi(\mathrm{Gr}_{\mathbf{e}}(M)) = \begin{cases} 1 & \text{if supp}(\mathbf{e}) \text{ is terminal in supp(M)}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that for every M indecomposable, we have

$$f_M = \left(\prod_{i \in \mathrm{supp(M)}} \frac{1}{t_i} \right) \left[\sum_{I \lhd \mathrm{supp(M)}} \left(\prod_{i \in I} q_i \right) \left(\prod_{j \in \mathrm{supp(M)} \setminus \mathrm{I}} t_j t_{j+1} \right) \right] \tag{1}$$

We now prove an almost split multiplication formula analogous to [CC06, Proposition 3.10]:

Proposition 3.6. *Let M be an indecomposable object in* $\mathrm{rep}(\mathbb{A}_\infty^\infty)$ *and*

$$0 \longrightarrow N \longrightarrow B \longrightarrow M \longrightarrow 0$$

be an almost split sequence in $\mathrm{rep}(\mathbb{A}_\infty^\infty)$. *Then,*

$$f_M f_N = f_B + \prod_{i \in \mathrm{supp(M)}} q_i.$$

Proof. Let ϵ be any almost split sequence in $\mathrm{rep}(\mathbb{A}_\infty^\infty)$. Then there exists $n \geq 1$ and $k \in \mathbb{Z}$ such that ϵ is

$$0 \longrightarrow \Sigma_k^{(n)} \longrightarrow \Sigma_k^{(n+1)} \oplus \Sigma_{k+1}^{(n-1)} \longrightarrow \Sigma_{k+1}^{(n)} \longrightarrow 0.$$

We thus write $N = \Sigma_k^{(n)}$, $B = \Sigma_k^{(n+1)} \oplus \Sigma_{k+1}^{(n-1)}$ and $M = \Sigma_{k+1}^{(n)}$. We know that $\Sigma_k^{(n)}$ is the representation given by

$$\cdots \longrightarrow 0 \longrightarrow \underset{k+n-1}{\mathbf{k}} \xrightarrow{\ 1\ } \cdots \xrightarrow{\ 1\ } \underset{k}{\mathbf{k}} \xrightarrow{\ 0\ } 0 \longrightarrow \cdots$$

Thus, using the expression given in equation (1), we get

$$f_{\Sigma_k^{(n)}} = \left(\prod_{i=k}^{k+n-1} \frac{1}{t_i} \right) \left[\sum_{I \lhd [k, k+n-1]} \left(\prod_{i \in I} q_i \right) \left(\prod_{j \in [k, k+n-1] \setminus I} t_j t_{j+1} \right) \right].$$

Since proper terminal intervals $I \lhd [k, k+n-1]$ are terminal intervals $I \lhd [k, k+n-2] = \mathrm{supp}(\Sigma_{\mathrm{k}}^{(\mathrm{n}-1)})$, we obtain:

$$f_{\Sigma_k^{(n)}} = \left(\prod_{i=k}^{k+n-1} \frac{1}{t_i}\right) \left[\prod_{i=k}^{k+n-1} q_i + \sum_{I \lhd [k,k+n-2]} \left(\prod_{i \in I} q_i\right) t_{k+n} t_{k+n-1} \left(\prod_{j \in [k,k+n-2] \setminus I} t_j t_{j+1}\right)\right]$$
$$= \left(\prod_{i=k}^{k+n-1} \frac{q_i}{t_i}\right) + \frac{t_{k+n} t_{k+n-1}}{t_{k+n-1}} f_{\Sigma_k^{(n-1)}}$$

In order to simplify notations, we set $\sigma_k^{(n)} = \mathbf{dim}\, \Sigma_k^{(n)}$.

Thus, we proved

$$f_{\Sigma_k^{(n)}} = \frac{\mathbf{q}^{\sigma_k^{(n)}}}{\mathbf{t}^{\sigma_k^{(n)}}} + t_{k+n} f_{\Sigma_k^{(n-1)}}$$

We get

$$f_M f_N = f_{\Sigma_k^{(n)}} f_{\Sigma_{k+1}^{(n)}}$$
$$= f_{\Sigma_k^{(n)}} \left(\frac{\mathbf{q}^{\sigma_{k+1}^{(n)}}}{\mathbf{t}^{\sigma_{k+1}^{(n)}}} + t_{k+1+n} f_{\Sigma_{k+1}^{(n-1)}}\right)$$

and

$$f_B = f_{\Sigma_{k+1}^{(n-1)}} f_{\Sigma_k^{(n+1)}}$$
$$= f_{\Sigma_{k+1}^{(n-1)}} \left(\frac{\mathbf{q}^{\sigma_k^{(n+1)}}}{\mathbf{t}^{\sigma_k^{(n+1)}}} + t_{k+1+n} f_{\Sigma_k^{(n)}}\right).$$

Thus,

$$f_M f_N - f_B = \frac{\mathbf{q}^{\sigma_{k+1}^{(n)}}}{\mathbf{t}^{\sigma_{k+1}^{(n)}}} \left(f_{\Sigma_k^{(n)}} - \frac{q_k}{t_k} f_{\Sigma_{k+1}^{(n-1)}}\right).$$

Now, since every non-empty $I \lhd \mathrm{supp}(\Sigma_{\mathrm{k}}^{(\mathrm{n})})$ contains k, it follows that $(I \setminus \{k\}) \lhd \mathrm{supp}(\Sigma_{\mathrm{k+1}}^{(\mathrm{n}-1)})$ so that:

$$f_{\Sigma_k^{(n)}} = \frac{1}{\mathbf{t}^{\sigma_k^{(n)}}} \left[\prod_{i=k}^{k+n-1} t_i t_{i+1} + q_k \sum_{I \lhd [k+1,k+n-1]} \left(\prod_{i \in I} q_i\right) \left(\prod_{j \in [k+1,k+n-1] \setminus I} t_j t_{j+1}\right)\right]$$
$$= \prod_{i=k+1}^{k+n} t_i + \frac{q_k}{t_k} f_{\Sigma_{k+1}^{(n-1)}}.$$

Thus, we can compute the difference

$$f_M f_{\tau M} - f_B = \frac{\mathbf{q}^{\sigma_{k+1}^{(n)}}}{\mathbf{t}^{\sigma_{k+1}^{(n)}}} \left(f_{\Sigma_k^{(n)}} - \frac{q_k}{t_k} f_{\Sigma_{k+1}^{(n-1)}}\right)$$
$$= \frac{\mathbf{q}^{\sigma_{k+1}^{(n)}}}{\mathbf{t}^{\sigma_{k+1}^{(n)}}} \left(\mathbf{t}^{\sigma_{k+1}^{(n)}} + \frac{q_k}{t_k} f_{\Sigma_{k+1}^{(n-1)}} - \frac{q_k}{t_k} f_{\Sigma_{k+1}^{(n-1)}}\right)$$
$$= \mathbf{q}^{\sigma_{k+1}^{(n)}}$$
$$= \prod_{i \in \mathrm{suppM}} q_i$$

and the proposition is proved. □

Example 3.7. We consider the almost split sequence

$$0 \longrightarrow \Sigma_0^{(2)} \longrightarrow \Sigma_0^{(3)} \oplus \Sigma_1 \longrightarrow \Sigma_1^{(2)} \longrightarrow 0,$$

we compute directly

$$f_{\Sigma_1} = \frac{t_1 t_2 + q_1}{t_1}, \quad f_{\Sigma_0^{(3)}} = \frac{t_1^2 t_0 t_2^2 t_3 + q_0 t_2^2 t_1 t_3 + q_0 q_1 t_3 t_2 q_0 q_1 q_2}{t_0 t_1 t_2}$$

$$f_{\Sigma_0^{(2)}} = \frac{t_1^2 t_0 t_2 + q_0 t_2 t_1 + q_0 q_1}{t_1 t_0}, \quad f_{\Sigma_1^{(2)}} = \frac{t_2^2 t_1 t_3 + q_1 t_3 t_2 + q_1 q_2}{t_2 t_1}.$$

and we check by direct calculation that

$$f_{\Sigma_0^{(2)}} f_{\Sigma_1^{(2)}} = f_{\Sigma_0^{(3)}} f_{\Sigma_1} + q_1 q_2$$

and $\mathrm{supp}(\Sigma_1^{(2)}) = \{1, 2\}$.

Remark 3.8. In a recent paper, Holm and Jørgensen introduced a cluster structure in a 2-Calabi-Yau category **D** whose Auslander-Reiten quiver is isomorphic to $\mathbb{Z}\mathbb{A}_\infty$ and asked if it was possible to define a cluster algebra associated to this cluster structure [HJ09]. We defined the positive character $f_?$ on the category $\mathrm{rep}(\mathbb{A}_\infty^\infty)$ independently for positivity purposes. It would be interesting to understand if the $f_?$-map coincides with some cluster character in the sense of Palu on the category **D**.

4. Applications to generalized Chebyshev polynomials

We shall now study interactions between the f-map and positivity properties of generalized Chebyshev polynomials and Δ-polynomials.

4.1. f-map and Chebyshev polynomials.

The first motivation for introducing the f-map comes from its interaction with generalized Chebyshev polynomials. This allows to provide a substraction-free algebraic identity for generalized Chebyshev polynomials. This identity plays a fundamental role in the study of properties of generalized Chebyshev polynomials with respect to positivity.

Corollary 4.1. *For every $i \in \mathbb{Z}$ and $n \geq 1$, we have*

$$f_{\Sigma_i^{(n)}} = P_n(q_i, \ldots, q_{i+n-1}, f_{\Sigma_i}, \ldots, f_{\Sigma_{i+n-1}}).$$

Proof. This is a direct consequence of Proposition 3.6 and [Dup09d, Theorem 1]. □

Remark 4.2. We recall that the n-th normalized Chebyshev polynomials of the second kind S_n given by $S_n(x) = \underline{P}_n(x, \ldots, x)$. It is characterized by $S_n(t + t^{-1}) = \sum_{k=0}^{n} t^{2k-n}$, thus Corollary 4.1 can be viewed as a generalization of this characterization for quantized generalized Chebyshev polynomials.

Namely, quantized generalized Chebyshev polynomials can be characterized by

$$P_n(q_1, \ldots, q_n, \frac{q_1}{t_1} + t_2, \ldots, \frac{q_n}{t_n} + t_{n+1}) = \frac{1}{t_1 \ldots t_n} \sum_{k=1}^{n+1} \left(\prod_{i=1}^{k-1} q_i \prod_{j=k}^{n} t_j t_{j+1} \right)$$

where by convention, the empty products are equal to 1. For non-quantized generalized Chebyshev polynomials, the identity is thus

$$\underline{P}_n(\frac{1}{t_1}+t_2,\ldots,\frac{1}{t_n}+t_{n+1}) = \frac{1}{t_1\ldots t_n}\sum_{k=1}^{n+1}\left(\prod_{j=k}^{n} t_j t_{j+1}\right).$$

From Corollary 4.1 we deduce directly the following useful result :

Corollary 4.3. *For every $i \in \mathbb{Z}$ and $n \geq 1$,*

$$P_n(q_i,\ldots,q_{i+n-1},\frac{q_i}{t_i}+t_{i+1},\ldots,\frac{q_{i+n-1}}{t_{i+n-1}}+t_{i+n}) \in \mathbb{N}[\boldsymbol{q},\boldsymbol{t}^{\pm 1}].$$

In order to simplify notations, we will adopt the following convention. If $i \leq j$ are integers, we denote by $\mathbf{t}_{[i,j]} = \{t_i,\ldots,t_j\}$ and $\mathbf{q}_{[i,j]} = \{q_i,\ldots,q_j\}$.

In order to the positivity of the "extra elements" in their basis, Sherman and Zelevinsky used Taylor expansions and differential properties of normalized Chebyshev of the first kind [SZ04]. In our context, we will use analogous methods for generalized Chebyshev polynomials. For this, we need to establish some technical results concerning partial derivatives of generalized Chebyshev polynomials :

Lemma 4.4. *For every $n \geq 1$ and $1 \leq i \leq n$, we have*

$$\frac{\partial}{\partial t_i}P_n(\boldsymbol{q}_{[1,n]},\boldsymbol{t}_{[1,n]}) = P_{i-1}(\boldsymbol{q}_{[1,i-1]},\boldsymbol{t}_{[1,i-1]})P_{n-i}(\boldsymbol{q}_{[i+1,n]},\boldsymbol{t}_{[i+1,n]})$$

Proof. We recall that

$$P_n(\mathbf{q}_{[1,n]},\mathbf{t}_{[1,n]}) = \det\begin{bmatrix} t_n & 1 & & & (0) \\ q_n & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ (0) & & & q_2 & t_1 \end{bmatrix}$$

Expanding with respect to the i-th column, we get

$$P_n(\mathbf{q}_{[1,n]},\mathbf{t}_{[1,n]}) = t_i P_{i-1}(\mathbf{q}_{[1,i-1]},\mathbf{t}_{[1,i-1]})P_{n-i}(\mathbf{q}_{[i+1,n]},\mathbf{t}_{[i+1,n]}) + f(\mathbf{q},\mathbf{t})$$

where f does not depend on the variable t_i. Thus, taking the derivative in t_i, we get

$$\frac{\partial}{\partial t_i}P_n(\mathbf{q}_{[1,n]},\mathbf{t}_{[1,n]}) = P_{i-1}(\mathbf{q}_{[1,i-1]},\mathbf{t}_{[1,i-1]})P_{n-i}(\mathbf{q}_{[i+1,n]},\mathbf{t}_{[i+1,n]}).$$

□

We can now deduce the following technical lemma which will be essential in Section 5.

Lemma 4.5. *Let $\boldsymbol{u} = \{u_i \ : i \in \mathbb{Z}\}$ be a family of indeterminates over $\mathbb{Z}[\boldsymbol{q},\boldsymbol{t}]$, then for every $i \in \mathbb{Z}$, $n \geq 1$,*

$$P_n(q_i,\ldots,q_{i+n-1},\frac{q_i}{t_i}+u_i+t_{i+1},\ldots,\frac{q_{i+n-1}}{t_{i+n-1}}+u_{i+n-1}+t_{i+n}) \in \mathbb{N}[\boldsymbol{q},\boldsymbol{t}^{\pm 1},\boldsymbol{u}].$$

Proof. It suffices to prove it for $i=1$. Fix $n \geq 1$. We consider the function

$$(u_1, \ldots, u_n) \mapsto P_n(q_1, \ldots, q_n, \frac{q_1}{t_1} + t_2 + u_1, \ldots, \frac{q_n}{t_n} + t_{n+1} + u_n)$$

Taking the Taylor expansion in $(0, \ldots, 0)$, we get that $P_n(q_1, \ldots, q_n, \frac{q_1}{t_1} + t_2 + u_1, \ldots, \frac{q_n}{t_n} + t_{n+1} + u_n)$ is equal to

$$\sum_{i_1, \ldots, i_n} \frac{\partial^{i_1} \cdots \partial^{i_n}}{\partial t_1 \cdots \partial t_n} P_n(\frac{q_1}{t_1} + t_2, \ldots, \frac{q_n}{t_n} + t_{n+1}) u_1^{i_1} \cdots u_n^{i_n}$$

But according to Lemma 4.4, every partial derivative $\sum_{i_1, \ldots, i_n} \frac{\partial^{i_1} \cdots \partial^{i_n}}{\partial t_1 \cdots \partial t_n} P_n(\frac{q_1}{t_1} + t_2, \ldots, \frac{q_n}{t_n} + t_{n+1})$ is a product of $P_j(\frac{q_k}{t_k} + t_{k+1}, \ldots, \frac{q_{k+j-1}}{t_{k+j-1}} + t_{k+j})$ for some $j < n$. According to Corollary 4.3, each of these $P_j(\frac{q_k}{t_k} + t_{k+1}, \ldots, \frac{q_{k+j-1}}{t_{k+j-1}} + t_{k+j})$ is in $\mathbb{N}[\mathbf{q}, \mathbf{t}^{\pm 1}]$ and thus each partial derivative is in $\mathbb{N}[\mathbf{q}, \mathbf{t}^{\pm 1}]$. □

4.2. **f-map and Δ-polynomials.** We now introduce a family of polynomials called Δ-polynomials for which the motivation is coming from the difference properties introduced in [Dup08, Dup09c]. These polynomials will be of particular interest in Section 6.

Definition 4.6. Let $\mathbf{q} = \{q_i | i \in \mathbb{Z}\}$ be a family of indeterminates over $\mathbb{Z}$ and $\mathbf{t} = \{t_i | i \in \mathbb{Z}\}$ be a family of indeterminates over $\mathbb{Z}[\mathbf{q}]$. For any $p \geq 1$, and $l \geq 1$, we set

$$\Delta_{l,p}(\mathbf{q}_{[1,lp]}, \mathbf{t}_{[1,lp]}) = P_{lp}(\mathbf{q}_{[1,lp]}, \mathbf{t}_{[1,lp]}) - q_1 P_{lp-2}(\mathbf{q}_{[2,lp-1]}, \mathbf{t}_{[2,lp-1]})$$

and

$$\begin{aligned} \underline{\Delta}_{l,p}(\mathbf{t}_{[1,lp]}) &= \Delta_{l,p}(1, \ldots, 1, \mathbf{t}_{[1,lp]}) \\ &= \underline{P}_{lp}(\mathbf{t}_{[1,lp]}) - \underline{P}_{lp-2}(\mathbf{t}_{[2,lp-1]}) \end{aligned}$$

We now prove an analogue of Corollary 4.5 for Δ-polynomials :

Lemma 4.7. *Let $\boldsymbol{q} = \{q_i | i \in \mathbb{Z}\}$ be a family of indeterminates over $\mathbb{Z}$ and $\boldsymbol{t} = \{t_i | i \in \mathbb{Z}\}$, $\boldsymbol{u} = \{u_i | i \in \mathbb{Z}\}$ be families of indeterminates over $\mathbb{Z}[\boldsymbol{q}]$. Then*

$$\Delta_{l,p}(q_1, \ldots, q_p, \frac{q_1}{t_1} + u_1 + t_2, \ldots, \frac{q_{lp}}{t_{lp}} + u_p + t_1) \in \mathbb{N}[\boldsymbol{q}, \boldsymbol{t}^{\pm 1}, \boldsymbol{u}].$$

Proof. The idea of the proof is again based on a Taylor expansion of $\Delta_{l,p}$ in $\mathbf{u}$. In order to shorten notations, for every $n \geq 1$ and any $i \in \mathbb{Z}$, we set

$$P_n([i, i+n-1]) = P_n(\mathbf{q}_{[i,i+n-1]}, \mathbf{t}_{[i,i+n-1]})$$

and

$$\Delta_{l,p}([i, i+lp-1]) = \Delta_{l,p}(\mathbf{q}_{[i,i+lp-1]}, \mathbf{t}_{[i,i+lp-1]}).$$

Let $p \geq 2$ be an integer, we claim that for every $i = 1, \ldots, p$,

$$\frac{\partial}{\partial t_i} \Delta_{l,p}([1,p]) = P_{lp-1}(q_{i+1}, \ldots, q_{lp}, q_1, \ldots, q_{i-1}, t_{i+1}, \ldots, t_{lp}, t_1, \ldots, t_{i-1}).$$

Indeed, we have

$$\frac{\partial}{\partial t_i} \Delta_{l,p}([1,lp]) = \frac{\partial}{\partial t_i} P_{lp}([1,lp]) - q_1 \frac{\partial}{\partial t_i} P_{lp-2}([2,lp-1]).$$

If $i = 1$, the result clearly holds. Now if $i > 1$, using Lemma 4.4 and the following three terms relations for generalized Chebyshev polynomials (obtained by expanding the determinental expression of P_{i-1})

$$P_{i-1}([1, i-1]) = t_1 P_{i-2}([2, i-1]) - q_2 P_{i-3}([3, i-1]),$$

we get :

$$\begin{aligned}\frac{\partial}{\partial t_i}\Delta_{l,p}([1,p]) &= -q_2 P_{i-3}([3, i-1]) P_{lp-i}([i+1, lp]) \\ &+ t_1 P_{i-2}([2, i-1]) P_{lp-i}([i+1, lp]) \\ &- q_1 P_{i-2}([2, i-1]) P_{lp-i-1}(i+1, lp-1)\end{aligned}$$

On the other hand, $P_{lp-1}(q_{i+1}, \ldots, q_{lp}, q_1, \ldots, q_{i-1}, t_{i+1}, \ldots, t_{lp}, t_1, \ldots, t_{i-1})$ is :

$$D = \det \begin{bmatrix} t_{i-1} & 1 & & & & & & (0) \\ q_{i-1} & \ddots & \ddots & & & & & \\ & \ddots & \ddots & 1 & & & & \\ & & \ddots & t_1 & 1 & & & \\ & & & q_1 & t_{lp} & \ddots & & \\ & & & & q_{lp} & \ddots & \ddots & \\ & & & & & \ddots & \ddots & 1 \\ (0) & & & & & & q_{i+2} & t_{i+1} \end{bmatrix}$$

Expanding with respect to the column containing q_1, we get:

$$D = (-1)A + t_1 B - q_1 C$$

where A, B, C are $n-1 \times n-1$ minors. Computing these minors, we get:

$$A = q_2 P_{i-3}([3, i-1]) P_{lp-i}([i+1, lp]) \ ;$$

$$B = P_{i-2}([2, i-1]) P_{lp-i}([i+1, lp]) \ ;$$

$$C = P_{i-2}([2, i-1]) P_{lp-i-1}([i+1, lp-1])$$

so that

$$\frac{\partial}{\partial t_i}\Delta_{l,p}([1, lp]) = P_{lp-1}(q_{i+1}, \ldots, q_{lp}, q_1, \ldots, q_{i-1}, t_{i+1}, \ldots, t_{lp}, t_1, \ldots, t_{i-1})$$

and the claim is proved.

Considering the Taylor expansion in $\mathbf{u}$, we get that $\Delta_{l,p}(q_1, \ldots, q_{lp}, \frac{q_1}{t_1} + u_1 + t_2, \ldots, \frac{q_{lp}}{t_{lp}} + u_{lp} + t_1)$ is equal to

$$\sum_{i_1, \ldots, i_p} \left(\frac{\partial^{i_1} \cdots \partial^{i_p}}{\partial t_1 \cdots \partial t_p}\right) \Delta_{l,p}(q_1, \ldots, q_{lp}, \frac{q_1}{t_1} + t_2, \ldots, \frac{q_{lp}}{t_{lp}} + t_1) u_1^{i_1} \cdots u_p^{i_p}$$

But for every $i_1, \ldots, i_p$, it follows from the claim that

$$\left(\frac{\partial^{i_1} \cdots \partial^{i_p}}{\partial t_1 \cdots \partial t_p}\right) \Delta_{l,p}(q_1, \ldots, q_{lp}, \frac{q_1}{t_1} + t_2, \ldots, \frac{q_{lp}}{t_{lp}} + t_1)$$

is a derivative of a generalized Chebyshev polynomial in consecutive variables. Thus, by Lemma 4.4, this is again a product of generalized Chebyshev polynomial in consecutive variables. Then, Lemma 4.5 implies that

$$\left(\frac{\partial^{i_1}\cdots\partial^{i_p}}{\partial t_1\cdots\partial t_p}\right)\Delta_{l,p}(q_1,\ldots,q_{lp},\frac{q_1}{t_1}+t_2,\ldots,\frac{q_{lp}}{t_{lp}}+t_1)\in\mathbb{N}[\mathbf{q},\mathbf{t}^{\pm 1}].$$

so that finally,

$$\Delta_{l,p}(q_1,\ldots,q_{lp},\frac{q_1}{t_1}+u_1+t_2,\ldots,\frac{q_{lp}}{t_{lp}}+u_{lp}+t_1)\mathbb{N}[\mathbf{q},\mathbf{t}^{\pm 1},\mathbf{u}]$$

and the lemma is proved. □

Remark 4.8. We will see in Section 6 that the Δ-polynomials are motivated by the so-called *difference properties* introduced in [Dup08, Dup09c]. Note that the positivity of the $\Delta_{l,p}(q_1,\ldots,q_p,\frac{q_1}{t_1}+u_1+t_2,\ldots,\frac{q_{lp}}{t_{lp}}+u_{lp}+t_1)$ really comes from the fact that the last variable is t_1 and not some t_{lp+1}. This is an illustration of the fact that the Δ-polynomials arise from tubes, that is, periodic regular components. Indeed, in general, the polynomial $\Delta_{l,p}(q_1,\ldots,q_{lp},\frac{q_1}{t_1}+u_1+t_2,\ldots,\frac{q_{lp}}{t_{lp}}+u_{lp}+t_{lp+1})$ is not substraction-free in $\mathbb{Z}[\mathbf{q},\mathbf{u},\mathbf{t}^{\pm 1}]$. For example, if we consider $l=1,p=2$ and we specialize $\mathbf{u}$ to $\{0\}$, we get

$$\Delta_{1,2}(q_1,q_2,\frac{q_1}{t_1}+t_2,\frac{q_2}{t_2}+t_1)=\frac{t_1^2t_2^2+q_1q_2}{t_1t_2}\in\mathbb{N}[q_1,q_2,t_1^{\pm 1},t_2^{\pm 2}]$$

whereas

$$\Delta_{1,2}(q_1,q_2,\frac{q_1}{t_1}+t_2,\frac{q_2}{t_2}+t_3)=\frac{t_1t_2^2t_3+q_1t_1t_2+q_1q_2-q_1t_1t_2}{t_1t_2}$$
$$\in\mathbb{Z}[q_1,q_2,t_1^{\pm 1},t_2^{\pm 2},t_3^{\pm 1}]\setminus\mathbb{N}[q_1,q_2,t_1^{\pm 1},t_2^{\pm 2},t_3^{\pm 1}].$$

5. Positivity coming from quasi-composition factors

We now investigate positivity of cluster characters associated to regular modules over the path algebra of an acyclic quiver Q. This section is mainly devoted to the proof of the fact that if M is an indecomposable regular $\mathbf{k}Q$-module, then the positivity of the cluster character X_M in a given cluster can generally be reduced to the study of the cluster characters of the quasi-composition factors of M in this cluster.

5.1. Graded components of cluster characters.

This short section is devoted to an easy but key lemma on cluster characters which enables to use positivity properties of generalized Chebyshev polynomials.

We fix an acyclic quiver Q, a cluster-tilting object T in $\mathcal{C}_Q$. We denote by $B=\mathrm{End}_{\mathcal{C}_Q}(T)$, $F=\mathrm{Hom}_{\mathcal{C}_Q}(T,-)$ and $\mathbf{c}_T=\{c_i|i\in Q_0\}$ the corresponding cluster in $\mathcal{A}(Q)$.

Lemma 5.1. *With the above notations, for any object M in $\mathcal{C}_Q$ such that M and $M[1]$ are not in* $\mathrm{add}(T[1])$*, we have*

$$X^T_{M[1]}([F_TM[1]])=\frac{1}{X^T_M(0)}$$

where $[FM]$ denotes the class of FM in $K_0(B\text{-mod})$.

Proof. The suspension functor in $\mathcal{C}_Q$ being given by the shift functor [1], it follows from [Pal08] that for any object X in $\mathcal{C}_Q$, there are equalities

$$\prod_{i \in Q_0} c_i^{\langle S_i, FX\rangle_a} = \mathbf{c}^{\mathrm{ind}_T X - \mathrm{coind}_T X}$$

so that

$$\mathrm{coind} X = -\mathrm{ind} X[1]$$

where ind_T (resp. coind_T) denotes the index (resp. coindex) in $\mathcal{C}_Q$ with respect to the cluster-tilting object T (see [Pal08] for details).

It follows that

$$X_M^T(0) = \mathbf{c}^{-\mathrm{coind}_T M} = \mathbf{c}^{\mathrm{ind}_T M[1]}$$

and

$$\begin{aligned} X_{M[1]}^T([F_T M[1]]) &= \mathbf{c}^{-\mathrm{coind}_T M[1]} \mathbf{c}^{\mathrm{coind}_T M[1] - \mathrm{ind}_T M[1]} \\ &= \frac{1}{\mathbf{c}^{\mathrm{ind}_T M[1]}} \\ &= \frac{1}{X_M^T(0)}. \end{aligned}$$

□

Example 5.2. Consider the quiver of Dynkin type $\mathbb{A}_2$:

$$Q: \qquad 1 \longrightarrow 2$$

For any $i = 0, 1$, we denote by S_i the simple $\mathbf{k}Q$-module associated to vertex i so that $S_1 \simeq \tau S_2 \simeq S_2[1]$. We consider the cluster-tilting object $T = \mathbf{k}Q$, then $\mathrm{mod}{-}B \simeq \mathrm{mod}{-}\mathbf{k}Q^{\mathrm{op}} \simeq \mathbf{k}Q{-}\mathrm{mod}$ so that we identify $K_0(\mathrm{mod}{-}B)$ with $K_0(\mathbf{k}Q{-}\mathrm{mod})$ which is identified with $\mathbb{Z}^2$ using the dimension vector map.

We have

$$X_{S_1} = \frac{1+u_2}{u_1} \text{ and } X_{S_2} = \frac{1+u_1}{u_2}.$$

and the graded components are

$$X_{S_1}([00]) = \frac{1}{u_1},\ X_{S_1}([10]) = \frac{u_1}{u_2},$$

$$X_{S_2}([00]) = \frac{u_2}{u_1},\ X_{S_2}([01]) = \frac{1}{u_2}$$

illustrating Lemma 5.1.

5.2. **Positivity for higher quasi-lengths.** We now investigate positivity of cluster characters associated to regular $\mathbf{k}Q$-modules with quasi-length ≥ 2.

Theorem 5.3. *Let Q be an acyclic quiver and T be a cluster-tilting object in $\mathcal{C}_Q$. Let M be an indecomposable regular $\mathbf{k}Q$-module. Assume that, for any quasi-composition factor N of M, N is rigid, not in $\mathrm{add}(T[1])$ and $\mathrm{int}(X_N^T) \in \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$. Then,*

$$X_M \in \mathbb{N}[\boldsymbol{c}_T^{\pm 1}] \cap \mathcal{A}(Q).$$

Proof. Let M be an indecomposable regular $\mathbf{k}Q$-module and $\mathcal{R}$ be the regular component in $\Gamma(\mathbf{k}Q)$ containing M. Then $\mathcal{R} \simeq \mathbb{Z}\mathbb{A}_\infty/(p)$ for some $p \geq 0$. Assume that M has quasi-length $n \geq 2$. We denote by $M_1, \ldots, M_n$ its quasi-composition factors indexed by $\mathbb{Z}/p\mathbb{Z}$ such that $\tau M_i \simeq M_{i-1}$ for every $i \in \mathbb{Z}/p\mathbb{Z}$.

Viewing elements in $\mathcal{A}(Q)$ as elements in $\mathbb{Z}[\mathbf{u}^{\pm 1}]$ or as elements in $\mathbb{Z}[\mathbf{c}_T^{\pm 1}]$, it follows from [Pal08, Section 5.2] that $X_N = X_N^T$ for every rigid object N in $\mathcal{C}_Q$. Thus,

$$\begin{aligned} X_M &= \underline{P}_n(X_{M_1}, \ldots, X_{M_n}) \\ &= \underline{P}_n(X^T_{M_1}, \ldots, X^T_{M_n}) \end{aligned}$$

where the last equality holds since each M_i is rigid. For every $i \in \{1, \ldots, n\}$, M_i is indecomposable rigid in $\mathcal{C}_Q$ so that X_{M_i} is a cluster variable in $\mathcal{A}(Q)$. In particular, X_M^T is a polynomial in cluster variables and thus belongs to $\mathcal{A}(Q)$.

Now for every $i = 1, \ldots, n$, we set :

$$\tau_i^T = \frac{1}{X^T_{M_i}(0)} \text{ and } \nu_i^T = \mathrm{int}(X^T_{M_i}).$$

Thus, Lemma 5.1 implies that for every $i = 1, \ldots, n$,

$$X^T_{M_i} = \frac{1}{\tau_i^T} + \nu_i^T + \tau_{i+1}^T$$

and

$$X_M^T = \underline{P}_n\left(\frac{1}{\tau_1^T} + \nu_1^T + \tau_2^T, \ldots, \frac{1}{\tau_n^T} + \nu_n^T + \tau_{n+1}^T\right) \in \mathbb{N}[\nu_i^T, {\tau_i^T}^{\pm 1} | i = 1, \ldots n]$$

by Lemma 4.5. Since, $\nu_i^T = \mathrm{int}(X^T_{M_i})$ belongs to $\mathbb{N}[\mathbf{c}_T^{\pm 1}]$ by hypothesis and τ_i^T is an unitary monomial in $\mathbb{N}[\mathbf{c}_T^{\pm 1}]$, it follows that

$$\mathbb{N}[\nu_i^T, {\tau_i^T}^{\pm 1} | i = 1, \ldots n] \subset \mathbb{N}[\mathbf{c}_T^{\pm 1}]$$

and thus $X_M \in \mathbb{N}[\mathbf{c}_T^{\pm 1}]$. □

The assumption concerning the positivity of the interior of X_M may seem difficult to check in practice. Nevertheless, if the positivity conjecture is proved, then it is automatically satisfied for rigid objects. For example, rephrasing a result of Musiker-Schiffler-Williams, we get :

Theorem 5.4 ([MSW09])**.** *Let Q be a quiver of affine type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$. Let T be any cluster-tilting object in $\mathcal{C}_Q$ and M an indecomposable rigid object not in* $\mathrm{add}(T[1])$*. Then, for any $\boldsymbol{e} \in K_0(\mathrm{mod}{-}B)$, we have ;*

$$\chi(\mathrm{Gr}_{\boldsymbol{e}}(F_T M)) \geq 0.$$

In particular,$\mathrm{int}(X_M^T) \in \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$.

Combining Theorem 5.3 and Theorem 5.4, we get

Corollary 5.5. *Let Q be a quiver of affine type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$ and M be an indecomposable regular $\boldsymbol{k}Q$-module in an exceptional tube $\mathcal{T}$. Then,*

$$X_M \in \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}^{\pm 1}]$$

for any cluster $\mathbf{c}$ in $\mathcal{A}(Q)$ compatible with $\mathcal{T}$.

6. Positivity, homogeneous tubes and bases

In this section, we investigate the case of regular modules in homogeneous tubes. In this case, Q is necessarily an affine quiver. We denote by δ the minimal positive imaginary root of Q. It $\mathcal{T}$ is a homogeneous tube and R is its quasi-simple module, it is known that $\mathbf{dim}\, R^{(n)} = n\delta$ for any $n \geq 1$.

6.1. Positivity and homogeneous tubes.

It is known that for any two quasi-simple representations M, N in possibly distinct homogeneous tubes, the characters X_M and X_N coincide. This common value is denoted by X_δ and is called the *generic value of dimension* δ following the terminology of [Dup08].

It is also known that for any $n \geq 1$, normalized Chebyshev polynomials of first and second kinds satisfy the following identities in $\mathbb{Z}[t, t^{-1}]$:

$$F_n(t + t^{-1}) = t^n + t^{-n}$$

$$S_n(t + t^{-1}) = \sum_{k=0}^{n} t^{n-2k}$$

so that F_n and S_n are related by

$$S_n = \sum_{k=0}^{n} F_{n-2k}$$

for every $n \geq 1$ with the convention that $F_i = 0$ if $i < 0$.

Theorem 6.1. *Let Q be an affine quiver with at least three vertices. Let T be a cluster-tilting object such that there exists an exceptional tube $\mathcal{T}$ compatible with T. Assume that $\mathrm{int}(X_M) \in \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$ for any quasi-simple module M in $\mathcal{T}$. Then, for any $l \geq 1$, the following hold :*

(1) $F_l(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$;
(2) $S_l(X_\delta) \in \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$;
(3) $X_\delta^l \in \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$.

Proof. We denote by $p \geq 2$ the rank of the tube $\mathcal{T}$. We denote by $R_i, i \in \mathbb{Z}$ the quasi-simple modules in $\mathcal{T}$ ordered such that $R_i \simeq R_{i+p}$ and $\tau R_i \simeq R_{i-1}$ for any $i \in \mathbb{Z}$. It is proved in [Dup09c, Proposition 3.3] that for any $l \geq 1$,

$$F_l(X_\delta) = \underline{\Delta}_{l,p}(X_{R_0}, \ldots, X_{R_{lp-1}}).$$

As before, for every $i = 0, \ldots, p-1$, we set

$$\tau_i^T = \frac{1}{X_{R_i}^T(0)} \text{ and } \nu_i^T = \mathrm{int}(X_{R_i}^T).$$

Thus, Lemma 5.1 implies that

$$X_{R_i}^T = \frac{1}{\tau_i^T} + \nu_i^T + \tau_{i+1}^T$$

for any $i \in \mathbb{Z}/p\mathbb{Z}$. Thus,

$$F_l(X_\delta) = \underline{\Delta}_{l,p}(\frac{1}{\tau_1^T} + \nu_1^T + \tau_2^T, \ldots, \frac{1}{\tau_{lp}^T} + \nu_{lp}^T + \tau_1^T) \in \mathbb{N}[\nu_i^T, {\tau_i^T}^{\pm 1} | i = 1, \ldots p]$$

by Lemma 4.7. Since, $\nu_i^T = \mathrm{int}(X_{M_i}^T)$ belongs to $\mathbb{N}[\mathbf{c}_T^{\pm 1}]$ by hypothesis and τ_i^T is an unitary monomial in $\mathbb{N}[\mathbf{c}_T^{\pm 1}]$, it follows that

$$\mathbb{N}[\nu_i^T, \tau_i^{T^{\pm 1}} | i = 1, \ldots n] \subset \mathbb{N}[\mathbf{c}_T^{\pm 1}]$$

and thus $F_l(X_\delta) \in \mathbb{N}[\mathbf{c}_T^{\pm 1}]$. The fact that X_δ, and thus $F_l(X_\delta)$ and $S_l(X_\delta)$, belong to the cluster algebra $\mathcal{A}(Q)$ is proved in [Dup08]. This proves the first point.

The second point follows from the first. Indeed, for any $l \geq 1$, $S_l(X_\delta) = \sum_{k=0}^{l} F_{l-2k}(X_\delta) \in \mathbb{N}[\mathbf{c}_T^{\pm 1}]$.

Similarly, it is known that for any $l \geq 1$, X_δ^l can be written as a $\mathbb{N}$-linear combination of $S_k(X_\delta), k \leq l$ (see e.g. [Dup08]) so that $X_\delta^l \in \mathbb{N}[\mathbf{c}_T^{\pm 1}]$. □

As an immediate corollary we get :

Corollary 6.2. *With the above hypotheses, $X_M \in \mathcal{A}(Q) \cap \mathbb{N}[\mathbf{c}_T^{\pm 1}]$ for any indecomposable $\mathbf{k}Q$-module M in a homogeneous tube.*

Proof. Let M be an indecomposable $\mathbf{k}Q$-module in a homogeneous tube $\mathcal{T}$ of $\Gamma(\mathbf{k}Q)$. Then, $M \simeq R^{(n)}$ for some integer $n \geq 1$ where R denotes the quasi-simple module in $\mathcal{T}$. It follows that

$$X_M = \underline{P}_n(X_R, \cdots, X_R) = S_n(X_R) = S_n(X_\delta).$$

The second point of Theorem 6.1 implies that $X_M \in \mathcal{A}(Q) \cap \mathbb{N}[\mathbf{c}_T^{\pm 1}]$. □

Corollary 6.3. *Let Q be a quiver of affine type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$. Then,*

$$X_M \in \mathcal{A}(Q) \cap \mathbb{N}[\mathbf{c}^{\pm 1}]$$

for any concealed cluster $\mathbf{c}$ $\mathcal{A}(Q)$ and any object M in $\mathcal{C}_Q$.

Remark 6.4. Note that we only use polynomials $\underline{\Delta}$ in the context of coefficient-free cluster algebras. The Δ-polynomials are an analogue for the context of cluster algebras with principal coefficients at the initial seed. For example, if one uses the cluster character with coefficients $X_?^{\mathbf{y}}$ introduced in [Dup09d] (see also [FK10] for a more general definition), one can prove, using the methods in [Dup08], that for any quiver Q of affine type $\widetilde{\mathbb{A}}$,

$$X_\delta^{\mathbf{y}} = \Delta_{1,p}(\mathbf{y}^{\mathbf{dim}\, R_0}, \ldots, \mathbf{y}^{\mathbf{dim}\, R_{p-1}}, X_{R_0}^{\mathbf{y}}, \ldots, X_{R_{p-1}}^{\mathbf{y}})$$

where $R_0, \ldots, R_{p-1}$ are the quasi-simple modules of an exceptional tube ordered such that $\tau R_i \simeq R_{i-1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. This is the reason why we proved results for the polynomials Δ and P_n rather than just $\underline{\Delta}$ and $\underline{P_n}$. Nevertheless, several results we use for coefficient-free cluster characters, especially in the proof of [Dup09c, Proposition 3.3], need a lot of work in order to be generalized to cluster characters with coefficients. This is the reason why we decided to present only the results for the coefficient-free case.

6.2. **Positivity and bases.** From now on, we assume that $\mathbf{k} = \mathbb{C}$ is the field of complex numbers. In this case, there are three $\mathbb{Z}$-bases of particular interest in a cluster algebra $\mathcal{A}(Q)$ associated to a quiver Q of affine type. The precise definitions of these bases are given in Section 2.4. It is conjectured that these three bases are constituted of positive elements of the cluster algebra. Using our results, we can partially prove this conjecture.

Corollary 6.5. *Let Q be a quiver of affine type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$ with at least three vertices. Let T be a cluster-tilting object in $\mathcal{C}_Q$ such that there exists an exceptional tube $\mathcal{T}$ compatible with T. Then, the following hold :*

1. $\mathcal{B}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$;
2. $\mathcal{C}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$;
3. $\mathcal{G}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}_T^{\pm 1}]$.

Proof. According to Theorem 5.4, for any cluster-tilting object T, $\mathcal{M}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\mathbf{c}_T^{\pm 1}]$ and $X_R \in \mathcal{A}(Q) \cap \mathbb{N}[\mathbf{c}_T^{\pm 1}]$ for any regular rigid $\mathbf{k}Q$-module R. Since $\mathcal{A}(Q) \cap \mathbb{N}[\mathbf{c}_T^{\pm 1}]$ is stable under multiplication, it only remains to prove that $F_l(X_\delta), S_l(X_\delta)$ and X_δ^l belong to $\mathcal{A}(Q) \cap \mathbb{N}[\mathbf{c}_T^{\pm 1}]$ for any cluster-tilting object satisfying the hypotheses. This is a direct consequence of Theorem 6.1. □

Remark 6.6. The only affine quiver with two vertices is the Kronecker quiver. In this case, Corollary 6.5 was established by [SZ04] for $\mathcal{B}(Q)$. The cases of $\mathcal{C}(Q)$ and $\mathcal{G}(Q)$ follow easily.

Finally, we proved :

Corollary 6.7. *Let Q be a quiver of affine type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$. Let $\mathbf{c}$ be any unmixed cluster. Then, the following hold :*

1. $\mathcal{B}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}^{\pm 1}]$;
2. $\mathcal{C}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}^{\pm 1}]$;
3. $\mathcal{G}(Q) \subset \mathcal{A}(Q) \cap \mathbb{N}[\boldsymbol{c}^{\pm 1}]$.

Remark 6.8. Let Q be an acyclic quiver, T be any cluster-tilting object in $\mathcal{C}_Q$, $B = \mathrm{End}_{\mathcal{C}_Q}(T)$ its endomorphism ring and M be any indecomposable $\mathbf{k}Q$-module. In order to prove the positivity of X_M in the cluster $\mathbf{c}_T$, it is natural to study the positivity of $\chi(\mathrm{Gr}_{\mathbf{e}}(F_T M))$ for any $\mathbf{e} \in K_0(B)$. There are two obstacles for using this method for our purpose. The first one is that when M is not rigid, it may happen that $\chi(\mathrm{Gr}_{\mathbf{e}}(F_T M)) < 0$ (see [DWZ09, Example 3.5]) and we precisely want to consider non-rigid objects in our case. The second one is that, when Q is an affine quiver with positive minimal imaginary root δ, it is known that, for any $l \geq 2$, $F_l(X_\delta)$ may not be expressed as a generating series of Euler characteristics of quiver grassmannians. Thus, the approach using positive characteristics of quiver grassmannians would not help in order to prove that Sherman-Zelevinsky type bases are constituted of positive elements. Nevertheless, it was observed in [Dup09c] that $F_l(X_\delta)$ may be expressed (in the initial cluster) as a generating series of Euler characteristics of a certain constructible subset $\mathrm{Tr}(M) \subset \mathrm{Gr}(M)$ in the quiver grassmannian, called *transverse quiver grassmannian*. Thus, in this context, it would be interesting to study the Euler characteristics of these transverse quiver grassmannians.

Remark 6.9. In order to conclude, we would like to make a remark concerning a possible use of Theorem 5.3 in the wider context of bases in acyclic cluster algebras. If Q is any acyclic quiver, Geiss, Leclerc and Schröer constructed a $\mathbb{C}$-basis $\mathcal{S}^*(Q)$ of $\mathbb{C} \otimes_{\mathbf{Z}} \mathcal{A}(Q)$, called the *dual semicanonical basis of* $\mathcal{A}(Q)$ [GLS08]. It was recently observed by Geiss, Leclerc and Schröer that this basis coincides with the set of generic variables introduced in [Dup08] (coinciding in particular with $\mathcal{G}(Q)$ when Q is an affine quiver). It follows that $\mathcal{S}^*(Q)$ can be realized as products of cluster characters evaluated at Schur representations of Q. Theorem 5.3 may thus be useful in order to establish positivity properties of the elements in $\mathcal{S}^*(Q)$. Indeed, if M is a Schur representation, it may be rigid or not. If M is rigid, X_M is a cluster variable whose positivity may be studied from classical point of views (e.g. positive characteristics of grassmannians, arcs in triangulations). If M is not rigid, it is necessarily regular and if M is not quasi-simple, it is known that the quasi-composition factors of M are rigid $\mathbf{k}Q$-modules (see e.g. [SS07, Chapter XVIII, Proposition 2.14]). Then, their positivity may be studied from classical points of view and Theorem 5.3 would thus imply positivity properties for the cluster character X_M.

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