TRANSVERSE QUIVER GRASSMANNIANS AND BASES IN AFFINE CLUSTER ALGEBRAS

G. DUPONT

Abstract. Sherman-Zelevinsky and Cerulli constructed canonically positive bases in cluster algebras associated to affine quivers having at most three vertices. These constructions involve cluster monomials and Chebyshev polynomials of the first kind evaluated at a certain "imaginary" element in the cluster algebra.

For any affine quiver Q, we provide a geometric realization of these elements in terms of the representation theory of Q. This is done by introducing an analogue of the Caldero-Chapoton cluster character where the usual quiver grassmannian is replaced by a constructible subset called *transverse quiver grassmannian*.

Contents

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in order to define a combinatorial framework for studying positivity in algebraic groups and canonical bases in quantum groups [FZ02, FZ03, BFZ05, FZ07]. Since then, cluster algebras found applications in various areas of mathematics like Lie theory, combinatorics, Teichmüller theory, Poisson geometry or quiver representations.

A (coefficient-free) cluster algebra $\mathcal{A}$ is a commutative $\mathbb{Z}$-algebra equipped with a distinguished set of generators, called *cluster variables*, gathered into possibly overlapping sets of fixed cardinality, called *clusters*. Monomials in variables belonging to a same cluster are called *cluster monomials*. According to the so-called *Laurent phenomenon* [FZ02], it is known that $\mathcal{A} \subset \bigcap_{\mathbf{c}} \mathbb{Z}[\mathbf{c}^{\pm 1}]$ where $\mathbf{c}$ runs over the clusters in $\mathcal{A}$. An element $y \in \mathcal{A}$ is called *positive* if $y \in \bigcap_{\mathbf{c}} \mathbb{Z}_{\geq 0}[\mathbf{c}^{\pm 1}] \setminus \{0\}$ where $\mathbf{c}$ runs over

2000 *Mathematics Subject Classification.* 13F99, 16G99, 16G20 .

Key words and phrases. Cluster algebras, Canonically positive basis, Chebyshev Polynomials, Cluster categories, Cluster characters.

the clusters in $\mathcal{A}$. A $\mathbb{Z}$-basis $\mathcal{B} \subset \mathcal{A}$ is called *canonically positive* if the semi-ring of positive elements in $\mathcal{A}$ coincides with the set of $\mathbb{Z}_{\geq 0}$-linear combinations of elements of $\mathcal{B}$. Note that if such a basis exists, it is unique.

The problems of both the existence and the description of a canonically positive basis in an arbitrary cluster algebra are still widely open. This was first solved in the particular case of cluster algebras of finite type $\mathbb{A}_2$ and affine type $\tilde{\mathbb{A}}_{1,1}$ by Sherman and Zelevinsky [SZ04]. It was later extended by Cerulli for cluster algebras of affine type $\tilde{\mathbb{A}}_{2,1}$ [Cer09]. At the best of our knowledge, these are the only known constructions of canonically positive bases in cluster algebras.

Using categorifications of acyclic cluster algebras with cluster categories and cluster characters, it is possible to rephrase Sherman-Zelevinsky and Cerulli constructions in order to fit into the more general context of acyclic cluster algebras associated to arbitrary affine quivers.

If Q is an acyclic quiver and $\mathbf{u}$ is a Q_0-tuple of indeterminates over $\mathbb{Z}$, we denote by $\mathcal{A}(Q)$ the acyclic cluster algebra with initial seed $(Q, \mathbf{u})$. We denote by $\mathcal{C}_Q$ the associated *cluster category* and by $X_? : \mathrm{Ob}(\mathcal{C}_Q) \longrightarrow \mathbb{Z}[\mathbf{u}^{\pm 1}]$ the *Caldero-Chapoton map* on $\mathcal{C}_Q$, also called *cluster character* (see Section 2 for details). When Q is an affine quiver with positive minimal imaginary root δ, we set

$$\mathcal{B}(Q) = \mathcal{M}(Q) \sqcup \{F_n(X_\delta)X_R | n \geq 1,\ R \text{ is regular and rigid in } kQ\text{-mod}\}$$

where $\mathcal{M}(Q)$ denotes the set of cluster monomials in $\mathcal{A}(Q)$, F_n denotes the n-th *normalized Chebyshev polynomials of the first kind* and X_δ is the evaluation of $X_?$ at any quasi-simple module in an homogeneous tube of the Auslander-Reiten quiver $\Gamma(kQ)$ of kQ-mod.

If Q is of type $\tilde{\mathbb{A}}_{1,1}$ (respectively $\tilde{\mathbb{A}}_{2,1}$), the set $\mathcal{B}(Q)$ coincides with the canonically positive basis constructed in [SZ04] (respectively [Cer09]). It was conjectured in [Dup09b, Conjecture 7.10] that, for any affine quiver Q, the set $\mathcal{B}(Q)$ is the canonically positive basis of $\mathcal{A}(Q)$. Using the so-called *generic basis*, it is possible to prove that, for any affine quiver Q, the set $\mathcal{B}(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$ [Dup08, DXX09]. Nevertheless, it is not known if this basis is the canonically positive basis in general.

An essential problem for investigating this question is due to the fact that the elements of the form $F_n(X_\delta)X_R$ are defined combinatorially and have no representation-theoretic or geometric interpretation. The aim of this article is to provide such an interpretation.

Extending the idea of Caldero and Chapoton [CC06], for any integrable bundle $\mathcal{F}$ on $\mathrm{rep}(Q)$ (see Section 4 for detailed definitions), we define a character $\theta_{\mathcal{F}}$ from the set of objects in $\mathcal{C}_Q$ to the set $\mathbb{Z}[\mathbf{u}^{\pm 1}]$. With this terminology, the Caldero-Chapoton map $X_?$ corresponds to the character θ_{Gr} where $\mathrm{Gr} : M \mapsto \mathrm{Gr}(M)$ denotes the integrable bundle of quiver grassmannians.

We introduce a constructible subset $\mathrm{Tr}(M) \subset \mathrm{Gr}(M)$, called *transverse quiver grassmannian*. We prove that the bundle $M \mapsto \mathrm{Tr}(M)$ is integrable on $\mathrm{rep}(Q)$ and that the elements in $\mathcal{B}(Q)$ can be described using the associated character θ_{Tr}. More precisely, we prove that

$$F_l(X_\delta) = \theta_{\mathrm{Tr}}(M)$$

for any indecomposable object M with dimension vector $l\delta$. As opposed to θ_{Gr}, it turns out that θ_{Tr} is independent on the tube containing M. In particular, it takes the same values if M belongs to an homogeneous or to an exceptional tube. This is surprising since the usual quiver grassmannians of two indecomposable modules of dimension $l\delta$ belonging to tubes of different rank are in general completely different.

Moreover, if R is an indecomposable regular rigid kQ-module, then

$$F_l(X_\delta)X_R = \theta_{\mathrm{Tr}}(M)$$

where M is the unique indecomposable kQ-module of dimension $l\delta + \mathbf{dim}\, R$.

As a consequence, we obtain the following description of the set $\mathcal{B}(Q)$:

$$\mathcal{B}(Q) = \Big\{\theta_{\mathrm{Tr}}(M \oplus R) | M \text{ is an indecomposable (or zero) regular } kQ\text{-module,} \\ R \text{ is any rigid object in } \mathcal{C}_Q \text{ such that } \mathrm{Ext}^1_{\mathcal{C}_Q}(M, R) = 0\Big\}.$$

The paper is organized as follows. In Section 2, we start by recalling several results concerning Chebyshev and generalized Chebyshev polynomials. Then, we recall necessary background on cluster categories and cluster characters associated to acyclic and especially affine quiver. Finally, we recall the known results concerning construction of bases in affine cluster algebras.

In Section 3, we use the combinatorics of generalized Chebyshev polynomials in order to prove relations for cluster characters associated to regular kQ-modules when Q is an affine quiver with minimal imaginary root δ. These relations are generalizations of the so-called *difference property*, introduced previously in [Dup08] in order to compute the difference between cluster characters evaluated at indecomposable modules of dimension vector δ in different tubes.

Section 4 introduces the notions of integrable bundles on $\mathrm{rep}(Q)$ and associated characters for any acyclic quiver. In this terminology, the Caldero-Chapoton map is the character associated to the quiver grassmannian bundle. For affine quivers, we introduce the integrable bundle Tr of grassmannian of transverse submodules and prove that it coincides with the Caldero-Chapoton map on rigid objects in the cluster category.

In Section 5, we prove that the elements in $\mathcal{B}(Q)$ can be expressed as values of the character θ_{Tr} associated to the integrable bundle Tr of $\mathrm{rep}(Q)$. This provides a geometrization of the set $\mathcal{B}(Q)$.

In Section 6, we illustrate some of our results for quivers of affine types $\tilde{\mathbb{A}}_{1,1}$ and $\tilde{\mathbb{A}}_{2,1}$, putting into context the results of [SZ04] and [Cer09].

2. Background, notations and terminology

Throughout the paper, k will denote the field $\mathbb{C}$ of complex numbers. Given a quiver Q, we denote by Q_0 its set of arrows, Q_1 its set of vertices. We always assume that Q_0, Q_1 are finite sets and that the underlying unoriented graph of Q is connected. A quiver is called *acyclic* if it does not contain any oriented cycles.

We now fix an acyclic quiver Q and a Q_0-tuple $\mathbf{u} = (u_i | i \in Q_0)$ of indeterminates over $\mathbb{Z}$. We denote by $\mathcal{A}(Q)$ the coefficient-free cluster algebra with initial seed $(Q, \mathbf{u})$.

2.1. **Chebyshev polynomials and their generalizations.** Chebyshev and generalized Chebyshev polynomials are orthogonal polynomials in one or several variables playing an important role in the context of cluster algebras associated to representation-infinite quivers [SZ04, CZ06, Dup09a, Dup09b]. Before introducing cluster categories and cluster characters, we recall some basic results concerning these polynomials.

For any $l \geq 0$, the *l-th normalized Chebyshev polynomial of the first kind* is the polynomial F_l in $\mathbb{Z}[x]$ defined inductively by

$$F_0(x) = 2,\ F_1(x) = x,$$
$$F_l(x) = xF_{l-1}(x) - F_{l-2}(x) \text{ for any } l \geq 2.$$

F_l is characterized by the following identity in $\mathbb{Z}[t, t^{-1}]$:

$$F_l(t + t^{-1}) = t^l + t^{-l}.$$

These polynomials first appeared in the context of cluster algebras in [SZ04].

For any $l \geq 0$, the *l-th normalized Chebyshev polynomial of the second kind* is the polynomial S_l in $\mathbb{Z}[x]$ defined inductively by

$$S_0(x) = 1,\ S_1(x) = x,$$
$$S_l(x) = xS_{l-1}(x) - S_{l-2}(x) \text{ for any } l \geq 2.$$

S_l is characterized by the following identity in $\mathbb{Z}[t, t^{-1}]$:

$$S_l(t + t^{-1}) = \sum_{k=0}^{n} t^{n-2k}.$$

Second kind Chebyshev polynomials first appeared in the context of cluster algebras in [CZ06]. For any $l \geq 1$, $S_l(x)$ is the polynomial given by

$$S_l(x) = \det \begin{bmatrix} x & 1 & & & (0) \\ 1 & x & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ (0) & & & 1 & x \end{bmatrix}$$

where the matrix is tridiagonal in $M_l(\mathbb{Z}[x])$. First kind and second kind Chebyshev polynomials are related by :

$$F_l(x) = S_l(x) - S_{l-2}(x)$$

for any $l \geq 1$ with the convention that $S_{-1}(x) = 0$.

Fix $\{x_i | i \geq 1\}$ a family of indeterminates over $\mathbb{Z}$. For any $l \geq 0$, the *l-th generalized Chebyshev polynomial* is the polynomial in $\mathbb{Z}[x_1, \ldots, x_l]$ defined inductively by

$$P_0 = 1,\ P_1(x_1) = x_1,$$
$$P_l(x_1, \ldots, x_l) = x_l P_{l-1}(x_1, \ldots, x_{l-1}) - P_{l-2}(x_1, \ldots, x_{l-2}) \text{ for any } l \geq 2.$$

Equivalently,

$$P_l(x_1, \ldots, x_l) = \det \begin{bmatrix} x_l & 1 & & & (0) \\ 1 & x_{l-1} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ (0) & & & 1 & x_1 \end{bmatrix}$$

where the matrix is tridiagonal in $M_l(\mathbb{Z}[x_1, \ldots, x_l])$. These polynomials first appeared in the context of cluster algebras in [Dup09a] under the name of *generalized*

Chebyshev polynomials of infinite rank and similar polynomials also arose in the context of cluster algebras in [YZ08, Dup09b].

2.2. **Cluster categories and cluster characters.** Let kQ-mod be the category of finitely generated left-modules over the path algebra kQ of Q. As usual, this category will be identified with the category $\mathrm{rep}(Q)$ of finite dimensional representations of Q over k.

For any vertex $i \in Q_0$, we denote by S_i the simple module associated to i, by P_i its projective cover and by I_i its injective hull. We denote by $\langle -,- \rangle$ the homological Euler form defined on kQ-mod by

$$\langle M, N \rangle = \dim \mathrm{Hom}_{kQ}(M,N) - \dim \mathrm{Ext}^1_{kQ}(M,N)$$

for any two kQ-modules M, N. Since Q is acyclic, kQ is a finite dimensional hereditary algebra so that $\langle -,- \rangle$ is well defined on the Grothendieck group $K_0(kQ)$.

For any kQ-module M, the dimension vector of M is

$$\mathbf{dim}\, M = (\dim \mathrm{Hom}_{kQ}(P_i, M))_{i \in Q_0} \in \mathbb{N}^{Q_0}.$$

Viewed as a representation of Q, $\mathbf{dim}\, M = (\dim M(i))_{i \in Q_0}$ where $M(i)$ is the k-vector space at vertex i in the representation M of Q. The dimension vector map $\mathbf{dim}$ induces an isomorphism of abelian group

$$\mathbf{dim} : K_0(kQ) \xrightarrow{\sim} \mathbb{Z}^{Q_0}$$

sending the simple S_i to the i-th vector of the canonical basis of $\mathbb{Z}^{Q_0}$.

The *cluster category* was introduced in [BMR+06] (see also [CCS06]) in order to define a categorical framework for studying the cluster algebra $\mathcal{A}(Q)$. Let $D^b(kQ)$ be the bounded derived category of kQ-mod with shift functor [1] and Auslander-Reiten translation τ. The *cluster category* is the orbit category $\mathcal{C}_Q$ of the auto-functor $\tau^{-1}[1]$ in $D^b(kQ)$. It is a 2-Calabi-Yau triangulated category whose indecomposable objects are either indecomposable kQ-modules or shifts of indecomposable projective modules [Kel05, BMR+06]. In particular, every object M in $\mathcal{C}_Q$ can be decomposed into

$$M = M_0 \oplus P_M[1]$$

where M_0 is a kQ-module and P_M is a projective kQ-module.

Given a representation M of Q, the *quiver grassmannian of* M is the set $\mathrm{Gr}(M)$ of subrepresentations of M. For any element $\mathbf{e} \in \mathbb{Z}^{Q_0}$, the set

$$\mathrm{Gr}_{\mathbf{e}}(M) = \{N \subset M | \mathbf{dim}\, N = \mathbf{e}\}$$

is a projective variety. We denote by $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$ its Euler characteristic with respect to the simplicial cohomology.

Definition 2.1 ([CC06])**.** The *Caldero-Chapoton map* is the map

$$X_? : \mathrm{rep}(Q) \longrightarrow \mathbb{Z}[\mathbf{u}^{\pm 1}]$$

defined by :

(a) If M is an indecomposable kQ-module, then

$$X_M = \sum_{\mathbf{e} \in \mathbb{N}^{Q_0}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) \prod_{i \in Q_0} u_i^{-\langle \mathbf{e}, S_i \rangle - \langle S_i, \mathbf{dim}\, M - \mathbf{e} \rangle} \; ; \qquad (1)$$

where $\mathbf{u}^{\mathbf{d}} = \prod_{i \in Q_0} u_i^{d_i}$ for any $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$.

(b) For any $i \in Q_0$,

$$X_{P_i[1]} = u_i \ ;$$

(c) For any two objects M, N in $\mathcal{C}_Q$,

$$X_{M \oplus N} = X_M X_N.$$

Note that equation (1) also holds for decomposable modules.

Caldero and Keller proved that $X_?$ induces a 1-1 correspondence between indecomposable rigid objects in $\mathcal{C}_Q$ and cluster variables in $\mathcal{A}(Q)$. Moreover, $X_?$ induces a 1-1 correspondence between cluster-tilting objects in $\mathcal{C}_Q$ and clusters in $\mathcal{A}(Q)$ [CK06, Theorem 4]. In particular, it is possible to have the following description of cluster monomials in $\mathcal{A}(Q)$:

$$\mathcal{M}(Q) = \{X_M | M \text{ is rigid in } \mathcal{C}_Q\}.$$

For any Laurent polynomial $L \in \mathbb{Z}[\mathbf{u}^{\pm 1}]$, the *denominator vector* of L it the Q_0-tuple $\mathrm{den}\,(L) \in \mathbb{Z}^{Q_0}$ such that there exists a polynomial $P(u_i | i \in Q_0)$ not divisible by any u_i such that

$$L = \frac{P(u_i | i \in Q_0)}{\mathbf{u}^{\mathrm{den}\,(L)}}.$$

We define the dimension vector map $\mathbf{dim}_{\mathcal{C}_Q}$ on $\mathcal{C}_Q$ by setting $\mathbf{dim}_{\mathcal{C}_Q} M = \mathbf{dim}\, M$ if M is a kQ-module and $\mathbf{dim}_{\mathcal{C}_Q} P_i[1] = -\mathbf{dim}\, S_i$ and extending by additivity. Note that, for any kQ-module M, we have $\mathbf{dim}\, M = \mathbf{dim}_{\mathcal{C}_Q}(M)$ so that, we will abuse notations and write $\mathbf{dim}\, M$ for any object in $\mathcal{C}_Q$. Caldero-Keller's denominators theorem [CK06, Theorem 3] relates the denominator vector of the character with the dimension vector of the corresponding object in the cluster category, namely

$$\mathrm{den}\,(X_M) = \mathbf{dim}\, M$$

for any object M in $\mathcal{C}_Q$.

Moreover, it was proved in [CK08, CK06] that $X_?$ induces on $\mathcal{A}(Q)$ a structure of Hall algebra on the cluster category $\mathcal{C}_Q$. We will use a particular case of this Hall structure in the following (Theorem 2.2).

2.3. **Representation theory of affine quivers.** We shall briefly recall some well-known facts concerning the representation theory of affine quivers. We refer the reader to [SS07, Rin84] for details.

We now fix an affine quiver, that is, an acyclic quiver of type $\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$ of $\tilde{\mathbb{E}}_8$ where $n \geq 1$. We will sometimes say that a quiver is of affine type $\tilde{\mathbb{A}}_{r,s}$ if it is an orientation of an affine diagram of affine type $\tilde{\mathbb{A}}_{r+s-1}$ with r arrows going clockwise and s arrows going counterclockwise. Let $\mathfrak{g}_Q$ denote the Kac-Moody algebra associated to Q.

We denote by $\Phi_{>0}$ the set of *positive roots* of $\mathfrak{g}_Q$, by $\Phi^{\mathrm{re}}_{>0}$ the set of positive *real roots* and by $\Phi^{\mathrm{im}}_{>0}$ the set of positive *imaginary roots*. There exists a unique $\delta \in \Phi_{>0}$ such that $\Phi^{\mathrm{im}}_{>0} = \mathbb{Z}_{>0}\delta$. We always identify the root lattice of $\mathfrak{g}_Q$ with $\mathbb{Z}^{Q_0}$ by sending the i-th simple root of $\mathfrak{g}_Q$ to the i-th vector of the canonical basis of $\mathbb{Z}^{Q_0}$.

According to Kac's theorem, for any $\mathbf{d} \in \mathbb{N}^{Q_0}$, there exists an indecomposable representation M such that $\mathbf{dim}\, M = \mathbf{d}$ if and only if $\mathbf{d} \in \Phi_{>0}$. Moreover, this representation is unique up to isomorphism if and only if $\mathbf{d} \in \Phi^{\mathrm{re}}_{>0}$. A positive root $\mathbf{d}$ is called a *Schur root* if there exists a (necessarily indecomposable) representation M of Q such that $\mathbf{dim}\, M = \mathbf{d}$ and $\mathrm{End}_{kQ}(M) \simeq k$.

We define a partial order $\leq$ on the root lattice by setting

$$\mathbf{e} \leq \mathbf{f} \Leftrightarrow e_i \leq d_i \text{ for any } i \in Q_0.$$

and we set

$$\mathbf{e} \lneq \mathbf{f} \text{ if } \mathbf{e} \leq \mathbf{f} \text{ and } \mathbf{e} \neq \mathbf{f}.$$

The Auslander-Reiten quiver $\Gamma(kQ)$ of kQ-mod contains infinitely many connected components. There exists a connected component containing all the projective (resp. injective) modules, called *preprojective* (resp. *preinjective*) component of $\Gamma(kQ)$ and denoted by $\mathcal{P}$ (resp. $\mathcal{I}$). The other components are called *regular*. A kQ-module M is called *preprojective* (resp. *preinjective*, *regular*) if each indecomposable direct summand of M belong to a preprojective (resp. preinjective, regular) component.

It is convenient to introduce the so-called *defect form* on $\mathbb{Z}^{Q_0}$. It is given by

$$\partial_? : \begin{cases} \mathbb{Z}^{Q_0} & \longrightarrow & \mathbb{Z} \\ \mathbf{e} & \mapsto & \partial_{\mathbf{e}} = \langle \delta, \mathbf{e} \rangle \end{cases}$$

By definition, the defect ∂_M of a kQ-module M is the defect $\partial_{\mathbf{dim}\, M}$ of its dimension vector. It is well-known that an indecomposable kQ-module M is preprojective (resp. preinjective, regular) if and only if $\partial_M < 0$ (resp. > 0, $= 0$).

The regular components in $\Gamma(kQ)$ form a $\mathbb{P}^1(k)$-family of tubes. Thus, for every tube $\mathcal{T}$, there exists an integer $p \geq 1$, called *rank* of $\mathcal{T}$ such that $\mathcal{T} \simeq \mathbb{Z}\mathbb{A}_\infty/(\tau^p)$. The tubes of rank one are called *homogeneous*, the tubes of rank $p > 1$ are called *exceptional*. At most three tubes are exceptional in $\Gamma(kQ)$. It is well-known that the full subcategory of kQ-mod generated by the objects in any tube $\mathcal{T}$ is standard, that is, isomorphic to the mesh category of $\mathcal{T}$. It is also known that there are neither morphisms nor extensions between pairwise distinct tubes.

An indecomposable regular module M is called *quasi-simple* if it is at the mouth of the tube, or equivalently, if it does not contain any proper regular submodule. For any quasi-simple module R in a tube $\mathcal{T}$ and any integer $l \geq 1$, we denote by $R^{(l)}$ the unique indecomposable kQ-module with quasi-socle R and quasi-length l. For any indecomposable regular kQ-module $R^{(l)}$, we denote by

$$\text{q.soc} R^{(l)} = R$$

the *quasi-socle* of M and by

$$\text{q.rad} R^{(l)} = R^{(l-1)}$$

the quasi-radical of M with the convention that $R^{(0)} = 0$.

For any indecomposable regular kQ-module M, we have the following :

$$M \text{ is rigid } \Leftrightarrow \mathbf{dim}\, M \lneq \delta \text{ ;}$$

$$\text{End}_{kQ}(M) \simeq k \Leftrightarrow \mathbf{dim}\, M \leq \delta.$$

Cluster characters in tubes are known to be governed by the combinatorics of generalized Chebyshev polynomials. More precisely, it is proved in [Dup09a, Theorem 5.1] that for any quasi-simple module M in a tube $\mathcal{T}$, we have

$$X_{M^{(l)}} = P_l(X_M, X_{\tau^{-1}M}, \ldots, X_{\tau^{-l+1}M}).$$

In particular, if $\mathcal{T}$ is homogeneous, we get

$$X_{M^{(l)}} = S_l(X_M)$$

recovering a result of [CZ06].

In [Dup09a, Theorem 7.2], generalized Chebyshev polynomials provide multiplication formulas for cluster characters associated to indecomposable regular kQ-modules. The following theorem will be essential in the proofs :

Theorem 2.2 ([Dup09a]). *Let Q be an affine quiver and $\mathcal{T}$ be a tube of rank p in $\Gamma(kQ)$. Let $R_i, i \in \mathbb{Z}$ denote the quasi-simple modules in $\mathcal{T}$ ordered such that $\tau R_i \simeq R_{i-1}$ and $R_{i+p} \simeq R_i$ for any $i \in \mathbb{Z}$. Let $m, n > 0$ be integers and $j \in [0, p-1]$. Then, for every $k \in \mathbb{Z}$ such that $0 < j + kp < n$ and $m > n - j - kp$, we have the following identity :*

$$X_{R_j^{(m)}} X_{R_0^{(n)}} = X_{R_0^{(m+j+kp)}} X_{R_j^{(n-j-kp)}} + X_{R_0^{(j+kp-1)}} X_{R_{n+1}^{(m+j+kp-n-1)}}.$$

2.4. **Bases in affine cluster algebras.** We shall now review some results concerning the construction of $\mathbb{Z}$-bases in cluster algebras associated to affine quivers. In this section, Q still denotes a quiver of affine type with positive minimal imaginary root δ.

It is known that if M, N are quasi-simple modules in distinct homogeneous tubes, then $X_M = X_N$ (see e.g. [Dup08]). We denote by X_δ this common value. Following the terminology of [Dup08], X_δ is called the *generic variable of dimension* δ.

The following holds :

Theorem 2.3 ([Dup08, DXX09]). *Let Q be an affine quiver, then*

$$\mathcal{G}(Q) = \mathcal{M}(Q) \sqcup \left\{ X_\delta^l X_R | l \geq 1, R \text{ is regular and rigid in } kQ\text{-mod} \right\}.$$

Moreover, den *induces a 1-1 correspondence from $\mathcal{G}(Q)$ to $\mathbb{Z}^{Q_0}$.*

The set $\mathcal{G}(Q)$ is called the *generic basis* of $\mathcal{A}(Q)$.

Since F_l and S_l are monic polynomials of degree l, it follows that, for any affine quiver Q, the sets

$$\mathcal{B}(Q) = \mathcal{M}(Q) \sqcup \{F_l(X_\delta) X_R | l \geq 1, R \text{ is regular and rigid in } kQ\text{-mod}\}$$

and

$$\mathcal{C}(Q) = \mathcal{M}(Q) \sqcup \{S_l(X_\delta) X_R | l \geq 1, R \text{ is regular and rigid in } kQ\text{-mod}\}$$

are $\mathbb{Z}$-bases of the cluster algebra $\mathcal{A}(Q)$.

When Q is the Kronecker quiver $\mathcal{B}(Q)$ coincides with the canonically positive basis constructed by Sherman and Zelevinsky [SZ04] and $\mathcal{C}(Q)$ coincides with the basis constructed by Caldero and Zelevinsky [CZ06]. When Q is a quiver of affine type $\tilde{\mathbb{A}}_{2,1}$, the basis $\mathcal{B}(Q)$ is the canonically positive basis of $\mathcal{A}(Q)$ constructed by Cerulli [Cer09].

Since $X_\delta = X_M$ for any quasi-simple module in an homogeneous tube, it follows that $S_l(X_\delta) = X_{M^{(l)}}$ so that the set $\mathcal{C}(Q)$ has an interpretation in terms of the cluster character $X_?$. No such interpretation is known for the set $\mathcal{B}(Q)$. The aim of this paper is to provide one.

The map $\phi : \mathcal{G}(Q) \longrightarrow \mathcal{B}(Q)$ preserving cluster monomials and sending $X_\delta^l X_R$ to $F_l(X_\delta) X_R$ for any $l \geq 1$ and any rigid regular module R is a 1-1 correspondence. We denote by

$$\mathfrak{b}_? : \begin{cases} \mathbb{Z}^{Q_0} & \xrightarrow{1:1} & \mathcal{B}(Q) \\ \mathbf{d} & \mapsto & \mathfrak{b}_{\mathbf{d}} \end{cases}$$

the 1-1 correspondence obtained by composing the above bijection with the one provided in Theorem 2.3.

3. Difference properties of higher orders

In this section, Q still denotes an affine quiver with positive minimal imaginary root δ.

3.1. **The difference property.** In [Dup08] was introduced the *difference property* which relates the possibly different values of cluster characters evaluated at different indecomposable representations of dimension δ. This difference property is crucial in [Dup08]. It is also an essential ingredient in the present article since the transverse grassmannians will precisely arise from difference properties of higher orders.

This difference property was established in [Dup08] for affine type $\tilde{\mathbb{A}}$ and in [DXX09] for the other types. It can be expressed as follows :

Theorem 3.1 ([Dup08, DXX09]). *Let Q be an affine quiver and M be any indecomposable module of dimension δ, then*

$$\mathfrak{b}_\delta = X_\delta = X_M - X_{\mathrm{q.rad}M/\mathrm{q.soc}M}$$

with the convention that $X_{\mathrm{q.rad}M/\mathrm{q.soc}M} = 0$ if M is quasi-simple.

3.2. **Higher difference properties.** The aim of this section is to provide a analogous of Theorem 3.1 for $\mathfrak{b}_{\mathbf{d}}$ when $\mathbf{d}$ is any positive root with zero defect. We will first consider the imaginary roots and then the real roots of defect zero.

We fix a tube $\mathcal{T}$ in $\Gamma(kQ)$ of rank $p \geq 1$. The quasi-simples of $\mathcal{T}$ are denoted by $R_0, \ldots, R_{p-1}$ and are ordered such that $\tau R_i \simeq R_{i-1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. Note that for any $l \geq 1, 0 \leq k \leq p-1$ and any $i \in \mathbb{Z}/p\mathbb{Z}$, $\mathbf{dim}\, R_i^{(lp)} = l\delta \in \Phi_{>0}^{\mathrm{im}}$ and $\mathbf{dim}\, R_i^{(lp+k)} \in \Phi_{>0}^{\mathrm{re}}$ if $k \neq 0$.

The following technical lemma will be useful in the proof of Proposition 3.3 :

Lemma 3.2. *With the above notations, for any $l \geq 1$*

$$X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} = X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}}$$

Proof. We first notice that generalized Chebyshev polynomials are symmetric in the sense that for every $i \in \mathbb{Z}$ and $n \geq 1$,

$$P_n(x_i, \ldots, x_{i+n-1}) = P_n(x_{i+n-1}, \ldots, x_i).$$

If $l = 1$, the result is obvious. We thus fix some $l \geq 2$. For technical convenience, we denote by $R_i, i \in \mathbb{Z}$ the quasi-simple modules in $\mathcal{T}$ and we assume that $R_i \simeq R_{i+p}$ for every $i \in \mathbb{Z}$. Consider the morphism of $\mathbb{Z}$-algebras

$$\phi : \begin{cases} \mathbb{Z}[X_{R_0}, \ldots, X_{R_{lp-1}}] & \longrightarrow & \mathbb{Z}[X_{R_0}, \ldots, X_{R_{lp-1}}] \\ X_{R_i} & \mapsto & X_{R_{lp-1-i}} \end{cases} \qquad \text{for all } i = 0, \ldots, lp-1.$$

It is well defined since $X_{R_0}, \ldots, X_{R_{p-1}}$ are known to be algebraically independent over $\mathbb{Z}$ (see e.g. [Dup09a]).

According to Theorem 2.2, we have

$$X_{R_1^{(p-1)}} X_{R_0^{(lp-1)}} = X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_{p+1}^{((l-1)p-2)}}.$$

According to [Dup09a, Theorem 5.1], each of the $X_{R_j}^{(k)}$ appearing above lies in $\mathbb{Z}[X_{R_0}, \ldots, X_{R_{lp-1}}]$. We can thus apply ϕ and we get

$$(2) \qquad \phi(X_{R_1^{(p-1)}})\phi(X_{R_0^{(lp-1)}}) = \phi(X_{R_0^{(p)}})\phi(X_{R_1^{(lp-2)}}) - \phi(X_{R_{p+1}^{((l-1)p-2)}}).$$

We now compute these images under ϕ.

$$\begin{aligned}\phi(X_{R_1^{(p-1)}}) &= \phi(P_{p-1}(X_{R_1},\dots,X_{R_{p-1}}))\\ &= P_{p-1}(\phi(X_{R_1}),\dots,\phi(X_{R_{p-1}}))\\ &= P_{p-1}(X_{R_{lp-2}},\dots,X_{R_{(l-1)p}})\\ &= P_{p-1}(X_{R_{(l-1)p}},\dots,X_{R_{lp-2}})\\ &= X_{R_{(l-1)p}^{(p-1)}}\\ &= X_{R_0^{(p-1)}}\end{aligned}$$

$$\begin{aligned}\phi(X_{R_0^{(lp-1)}}) &= \phi(P_{lp-1}(X_{R_0},\dots,X_{R_{lp-2}}))\\ &= P_{lp-1}(\phi(X_{R_0}),\dots,\phi(X_{R_{lp-2}}))\\ &= P_{lp-1}(X_{R_{lp-1}},\dots,X_{R_1})\\ &= P_{lp-1}(X_{R_1},\dots,X_{R_{lp-1}})\\ &= X_{R_1^{(lp-1)}}\end{aligned}$$

$$\begin{aligned}\phi(X_{R_0^{(p)}}) &= \phi(P_p(X_{R_0},\dots,X_{R_{p-1}}))\\ &= P_p(\phi(X_{R_0}),\dots,\phi(X_{R_{p-1}}))\\ &= P_p(X_{R_{lp-1}},\dots,X_{R_{(l-1)p}})\\ &= P_p(X_{R_{(l-1)p}},\dots,X_{R_{lp-1}})\\ &= X_{R_{(l-1)p}^{(p)}}\\ &= X_{R_0^{(p)}}\end{aligned}$$

$$\begin{aligned}\phi(X_{R_1^{(lp-2)}}) &= \phi(P_{lp-2}(X_{R_1},\dots,X_{R_{lp-2}}))\\ &= P_{lp-2}(\phi(X_{R_1}),\dots,\phi(X_{R_{lp-2}}))\\ &= P_{lp-2}(X_{R_{lp-2}},\dots,X_{R_1})\\ &= P_{lp-2}(X_{R_1},\dots,X_{R_{lp-2}})\\ &= X_{R_1^{(lp-2)}}\end{aligned}$$

$$\begin{aligned}\phi(X_{R_{p+1}^{((l-1)p-2)}}) &= \phi(P_{(l-1)p-2}(X_{R_{p+1}},\dots,X_{R_{lp-2}}))\\ &= P_{(l-1)p-2}(\phi(X_{R_{p+1}}),\dots,\phi(X_{R_{lp-2}}))\\ &= P_{(l-1)p-2}(X_{R_{(l-1)p-2}},\dots,X_{R_1})\\ &= P_{(l-1)p-2}(X_{R_1},\dots,X_{R_{(l-1)p-2}})\\ &= X_{R_1^{(l-1)p-2}}\end{aligned}$$

Replacing in Equation (2), we get

$$\begin{aligned} X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} &= X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_1^{((l-1)p-2)}} \\ &= X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_{p+1}^{((l-1)p-2)}} \\ &= X_{R_1^{(p-1)}} X_{R_0^{(lp-1)}} \end{aligned}$$

which proves the lemma. □

We can now prove some higher difference properties for imaginary roots.

Proposition 3.3. *Fix $l \geq 1$. Then, for any indecomposable representation M in* $\operatorname{rep}(Q, l\delta)$*, we have*

$$\mathfrak{b}_{l\delta} = F_l(X_\delta) = X_M - X_{\text{q.rad}M/\text{q.soc}M}$$

with the convention that $X_{\text{q.rad}M/\text{q.soc}M} = 0$ if M is quasi-simple.

Proof. We first treat the case where M is an indecomposable representation of dimension $l\delta$ in an homogeneous tube. This is not necessary but in this particular case, the proof is straightforward. We write $M = R^{(l)}$ for some quasi-simple module R in an homogeneous tube. If $l = 1$, the proposition follows from Theorem 3.1. If $l \geq 2$, $\text{q.rad}M/\text{q.soc}M \simeq R^{(l-2)}$ so that

$$\begin{aligned} X_M - X_{\text{q.rad}M/\text{q.soc}M} &= X_{R^{(l)}} - X_{R^{(l-2)}} \\ &= S_l(X_R) - S_{l-2}(X_R) \\ &= F_l(X_R) \\ &= F_l(X_\delta). \end{aligned}$$

We now assume that M is an indecomposable representation of dimension $l\delta$ in an exceptional tube $\mathcal{T}$ of rank $p \geq 2$. We denote $R_0, \ldots, R_{p-1}$ the quasi-simples in $\mathcal{T}$ ordered such that $\tau R_i \simeq R_{i-1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. We can thus write $M \simeq R_i^{(lp)}$ for some $i \in \mathbb{Z}/p\mathbb{Z}$. Without loss of generality, we assume that $i = 0$. In order to simplify notations, for any $l \geq 1$, we denote by

$$\Delta_l = X_{R_0^{(lp)}} - X_{R_1^{(lp-2)}}.$$

We thus have to prove that for any

$$\Delta_l = F_l(X_\delta).$$

The central tool in this proof is Theorem 2.2. According to theorem 3.1, we have

$$X_\delta = X_{R_0^{(p)}} - X_{R_1^{(p-2)}}$$

so that the proposition holds for $l = 1$.

We now prove it for $l = 2$. We have

$$\begin{aligned} F_2(X_\delta) &= X_\delta^2 - 2 \\ &= (X_{R_0^{(p)}} - X_{R_1^{(p-2)}})^2 - 2 \\ &= X^2_{R_0^{(p)}} + X^2_{R_1^{(p-2)}} - 2X_{R_0^{(p)}} X_{R_1^{(p-2)}} - 2 \end{aligned}$$

but according to the almost split multiplication formula ([CC06, Proposition 3.10]), we have

$$X_{R_0^{(p-1)}} X_{R_1^{(p-1)}} = X_{R_0^{(p)}} X_{R_1^{(p-2)}}$$

so that

$$F_2(X_\delta) = X^2_{R_0^{(p)}} - 2X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} + X^2_{R_1^{(p-2)}}.$$

But, according to Theorem 2.2, we have

$$\begin{aligned} X^2_{R_0^{(p)}} &= X_{R_0^{(p)}}X_{R_0^{(p)}} \\ &= X_{R_0^{(2p)}} + X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} \end{aligned}$$

so that finally

$$F_2(X_\delta) = X_{R_0^{(2p)}} - X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} + X^2_{R_1^{(p-2)}}.$$

Thus,

$$\begin{aligned} &F_2(X_\delta) = \Delta_2 \\ \Leftrightarrow &- X_{R_1^{(2p-2)}} = X^2_{R_1^{(p-2)}} - X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} \\ \Leftrightarrow &X_{R_1^{(2p-2)}} + X^2_{R_1^{(p-2)}} - X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} = 0 \end{aligned}$$

But

$$X_{R_1^{(2p-2)}} = -X_{R_1^{(p-3)}}X_{R_0^{(p-1)}} + X_{R_1^{(p-2)}}X_{R_{p-1}^{(p)}}$$

So that,

$$\begin{aligned} &F_2(X_\delta) = \Delta_2 \\ \Leftrightarrow &X^2_{R_1^{(p-2)}} + X_{R_1^{(p-2)}}X_{R_{p-1}^{(p)}} - X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} - X_{R_0^{(p-1)}}X_{R_1^{(p-3)}} = 0 \\ \Leftrightarrow &X_{R_1^{(p-2)}}\left[X_{R_1^{(p-2)}} + X_{R_{p-1}^{(p)}}\right] - X_{R_0^{(p-1)}}\left[X_{R_1^{(p-3)}} + X_{R_1^{(p-1)}}\right] = 0. \end{aligned}$$

But Theorem 2.2 gives

$$\begin{aligned} X_{R_{p-1}}X_{R_0^{(p-1)}} &= X_{R_1^{(p-2)}} + X_{R_{p-1}^{(p)}} \\ X_{R_1^{(p-2)}}X_{R_{p-1}} &= X_{R_1^{(p-3)}} + X_{R_1^{(p-1)}} \end{aligned}$$

so that we get

$$F_2(X_\delta) = \Delta_2$$

and the proposition is proved for $l = 2$.

For $l > 2$, we will use the three terms relations for first kind Chebyshev polynomials

$$F_l(x) = xF_{l-1}(x) - F_{l-2}(x).$$

Thus, it is enough to prove that for any $l \geq 2$,

$$\Delta_{l+1} = \Delta_1\Delta_l - \Delta_{l-1}.$$

In order to simplify notations, we by (LHS) the left-hand side of the equality and by (RHS) the right-hand side of the above equality. We thus have

$$\begin{aligned} \text{(RHS)} &= (X_{R_0^{(p)}} - X_{R_1^{(p-2)}})(X_{R_0^{(lp)}} - X_{R_1^{(lp-2)}}) - (X_{R_0^{((l-1)p)}} - X_{R_1^{((l-1)p-2)}}) \\ &= X_{R_0^{(p)}}X_{R_0^{(lp)}} - X_{R_0^{(p)}}X_{R_1^{(lp-2)}} - X_{R_1^{(p-2)}}X_{R_0^{(lp)}} \\ &+ X_{R_1^{(p-2)}}X_{R_1^{(lp-2)}} - X_{R_0^{((l-1)p)}} + X_{R_1^{((l-1)p-2)}}. \end{aligned}$$

But, according to the multiplication theorem, we get

$$X_{R_0^{(p)}}X_{R_0^{(lp)}} = X_{R_0^{((l+1)p)}} + X_{R_0^{(p-1)}}X_{R_1^{(lp-1)}}$$

So that

$$\begin{aligned}\text{(LHS)} = \text{(RHS)} \Leftrightarrow\ & X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_1^{(p-2)}} X_{R_0^{(lp)}} \\ & + X_{R_1^{(p-2)}} X_{R_1^{(lp-2)}} - X_{R_0^{((l-1)p)}} + X_{R_1^{((l-1)p-2)}} + X_{R_1^{((l+1)p-2)}} = 0.\end{aligned}$$

Applying the multiplication theorem, we get

$$X_{R_0^{(lp-2)}} X_{R_{lp-2}^{(p)}} = X_{R_0^{((l+1)p-2)}} + X_{R_0^{(lp-3)}} X_{R_{lp-1}^{(p-1)}}$$

so that

$$X_{R_1^{((l+1)p-2)}} = X_{R_1^{(lp-2)}} X_{R_{p-1}^{(p)}} - X_{R_1^{(lp-3)}} X_{R_0^{(p-1)}}$$

and thus

$$\begin{aligned}\text{(LHS)} = \text{(RHS)} \Leftrightarrow\ & X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(p)}} X_{R_1^{(lp-2)}} \\ & - X_{R_1^{(p-2)}} X_{R_0^{(lp)}} + X_{R_1^{(p-2)}} X_{R_1^{(lp-2)}} \\ & - X_{R_0^{((l-1)p)}} + X_{R_1^{((l-1)p-2)}} \\ & + X_{R_1^{(lp-2)}} X_{R_{p-1}^{(p)}} - X_{R_1^{(lp-3)}} X_{R_0^{(p-1)}} = 0\end{aligned}$$

but

$$X_{R_0^{(p)}} X_{R_1^{(lp-2)}} = X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} + X_{R_1^{((l-1)p-2)}}.$$

Thus,

$$\begin{aligned}\text{(LHS)} = \text{(RHS)} \Leftrightarrow\ & X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} \\ & - X_{R_1^{(p-2)}} X_{R_0^{(lp)}} + X_{R_1^{(p-2)}} X_{R_1^{(lp-2)}} \\ & - X_{R_0^{((l-1)p)}} + X_{R_1^{(lp-2)}} X_{R_{p-1}^{(p)}} - X_{R_1^{(lp-3)}} X_{R_0^{(p-1)}} = 0 \\ \Leftrightarrow\ & X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} \\ & - X_{R_1^{(p-2)}} X_{R_0^{(lp)}} - X_{R_0^{((l-1)p)}} - X_{R_1^{(lp-3)}} X_{R_0^{(p-1)}} \\ & + X_{R_1^{(lp-2)}} \left(X_{R_1^{(p-2)}} + X_{R_{p-1}^{(p)}} \right) = 0.\end{aligned}$$

Theorem 2.2 gives

$$X_{R_{p-1}} X_{R_0^{(p-1)}} = X_{R_1^{(p-2)}} + X_{R_{p-1}^{(p)}}$$

so that

$$\begin{aligned}\text{(LHS)} = \text{(RHS)} \Leftrightarrow\ & X_{R_0^{(p-1)}} \left(X_{R_1^{(lp-1)}} - X_{R_1^{(lp-3)}} + X_{R_1^{(lp-2)}} X_{R_{p-1}} \right) \\ & - X_{R_0^{((l-1)p)}} - X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} - X_{R_1^{(p-2)}} X_{R_0^{(lp)}} = 0\end{aligned}$$

The three terms relation for generalized Chebyshev polynomials gives

$$X_{R_1^{(lp-1)}} = X_{R_{p-1}} X_{R_1^{(lp-2)}} - X_{R_1^{(lp-3)}}.$$

Thus

$$\begin{aligned}\text{(LHS)} = \text{(RHS)} \Leftrightarrow\ & 2X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} \\ & - \left(X_{R_0^{((l-1)p)}} + X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} + X_{R_0^{(lp)}} X_{R_1^{(p-2)}} \right) = 0\end{aligned}$$

Theorem 2.2 gives

$$X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} = X_{R_0^{(lp)}} X_{R_1^{(p-2)}} + X_{R_0^{((l-1)p)}}$$

so that we finally get

$$\text{(LHS)} = \text{(RHS)} \Leftrightarrow X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} = 0.$$

The second equality holds by lemma 3.2 so that we have proved that for any $l \geq 2$,

$$\Delta_{l+1} = \Delta_1 \Delta_l - \Delta_{l-1}.$$

Since we know that $\Delta_1 = X_\delta$ and $\Delta_2 = F_2(X_\delta)$, it follows that for any $l \geq 1$, $\Delta_l = F_l(X_\delta)$ which proves the proposition. □

We are now able to prove the general difference property :

Theorem 3.4. *Let Q be an affine quiver, $\mathcal{T}$ be a tube of rank $p \geq 1$ in $\Gamma(kQ)$. Then for any $l \geq 1$ and any $0 \leq k \leq p-1$, we have*

$$\mathfrak{b}_{l\delta + \boldsymbol{dim}\, R_0^{(k)}} = X_{R_0^{(k)}} F_l(X_\delta) = X_{R_0^{(lp+k)}} - X_{R_{k+1}^{(lp-k-2)}}.$$

Proof. The first equality follows from the fact that

$$\text{den}\,(X_{R_0^{(k)}} X_\delta^l) = \mathbf{dim}\, R_0^{(k)} + l\delta$$

so that

$$\mathfrak{b}_{l\delta + \mathbf{dim}\, R_0^{(k)}} = X_{R_0^{(k)}} F_l(X_\delta).$$

We now prove that

$$X_{R_0^{(k)}} F_l(X_\delta) = X_{R_0^{(lp+k)}} - X_{R_{k+1}^{(lp-k-2)}}.$$

We denote by (LHS) the left-hand side and by (RHS) the right-hand side.

$$\begin{aligned}
\text{(LHS)} &= X_{R_0^{(k)}} \left(X_{R_k^{(lp)}} - X_{R_{k+1}^{(lp-2)}} \right) \\
&= X_{R_0^{(k)}} X_{R_k^{(lp)}} - X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}} \\
&= X_{R_0^{(lp+k)}} + X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-1)}} - X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}}
\end{aligned}$$

so that (LHS) = (RHS) if and only if

$$(3) \qquad X_{R_{k+1}^{(lp-k-2)}} = X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}} - X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-1)}}$$

holds.

Using the three terms recurrence relations for generalized Chebyshev polynomials, we have

$$X_{R_0^{(k)}} = X_{R_{k-1}} X_{R_0^{(k-1)}} - X_{R_0^{(k-2)}}$$

and

$$X_{R_{k+1}^{(lp-1)}} = X_{R_{lp+k-1}} X_{R_{k+1}^{(lp-2)}} - X_{R_{k+1}^{(lp-3)}}$$

so that, replacing in the right-hand side of equality (3), we get :

$$X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}} - X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-1)}} = X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-3)}} - X_{R_0^{(k-2)}}$$

Thus, by induction, we get

$$X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}} - X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-1)}} = X_{R_0} X_{R_{k+1}^{(lp-k-1)}} - X_{R_{k+1}^{(lp-k)}}.$$

Now, the three terms recurrence relation gives

$$\begin{aligned} X_{R_{k+1}^{(lp-k)}} &= X_{R_{k+1+lp-k-1}} X_{R_{k+1}^{(lp-k-1)}} - X_{R_{k+1}^{(lp-k-2)}} \\ &= X_{R_0} X_{R_{k+1}^{(lp-k-1)}} - X_{R_{k+1}^{(lp-k-2)}} \end{aligned}$$

and thus

$$X_{R_0} X_{R_{k+1}^{(lp-k-1)}} - X_{R_{k+1}^{(lp-k)}} = X_{R_{k+1}^{(lp-k-2)}}$$

so that equality (3) holds. □

As a corollary, for any positive root $\mathbf{d}$ with defect zero, we obtain a description of $\mathfrak{b}_{\mathbf{d}}$ as a certain difference of cluster characters :

Corollary 3.5. *Let Q be an affine quiver, $\mathbf{d}$ be a positive root with defect zero. Let M be any indecomposable representation of dimension $\mathbf{d}$. Then, there exists a quasi-simple module R_0 in a tube of rank $p \geq 1$, an integer $0 \leq k \leq p-1$ and an integer $l \geq 0$ such that $\mathbf{d} = l\delta + \mathbf{dim}\, R_0^{(k)}$. Moreover, for any such R_0, k, l, we have*

$$\mathfrak{b}_{\mathbf{d}} = X_{R_0^{(k)}} F_l(X_\delta) = X_{R_0^{(lp+k)}} - X_{R_{k+1}^{(lp-k-2)}}.$$

where $R_0, \ldots, R_{p-1}$ are the quasi-simple modules in $\mathcal{T}$ ordered such that $\tau R_i \simeq R_{i-1}$ for every $i \in \mathbb{Z}/p\mathbb{Z}$.

4. Integrable bundle on rep(Q) and their characters

In the previous section, we obtained a realization of the elements $\mathfrak{b}_{\mathbf{d}}$ associated to defect zero roots as differences of cluster characters. The aim of this section is to introduce a new character θ_{Tr} such that these elements correspond precisely to values of θ_{Tr}.

Unless it is otherwise specified, Q denotes an arbitrary acyclic quiver in this section.

4.1. **Integrable bundles.** For any $\mathbf{d} \in \mathbb{N}^{Q_0}$, the representation variety $\mathrm{rep}(Q, \mathbf{d})$ of dimension $\mathbf{d}$ is the set of all representations M of Q with dimension vector $\mathbf{d}$. Note that

$$\mathrm{rep}(Q, \mathbf{d}) \simeq \prod_{i \longrightarrow j \in Q_1} \mathrm{Hom}_k(k^{d_i}, k^{d_j})$$

so that $\mathrm{rep}(Q, \mathbf{d})$ is an affine irreducible variety. We identify the set of objects in the category $\mathrm{rep}(Q)$ with

$$\mathrm{rep}(Q) = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathrm{rep}(Q, \mathbf{d}).$$

Definition 4.1. Let Q be any acyclic quiver. An *integrable bundle* on $\mathrm{rep}(Q)$ is a map

$$\mathcal{F} : M \mapsto \mathcal{F}(M)$$

from $\mathrm{rep}(Q)$ to the set of subsets in $\mathrm{rep}(Q)$, such that, for any $M \in \mathrm{rep}(Q)$:

1. the set $\mathcal{F}_{\mathbf{e}}(M) = \mathcal{F}(M) \cap \mathrm{rep}(Q, \mathbf{e})$ is a constructible subset of $\mathrm{rep}(Q, \mathbf{e})$ for any $\mathbf{e} \in \mathbb{N}^{Q_0}$;
2. the family $(\chi(\mathcal{F}_{\mathbf{e}}(M)))_{\mathbf{e} \in \mathbb{N}^{Q_0}}$ has finite support.

Example 4.2. The map $M \mapsto \mathrm{Gr}(M)$ is an integrable bundle called *quiver grassmannian bundle*.

For any kQ-module M and any submodule $U \subset M$, we set

$$\mathrm{Gr}^U(M) = \{N \in \mathrm{Gr}(M) | U \text{ is a submodule of } N\}$$

the set of submodules of M containing U as a submodule. This is a constructible subset in the quiver grassmannian $\mathrm{Gr}(M)$.

If Q is an affine quiver, we will define another integrable bundle Tr as follows. Let M be an indecomposable kQ-module. If M is rigid, we set $\mathrm{Tr}(M) = \mathrm{Gr}(M)$. If M is not rigid, it is regular and we can thus write $M = R_0^{(lp+k)}$ for some quasi-simple module R_0 in a tube of rank $p \geq 1$ and $0 \leq k \leq p-1$. In this case, we set

$$\mathrm{Tr}(M) = \mathrm{Gr}(M) \setminus \mathrm{Gr}^{R_0^{(k+1)}}(R_0^{(lp-1)}).$$

Note that if $l = 0$, M is rigid and $\mathrm{Tr}(M) = \mathrm{Gr}(M)$. Finally, if $M = \bigoplus_{i=1}^n M_i$ is decomposable, we define $\mathrm{Tr}(M)$ as the image of $\mathrm{Tr}(M_1) \times \cdots \times \mathrm{Tr}(M_n)$ under the algebraic isomorphism

$$\begin{array}{rcl} \mathrm{Gr}(M_1) \times \cdots \times \mathrm{Gr}(M_n) & \longrightarrow & \mathrm{Gr}(M) \\ (U_1, \cdots, U_n) & \mapsto & U_1 \oplus \cdots \oplus U_n. \end{array}$$

For every dimension vector $\mathbf{e} \in \mathbb{N}^{Q_0}$, we set $\mathrm{Tr}_{\mathbf{e}}(M) = \mathrm{Tr}(M) \cap \mathrm{rep}(Q, \mathbf{e})$. It follows directly from the definition that $\mathrm{Tr}_{\mathbf{e}}(M)$ is a constructible subset of $\mathrm{Gr}_{\mathbf{e}}(M)$ called *transverse quiver grassmannian of* M (of dimension $\mathbf{e}$). Since $\mathrm{Tr}_{\mathbf{e}}(M) \subset \mathrm{Gr}_{\mathbf{e}}(M)$ for any $\mathbf{e} \in \mathbb{N}^{Q_0}$, it follows that $\mathrm{Tr}_{\mathbf{e}}(M)$ is empty for all but finitely many $\mathbf{e} \in \mathbb{N}^{Q_0}$ so that

$$\mathrm{Tr} : M \mapsto \mathrm{Tr}(M)$$

is an integrable bundle on $\mathrm{rep}(Q)$.

4.2. **Character associated to an integrable bundle.** Extending an idea of Caldero and Chapoton, we associate to any integrable bundle on $\mathrm{rep}(Q)$ a map from the set of objects in $\mathcal{C}_Q$ to the ring $\mathbb{Z}[\mathbf{u}^{\pm 1}]$ of Laurent polynomials in the initial cluster of $\mathcal{A}(Q)$.

Definition 4.3. Let $\mathcal{F}$ be an integrable bundle on $\mathrm{rep}(Q)$. The *character associated to* $\mathcal{F}$ is the map

$$\theta_{\mathcal{F}}(?) : \mathrm{Ob}(\mathcal{C}_Q) \longrightarrow \mathbb{Z}[\mathbf{u}^{\pm 1}]$$

given by

$$\theta_{\mathcal{F}}(M) = \mathbf{u}^{-\mathbf{dim}\, P_M[1]} \left(\sum_{\mathbf{e} \in \mathbb{N}^{Q_0}} \chi(\mathcal{F}_{\mathbf{e}}(M_0)) \prod_{i \in Q_0} u_i^{-\langle \mathbf{e}, S_i \rangle - \langle S_i, \mathbf{dim}\, M_0 - \mathbf{e} \rangle} \right)$$

for any object $M = M_0 \oplus P_M[1]$ in $\mathcal{C}_Q$ where M_0 is a kQ-module and P_M is a projective kQ-module.

Example 4.4. In this terminology, the Caldero-Chapoton map is the character associated to the quiver grassmannian bundle. More precisely,

$$\theta_{\mathrm{Gr}}(M) = X_M$$

for any object M in $\mathcal{C}_Q$.

We thus set the following definition :

Definition 4.5. The *transverse character* is the map

$$\theta_{\mathrm{Tr}} : \left\{ \begin{array}{ccc} \mathrm{Ob}(\mathcal{C}_Q) & \longrightarrow & \mathbb{Z}[\mathbf{u}^{\pm 1}] \\ M & \mapsto & \theta_{\mathrm{Tr}}(M) \end{array} \right.$$

We first prove a multiplicative property of θ_{Tr}.

Lemma 4.6. *Let M, N be two objects in $\mathcal{C}_Q$, then*

$$\theta_{\mathrm{Tr}}(M \oplus N) = \theta_{\mathrm{Tr}}(M)\theta_{\mathrm{Tr}}(N).$$

Proof. For any $\mathbf{g} \in \mathbb{N}^{Q_0}$, there is an isomorphism of varieties

$$\begin{array}{ccc} \bigsqcup_{\mathbf{e}+\mathbf{f}=\mathbf{g}} \mathrm{Gr}_{\mathbf{e}}(M) \times \mathrm{Gr}_{\mathbf{f}}(N) & \longrightarrow & \mathrm{Gr}_{\mathbf{g}}(M \oplus N) \\ (U, V) & \mapsto & U \oplus V. \end{array}$$

By definition of Tr, this restricts to an isomorphism

$$\begin{array}{ccc} \bigsqcup_{\mathbf{e}+\mathbf{f}=\mathbf{g}} \mathrm{Tr}_{\mathbf{e}}(M) \times \mathrm{Tr}_{\mathbf{f}}(N) & \longrightarrow & \mathrm{Tr}_{\mathbf{g}}(M \oplus N) \\ (U, V) & \mapsto & U \oplus V. \end{array}$$

It follows easily that

$$\theta_{\mathrm{Tr}}(M \oplus N) = \theta_{\mathrm{Tr}}(M)\theta_{\mathrm{Tr}}(N).$$

□

We now prove that θ_{Tr} coincides with $X_?$ on the set of rigid objects in $\mathcal{C}_Q$. In particular, this will allow to realize cluster monomials in terms of θ_{Tr}.

Lemma 4.7. *Let Q be an affine quiver. Then, for any rigid object M in $\mathcal{C}_Q$, we have $\theta_{\mathrm{Tr}}(M) = X_M$. In particular,*

$$\mathcal{M}(Q) = \{\theta_{\mathrm{Tr}}(M) | M \text{ is rigid in } \mathcal{C}_Q\}.$$

Proof. We set $M = M_0 \oplus P_M[1]$. Since M is rigid in $\mathcal{C}_Q$, M_0 is a rigid kQ-module. Thus, it follows from the definition of the transverse grassmannian that $\mathrm{Tr}(M_0) = \mathrm{Gr}(M_0)$. Then,

$$\begin{aligned} \theta_{\mathrm{Tr}}(M) &= \mathbf{u}^{-\mathbf{dim}\, P_M[1]} \left(\sum_{\mathbf{e} \in \mathbb{N}^{Q_0}} \chi(\mathrm{Tr}_{\mathbf{e}}(M_0)) \prod_{i \in Q_0} u_i^{-\langle \mathbf{e}, S_i \rangle - \langle S_i, \mathbf{dim}\, M_0 - \mathbf{e} \rangle} \right) \\ &= X_{P_M[1]} \left(\sum_{\mathbf{e} \in \mathbb{N}^{Q_0}} \chi(\mathrm{Tr}_{\mathbf{e}}(M_0)) \prod_{i \in Q_0} u_i^{-\langle \mathbf{e}, S_i \rangle - \langle S_i, \mathbf{dim}\, M_0 - \mathbf{e} \rangle} \right) \\ &= X_{P_M[1]} \left(\sum_{\mathbf{e} \in \mathbb{N}^{Q_0}} \chi(\mathrm{Gr}_{\mathbf{e}}(M_0)) \prod_{i \in Q_0} u_i^{-\langle \mathbf{e}, S_i \rangle - \langle S_i, \mathbf{dim}\, M_0 - \mathbf{e} \rangle} \right) \\ &= X_{P_M[1]} X_{M_0} \\ &= X_{P_M[1] \oplus M_0} \\ &= X_M. \end{aligned}$$

The second assertion follows directly from Caldero-Keller's realization of cluster monomials :

$$\begin{aligned} \mathcal{M}(Q) &= \{X_M | M \text{ is rigid in } \mathcal{C}_Q\} \\ &= \{\theta_{\mathrm{Tr}}(M) | M \text{ is rigid in } \mathcal{C}_Q\}. \end{aligned}$$

□

5. A geometrization of $\mathcal{B}(Q)$

We will now relate the transverse character with the difference properties obtained in Section 3. This will provide a realization of the elements in $\mathcal{B}(Q)$ in terms of θ_{Tr}.

5.1. **From difference properties to θ_{Tr}.** Using Theorem 3.4, we can now deduce a realization in terms of θ_{Tr} of the elements in $\mathcal{B}(Q)$ corresponding to positive roots with zero defect :

Theorem 5.1. *Let $\mathbf{d}$ be any positive root. Then*

$$\mathfrak{b}_{\mathbf{d}} = \theta_{\mathrm{Tr}}(M)$$

where M is any indecomposable representation of dimension $\mathbf{d}$.

Proof. If $\mathbf{d}$ is a positive root with non-zero defect. Then, $\mathbf{d}$ is real and there exists a unique indecomposable representation M in $\mathrm{rep}(Q, \mathbf{d})$. Moreover, this representation has to be preprojective or preinjective. In both cases, it is rigid and thus $\mathfrak{b}_{\mathbf{d}} = X_M = \theta_{\mathrm{Tr}}(M)$. We can thus assume that $\mathbf{d} \in \mathbb{N}^{Q_0}$ is a root with zero defect.

Let M be an indecomposable representation in $\mathrm{rep}(Q, \mathbf{d})$. It is necessarily contained in a tube $\mathcal{T}$ of rank $p \geq 1$. We denote by $R_0, \ldots, R_{p-1}$ the quasi-simple modules in $\mathcal{T}$ ordered such that $\tau R_i \simeq R_{i-1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. We can write $\mathbf{d} = l\delta + \mathbf{n}$ where $\mathbf{n}$ is either a real Schur root or zero. If $\mathbf{n} \neq 0$, there exists a unique indecomposable representation N in $\mathrm{rep}(Q, \mathbf{n})$. More precisely, if $M \simeq R_0^{(lp+k)}$ with $l \geq 0$ and $0 \leq k \leq p-1$, N is the rigid representation $R_0^{(k)}$ (still with the convention that $R_0^{(0)} = 0$) and

$$\mathfrak{b}_{\mathbf{d}} = X_{R_0^{(k)}} F_l(X_\delta).$$

Now according to Theorem 3.4, we have

$$X_{R_0^{(k)}} F_l(X_\delta) = X_{R_0^{(lp+k)}} - X_{R_{k+1}^{(lp-k-2)}}$$

For any $\mathbf{e} \in \mathbb{N}^{Q_0}$, the map

$$\begin{cases} \mathrm{Gr}_{\mathbf{e}}^{R_0^{(k+1)}}(R_0^{(lp-1)}) & \longrightarrow & \mathrm{Gr}_{\mathbf{e}-\mathbf{dim}\, R_0^{(k+1)}}(R_{k+1}^{(lp-k-2)}) \\ U & \mapsto & U/R_0^{(k+1)} \end{cases}$$

is an algebraic isomorphism and we denote by $c_{\mathbf{e}} \in \mathbb{Z}$ the common value of the Euler characteristics of these constructible sets. Fix now some $\mathbf{e} \in \mathbb{N}^{Q_0}$, the monomial corresponding to $\mathbf{e}$ in $X_{R_0^{(lp+k)}}$ is

$$c_{\mathbf{e}} \prod_i u_i^{-\langle \mathbf{e}, S_i \rangle - \left\langle S_i, \mathbf{dim}\, R_0^{(lp+k)} - \mathbf{e} \right\rangle}$$

and the monomial corresponding to $\mathbf{e} - \mathbf{dim}\, R_0^{(k+1)}$ in $X_{R_{k+1}^{(lp-k-2)}}$ is

$$c_{\mathbf{e}} \prod_i u_i^{-\left\langle \mathbf{e} - \mathbf{dim}\, R_0^{(k+1)}, S_i \right\rangle - \left\langle S_i, \mathbf{dim}\, R_{k+1}^{(lp-k-2)} + \mathbf{dim}\, R_0^{(k+1)} - \mathbf{e} \right\rangle}.$$

We now prove that these monomials are the same. For any $i = 0, \ldots, p-1$, we set by $r_i = \mathbf{dim}\, R_i$ and we denote by c the Coxeter transformation on $\mathbb{Z}^{Q_0}$ induced by the Auslander-Reiten translation. We recall that for any $\beta, \gamma \in \mathbb{Z}^{Q_0}$, we have $\langle \gamma, c(\beta) \rangle = -\langle \beta, \gamma \rangle$. With these notations, we have

$$\begin{aligned}\mathbf{dim}\, R_0^{(k+1)} &= r_0 + \cdots + r_k \\ \mathbf{dim}\, R_{k+1}^{(lp-k-2)} &= (l-1)\delta + r_{k+1} + \cdots r_{p-2}\end{aligned}$$

so that

$$\mathbf{dim}\, R_0^{(k+1)} + \mathbf{dim}\, R_{k+1}^{(lp-k-2)} = l\delta - r_{p-1}.$$

We now compute the exponents :

$$\begin{aligned}&- \langle \mathbf{e}, S_i \rangle - \left\langle S_i, \mathbf{dim}\, R_0^{(lp+k)} - \mathbf{e} \right\rangle \\ &= - \langle \mathbf{e}, S_i \rangle - \langle S_i, l\delta + r_0 + \cdots + r_{k-1} - \mathbf{e} \rangle \\ &= - \langle \mathbf{e}, S_i \rangle - \langle S_i, l\delta - \mathbf{e} \rangle - \langle S_i, r_0 + \cdots + r_{k-1} \rangle \\ &= - \langle \mathbf{e}, S_i \rangle - \langle S_i, l\delta - \mathbf{e} \rangle - \langle r_1 + \cdots + r_k, S_i \rangle\end{aligned}$$

and

$$\begin{aligned}&- \left\langle \mathbf{e} - \mathbf{dim}\, R_0^{(k+1)}, S_i \right\rangle - \left\langle S_i, \mathbf{dim}\, R_{k+1}^{(lp-k-2)} + \mathbf{dim}\, R_0^{(k+1)} - \mathbf{e} \right\rangle \\ &= - \langle \mathbf{e}, S_i \rangle + \langle r_0 + \cdots + r_k, S_i \rangle + \langle S_i, r_{p-1} \rangle - \langle S_i, l\delta - \mathbf{e} \rangle \\ &= - \langle \mathbf{e}, S_i \rangle - \langle S_i, l\delta - \mathbf{e} \rangle + \langle r_1 + \cdots + r_k, S_i \rangle\end{aligned}$$

so that the two monomials are the same. Thus,

$$\begin{aligned}&X_{R_0^{(lp+k)}} - X_{R_{k+1}^{(lp-k-2)}} \\ &= \sum_{\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{e}}(R_0^{(lp+k)})) \prod_i u_i^{-\langle \mathbf{e}, S_i \rangle - \langle S_i, l\delta - \mathbf{e} \rangle - \langle r_1 + \cdots + r_k, S_i \rangle} \\ &- \sum_{\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{e} - \mathbf{dim}\, R_0^{(k+1)}}(R_{k+1}^{(lp-k-2)})) \prod_i u_i^{-\langle \mathbf{e}, S_i \rangle - \langle S_i, l\delta - \mathbf{e} \rangle - \langle r_1 + \cdots + r_k, S_i \rangle} \\ &= \sum_{\mathbf{e}} \chi\left(\mathrm{Gr}_{\mathbf{e}}(R_0^{(lp+k)}) \setminus \mathrm{Gr}_{\mathbf{e}}^{R_0^{(k+1)}}(R_0^{(lp-1)})\right) \prod_i u_i^{-\langle \mathbf{e}, S_i \rangle - \langle S_i, l\delta - \mathbf{e} \rangle - \langle r_1 + \cdots + r_k, S_i \rangle} \\ &= \sum_{\mathbf{e}} \chi(\mathrm{Tr}_{\mathbf{e}}(R_0^{(lp+k)})) \prod_i u_i^{-\langle \mathbf{e}, S_i \rangle - \left\langle S_i, \mathbf{dim}\, R_0^{(lp+k)} - \mathbf{e} \right\rangle}.\end{aligned}$$

This proves the corollary. $\square$

5.2. **Realization of $\mathcal{B}(Q)$ in terms of θ_{Tr}.** Summing up the previous results, we deduce the following geometric description of $\mathcal{B}(Q)$:

Theorem 5.2. *Let Q be an affine quiver, then*

$$\mathcal{B}(Q) = \{\theta_{\mathrm{Tr}}(M \oplus R) | M \text{ is an indecomposable (or zero) regular } kQ\text{-module}, \\ R \text{ is any rigid object in } \mathcal{C}_Q \text{ such that } \mathrm{Ext}^1_{\mathcal{C}_Q}(M, R) = 0\Big\}.$$

Proof. We denote by S the right-hand-side of the claimed equality. By definition, we have

$$\mathcal{B}(Q) = \mathcal{M}(Q) \sqcup \{F_l(X_\delta) X_R | l \geq 1, R \text{ is a regular rigid } kQ\text{-module}\}.$$

We first prove that every $S \subset \mathcal{B}(Q)$. Let R be a rigid object in $\mathcal{C}_Q$, then $\theta_{\mathrm{Tr}}(R)$ is a cluster monomial by Lemma 4.7. Fix now M to be an indecomposable regular kQ-module in a tube $\mathcal{T}$ such that $\mathrm{Ext}^1_{\mathcal{C}_Q}(M, R) = 0$. If M is rigid, then $M \oplus R$

is rigid in $\mathcal{C}_Q$ and $\theta_{\mathrm{Tr}}(M)\theta_{\mathrm{Tr}}(R) = \theta_{\mathrm{Tr}}(M \oplus R)$ is a cluster monomial by Lemma 4.7. Now, if M is non-rigid, then $\mathbf{d} = \mathbf{dim}\, M$ is a positive root of defect zero, thus, $\theta_{\mathrm{Tr}}(M) = \mathfrak{b}_{\mathbf{d}}$ by Theorem 5.1. Thus, there exists $l \geq 1$ and E a indecomposable rigid (or zero) module in $\mathcal{T}$ such that $\mathbf{d} = l\delta + \mathbf{dim}\, E$. According to Theorem 5.1, we have

$$\begin{aligned}\theta_{\mathrm{Tr}}(M \oplus R) &= \theta_{\mathrm{Tr}}(M)\theta_{\mathrm{Tr}}(R)\\ &= \mathfrak{b}_{l\delta + \mathbf{dim}\, E}\theta_{\mathrm{Tr}}(R)\\ &= F_l(X_\delta)X_E\theta_{\mathrm{Tr}}(R)\\ &= F_l(X_\delta)\theta_{\mathrm{Tr}}(E)\theta_{\mathrm{Tr}}(R)\\ &= F_l(X_\delta)\theta_{\mathrm{Tr}}(E \oplus R).\end{aligned}$$

Since $\mathrm{Ext}^1_{\mathcal{C}_Q}(M,R) = 0$, we have $\mathrm{Ext}^1_{kQ}(M,R) = 0$ and $\mathrm{Ext}^1_{kQ}(R,M) = 0$. Thus, it follows easily that $\mathrm{Ext}^1_{kQ}(E,R) = 0$ and $\mathrm{Ext}^1_{kQ}(R,E) = 0$ so that $E \oplus R$ is a rigid regular kQ-module. In particular, $\theta_{\mathrm{Tr}}(E \oplus R) = X_{E\oplus R}$ and thus

$$\theta_{\mathrm{Tr}}(M \oplus R) = F_l(X_\delta)X_{E\oplus R} \in \mathcal{B}(Q).$$

Conversely, fix an elements in $\mathcal{B}(Q)$. If x is a cluster monomial, then according to Lemma 4.7, there exists some rigid object M in $\mathcal{C}_Q$ such that $x = \theta_{\mathrm{Tr}}(M)$. Thus, $x \in S$. Fix now some regular rigid kQ-module R and some integer $l \geq 1$. Then the direct summands of R belong to exceptional tubes. We fix an indecomposable kQ-module M of dimension vector $l\delta$ in an homogeneous tube. Then, $\mathrm{Ext}^1_{\mathcal{C}_Q}(M,R) = 0$. According to Theorem 5.1, we have $F_l(X_\delta)X_R = \theta_{\mathrm{Tr}}(M)X_R$ but R is rigid so that $X_R\theta_{\mathrm{Tr}}(R)$. Thus,

$$F_l(X_\delta)X_R = \theta_{\mathrm{Tr}}(M)\theta_{\mathrm{Tr}}(R) = \theta_{\mathrm{Tr}}(M \oplus R) \in S.$$

This proves the theorem. □

6. Examples

We shall now study two examples corresponding to cases where it is known that $\mathcal{B}(Q)$ is the canonically positive basis in $\mathcal{A}(Q)$.

6.1. **The $\tilde{\mathbb{A}}_{1,1}$ case.** Let Q be the Kronecker quiver, that is, the affine quiver of type $\tilde{\mathbb{A}}_{1,1}$ with the following orientation :

$$Q: \qquad 1 \rightrightarrows 2$$

The minimal imaginary root of Q is $\delta = (11)$.

For any $\lambda \in k$, we set

$$M_\lambda: \qquad k \underset{\lambda}{\overset{1}{\rightrightarrows}} k$$

and

$$M_\infty: \qquad k \underset{1}{\overset{0}{\rightrightarrows}} k.$$

It is well-known that every tube in $\Gamma(kQ)$ is homogeneous and that the family $\{M_\lambda | \lambda \in k \sqcup \{\infty\}\}$ is a complete set of representatives of pairwise non-isomorphic quasi-simple kQ-modules.

For any $n \geq 1$, the indecomposable representations of quasi-length n are given by

$$M_\lambda^{(n)}: \qquad k^n \underset{J_n(\lambda)}{\overset{1}{\rightrightarrows}} k^n$$

for any $\lambda \in k$ and

$$M_\infty^{(n)}: \qquad k^n \underset{1}{\overset{J_n(0)}{\rightrightarrows}} k^n$$

where $J_n(\lambda) \in M_n(k)$ denotes the Jordan block of size n associated to the eigenvalue λ. Quiver grassmannians and transverse quiver grassmannians of indecomposable representations with quasi-length 2 are described in Figure 1 below.

Note that kQ-mod contains no regular rigid modules. It follows that in this case

$$\mathcal{B}(Q) = \mathcal{M}(Q) \sqcup \{\theta_{\mathrm{Tr}}(M) | M \text{ is an indecomposable (or zero) regular } kQ\text{-module}\}.$$

According to [SZ04], this set is the canonically positive basis of $\mathcal{A}(Q)$.

In Figure 1, we see that, for any $\lambda \in k \sqcup \{\infty\}$,

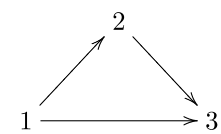
$$\theta_{Dr}(M_\lambda^{(2)}) = \theta_{Gr}(M_\lambda^{(2)}) - 1 = X_{M_\lambda^{(2)}} - 1 = S_2(X_{M_\lambda}) - 1 = F_2(X_{M_\lambda}) = \mathfrak{b}_{2\delta}.$$

This illustrates Theorem 5.1.

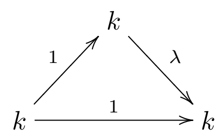
$\mathbf{e}$	$\mathrm{Gr}_{\mathbf{e}}(M_0^{(2)})$	$\mathrm{Tr}_{\mathbf{e}}(M_0^{(2)})$	$\mathrm{Gr}_{\mathbf{e}}(M_\lambda^{(2)})$	$\mathrm{Tr}_{\mathbf{e}}(M_\lambda^{(2)})$	$\mathrm{Gr}_{\mathbf{e}}(M_\infty^{(2)})$	$\mathrm{Tr}_{\mathbf{e}}(M_\infty^{(2)})$	$\mathbf{u}^{\langle -\mathbf{e}, S_i\rangle - \langle S_i, 2\delta - \mathbf{e}\rangle}$
(00)	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\frac{u_1^2}{u_2^2}$
(01)	$\mathbb{P}^1 \times \{S_2\}$	$\mathbb{P}^1 \times \{S_2\}$	$\mathbb{P}^1 \times \{S_2\}$	$\mathbb{P}^1 \times \{S_2\}$	$\mathbb{P}^1 \times \{S_2\}$	$\mathbb{P}^1 \times \{S_2\}$	$\frac{1}{u_2^2}$
(02)	$\{S_2 \oplus S_2\}$	$\{S_2 \oplus S_2\}$	$\{S_2 \oplus S_2\}$	$\{S_2 \oplus S_2\}$	$\{S_2 \oplus S_2\}$	$\{S_2 \oplus S_2\}$	$\frac{1}{u_1^2 u_2^2}$
(11)	$\{M_0\}$	$\emptyset$	$\{M_\lambda\}$	$\emptyset$	$\{M_\infty\}$	$\emptyset$	1
(12)	$\{P_1, M_0 \oplus S_2\}$	$\{P_1, M_0 \oplus S_2\}$	$\mathbb{P}^1 \times \{M_\lambda \oplus S_2\}$	$\mathbb{P}^1 \times \{M_\lambda \oplus S_2\}$	$\{P_1, M_\infty \oplus S_2\}$	$\{P_1, M_\infty \oplus S_2\}$	$\frac{1}{u_1^2}$
(22)	$\left\{M_0^{(2)}\right\}$	$\left\{M_0^{(2)}\right\}$	$\left\{M_\lambda^{(2)}\right\}$	$\left\{M_\lambda^{(2)}\right\}$	$\left\{M_\infty^{(2)}\right\}$	$\left\{M_\infty^{(2)}\right\}$	$\frac{u_2^2}{u_1^2}$

FIGURE 1. Grassmannians and transverse grassmannians of indecomposable modules of quasi-length 2

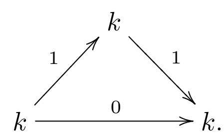
6.2. **The $\tilde{\mathbb{A}}_{2,1}$ case.** We now consider the quiver of affine type $\tilde{\mathbb{A}}_{2,1}$ equipped with the following orientation :

Q


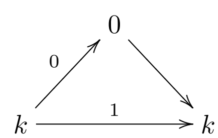
The minimal imaginary root of Q is $\delta = (111)$. For any $\lambda \in k$, we set

M_λ


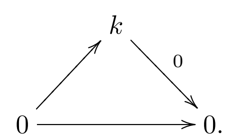
and

M_∞


$\Gamma(kQ)$ contains exactly one exceptional tube $\mathcal{T}$ of rank 2 whose quasi-simples are

R_0


and

$R_1 \simeq S_2$


The set $\{M_\lambda | \lambda \in k \sqcup \{\infty\}\}$ is a complete set of representatives of pairwise non-isomorphic indecomposable representations in $\mathrm{rep}(Q, \delta)$. For any $\lambda \neq 0, \infty$, M_λ is a quasi-simple kQ-module in an homogeneous tube. Moreover, $M_0 = R_1^{(2)}$ and $M_\infty = R_0^{(2)}$.

Quiver grassmannians and transverse quiver grassmannians of indecomposable representations of dimension δ are described in Figure 2 below. For simplicity, we only listed the dimension vectors corresponding giving non-empty quiver grassmannians.

$\mathbf{e}$	$\mathrm{Gr}_{\mathbf{e}}(M_\lambda)$	$\mathrm{Tr}_{\mathbf{e}}(M_\lambda)$	$\mathrm{Gr}_{\mathbf{e}}(M_0)$	$\mathrm{Tr}_{\mathbf{e}}(M_0)$	$\mathrm{Gr}_{\mathbf{e}}(M_\infty)$	$\mathrm{Tr}_{\mathbf{e}}(M_\infty)$	$\mathbf{u}^{\langle -\mathbf{e}, S_i\rangle - \langle S_i, \delta - \mathbf{e}\rangle}$
(000)	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\frac{u_1}{u_3}$
(001)	$\{S_3\}$	$\{S_3\}$	$\{S_3\}$	$\{S_3\}$	$\{S_3\}$	$\{S_3\}$	$\frac{1}{u_2 u_3}$
(010)	$\emptyset$	$\emptyset$	$\{S_2\}$	$\emptyset$	$\emptyset$	$\emptyset$	1
(011)	$\{P_2\}$	$\{P_2\}$	$\{S_2 \oplus S_3\}$	$\{S_2 \oplus S_3\}$	$\{P_2\}$	$\{P_2\}$	$\frac{1}{u_1 u_2}$
(101)	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{M_\infty\}$	$\emptyset$	1
(111)	$\{M_\lambda\}$	$\{M_\lambda\}$	$\{M_0\}$	$\{M_0\}$	$\{M_\infty\}$	$\{M_\infty\}$	$\frac{u_3}{u_1}$

FIGURE 2. Grassmannians and transverse grassmannians for quasi-length 2

In 2, we observe that $X_{M_\lambda} = X_{M_0} - 1 = X_{M_\infty} - 1$, illustrating Theorem 3.1. Also, we see that $\theta_{\mathrm{Tr}}(M_\lambda) = \theta_{\mathrm{Tr}}(M_0) = \theta_{\mathrm{Tr}}(M_\infty)$ for any $\lambda \in k \setminus \{0\}$ so that the transverse character does not depend on the chosen tube. Moreover,

$$\theta_{\mathrm{Tr}}(M_\lambda) = X_{M_\lambda} = F_1(X_\delta)$$

illustrating Theorem 5.1.

Remark 6.1. Figure 2 justifies in some sense the terminology "transverse submodule". Indeed, we see that, given two indecomposable regular modules M and N having the same dimension vectors, the submodules U in $\mathrm{Tr}(M)$ are those having a corresponding submodule in $\mathrm{Gr}(N)$. In some sense, we can see U as a submodule "common" to M and N. This is why we call it *transverse*.

As suggested by Bernhard Keller, this notion of transversality should have a more precise meaning in the context of deformation theory. Some connections are known at this time, this should be discussed in a forthcoming article.

Acknowledgements

The author is grateful to the CRM-ISM, the University of Sherbrooke, Ibrahim Assem, Thomas Brüstle and Virginie Charette for their financial support. He would also like to thank Giovanni Cerulli Irelli for motivating the investigation of higher difference properties during his stay at the University of Padova. This was the starting point of this work. He would also like to thank the rest of the algebra group of Padova for their kind hospitality. Finally, he would like to thank Bernhard Keller, Philippe Caldero and Frédéric Chapoton for interesting discussions on the topic.

References

[BFZ05] A. Berenstein, S. Fomin, and A. Zelevinsky. Cluster algebras III: Upper bounds and double Bruhat cells. *Duke Mathematical Journal*, 126(1):1–52, 2005. MR2110627 (2005i:16065).

[BMR+06] A. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov. Tilting theory and cluster combinatorics. *Advances in mathematics*, 204(2):572–618, 2006. MR2249625 (2007f:16033).

[CC06] P. Caldero and F. Chapoton. Cluster algebras as Hall algebras of quiver representations. *Commentarii Mathematici Helvetici*, 81:596–616, 2006. MR2250855 (2008b:16015).

[CCS06] P. Caldero, F. Chapoton, and R. Schiffler. Quivers with relations arising from clusters (A_n case). *Transactions of the AMS*, 358:1347–1354, 2006. MR2187656 (2007a:16025).

[Cer09] G. Cerulli Irelli. Canonically positive basis of cluster algebras of type $A_2^{(1)}$. *arXiv:0904.2543v1 [math.RT]*, 2009.

[CK06] P. Caldero and B. Keller. From triangulated categories to cluster algebras II. *Annales Scientifiques de l'Ecole Normale Supérieure*, 39(4):83–100, 2006. MR2316979 (2008m:16031).

[CK08] P. Caldero and B. Keller. From triangulated categories to cluster algebras. *Inventiones Mathematicae*, 172:169–211, 2008. MR2385670.

[CZ06] P. Caldero and A. Zelevinsky. Laurent expansions in cluster algebras via quiver representations. *Moscow Mathematical Journal*, 6:411–429, 2006. MR2274858 (2008j:16045).

[Dup08] G. Dupont. Generic variables in acyclic cluster algebras. *arXiv:0811.2909v1 [math.RT]*, 2008.

[Dup09a] G. Dupont. Cluster multiplication in regular components via generalized Chebyshev polynomials. *arXiv:0801.3964v2 [math.RT]*, 2009.

[Dup09b] G. Dupont. Quantized Chebyshev polynomials and cluster characters with coefficients. *Journal of Algebraic Combinatorics*, (to appear) 2009.

[DXX09] M. Ding, J. Xiao, and F. Xu. Integral bases of cluster algebras and representations of tame quivers. *arXiv:0901.1937v1 [math.RT]*, 2009.

[FZ02] S. Fomin and A. Zelevinksy. Cluster algebras I: Foundations. *J. Amer. Math. Soc.*, 15:497–529, 2002. MR1887642 (2003f:16050).

[FZ03] S. Fomin and A. Zelevinksy. Cluster algebras II: Finite type classification. *Inventiones Mathematicae*, 154:63–121, 2003. MR2004457 (2004m:17011).

[FZ07] S. Fomin and A. Zelevinksy. Cluster algebras IV: Coefficients. *Composition Mathematica*, 143(1):112–164, 2007. MR2295199 (2008d:16049).

[Kel05] B. Keller. On triangulated orbit categories. *Documenta Mathematica*, 10:551–581, 2005. MR2184464 (2007c:18006).

[Rin84] C.M. Ringel. Tame algebras and integral quadratic forms. *Lecture Notes in Mathematics*, 1099:1–376, 1984. MR0774589 (87f:16027).

[SS07] D. Simson and A. Skowroński. *Elements of the Representation Theory of Associative Algebras, Volume 2: Tubes and Concealed Algebras of Euclidean type*, volume 71 of *London Mathematical Society Student Texts*. Cambridge University Press, 2007.

[SZ04] P. Sherman and A. Zelevinsky. Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. *Mosc. Math. J.*, 4:947–974, 2004. MR2124174 (2006c:16052).

[YZ08] SW. Yang and A. Zelevinsky. Cluster algebras of finite type via Coxeter elements and principal minors. *Transform. Groups*, 13(3-4):855–895, 2008. MR2452619.

Université de Sherbrooke, 2500, Boul. de l'Université, J1K 2R1, Sherbrooke, QC, Canada.

E-mail address: gregoire.dupont@usherbrooke.ca

URL: http://pages.usherbrooke.ca/gdupont2