On the inadmissibility of various estimators of normal quantiles and on applications to two-sample problems with additional information

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Summary

We consider situations where a normal quantile $\mu + \eta\sigma$ is to be estimated other than the one where scale invariant squared error loss is used with unrestricted values of the population mean and standard deviation μ and σ, and for which Zidek (1971) established the inadmissibility of the MRE estimator for $\eta \neq 0$. Indeed, we explore: (i) the impact of the loss with the study of scale invariant absolute value loss, and (ii) situations where there is a parameter space restriction of a lower bounded mean μ. We establish

(i) the inadmissibility of the MRE estimator of $\mu + \eta\sigma$; $\eta \neq 0$; under scale invariant absolute value loss;

(ii) the inadmissibility of the Generalized Bayes estimator of $\mu + \eta\sigma$; $\eta \neq 0$; under scale invariant squared error loss, associated with the prior measure $1_{[0,\infty)}(\mu) 1_{(0,\infty)}(\sigma)$ which represents the truncation of the usual non-informative prior measure onto the restricted parameter space.

Both of these results are obtained through a conditional risk analysis and may be viewed as extensions of Zidek's result. Finally, we expand on further applications to two-sample problems under the presence of the additional information of ordered means.

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1. Introduction

Many practical situations and fields of study such as reliability, life testing, mortality date, insurance, and education, require efficient statistical methods for drawing inference upon percentiles or quantiles. For instance, Chakraborti and Li (2007) survey various interval estimation methods, while many papers have been dedicated to practical aspects of such problems, as described for instance in Keating and Tripathi (1985). In this regard, consider a random sample from a normal population with unknown mean and standard deviation μ and σ, and the problem of estimating a quantile $\mu+\eta\sigma$, with η known. A very interesting decision-theoretic result due to Zidek (1971), and expanded upon by Rukhin (1983), is the inadmissibility of the benchmark minimum risk equivariant (MRE) and minimax estimator δ_0 of $\mu+\eta\sigma$, for all $\eta \neq 0$, under scale and location invariant squared error loss

$$(\frac{\delta - \mu - \eta\sigma}{\sigma})^2. \tag{1}$$

This finding also represents a particular instance of an inadmissible Generalized Bayes estimator ($\eta \neq 0$), since δ_0 is indeed Bayes with respect to the non-informative prior measure

$$\pi(\mu,\sigma) = \frac{1}{\sigma} 1_{(-\infty,\infty)}(\mu)\, 1_{(0,\infty)}(\sigma)\,. \tag{2}$$

These findings contrasts as well with the case $\eta = 0$, that is the admissibility of the MRE estimator (i.e., the sample mean) for estimating the median of a normal population, as well as the admissibility of the Bayes estimator for a known variance with respect the flat prior $1_{(0,\infty)}(\mu)$ (Katz, 1961). Observe as well that the admissibility of the MRE estimator for $\eta = 0$ holds for other losses, such as absolute value loss.

It is of intrinsic interest to revisit Zidek's result, with the thought of (**i**) assessing the impact of the loss function and thus potentially gaining a more general understanding if similar results are found to be true, and **(ii)** investigating whether the inadmissibility result persists for other generalized Bayes estimators, such as for priors which incorporate parametric restrictions on (μ, σ). We will focus in **(i)** on scale invariant absolute value loss, which possesses attractive features of its own, and which has been recently considered for MRE estimators and for various models by Keating, Mason and Balakrishnan (2009). For **(ii)**, we will focus on the case of a lower bounded mean. Key findings of this paper include:

(a) the inadmissibility for $\eta \neq 0$ of the MRE (or Bayes with respect to the prior in (2)) estimator under scaled absolute value loss $|\frac{\delta-\mu-\eta\sigma}{\sigma}|$;

(b) the inadmissibility for $\eta > 0$ of the generalized Bayes estimator δ_{π_0} with respect to the prior

$$\pi_0(\mu, \sigma) = \frac{1}{\sigma} 1_{(0,\infty)}(\mu)\, 1_{(0,\infty)}(\sigma)\,, \tag{3}$$

which represents the truncation of the prior π in (2) onto the restricted parameter space. We expand further on the estimation context relative to **(b)** at the beginning of Section 4.

As in Zidek (1971) or Stein (1964), the inadmissibility results stem actually from more general complete class results, and are based on the study of scale invariant estimators and a conditional risk analysis on the maximal invariant (Section 2). Various other technical results and arguments, which break new ground in dealing with L_1 loss and the complex analytical properties of the Bayes estimator δ_{π_0}, are required and of independent interest (e.g., Lemma 2). With respect to other features, the treatment is unified in **(a)** with Zidek's result for L_2 loss (Section 3), while our findings in **(b)** are cast amongst a scarcity of work concerning the estimation of quantiles under a parametric restriction (Section 4). Finally in Section 5, we expand on further implications for two-sample problems with additional information on the means, by making use of a variant of the so-called "rotation" technique introduced in the late 60's by Blumenthal, Cohen and Sackrowitz.

2. Preliminaries and conditional risk analysis

Our results are derived for the canonical form :

$$X \sim N(\mu, \sigma^2),\ S^2 \sim \text{ Gamma}(\frac{n-1}{2}, 2\sigma^2) \text{ independent, } (n \geq 2), \tag{4}$$

where the objective is to estimate the quantile $\mu + \eta\sigma$ (known $\eta \neq 0$) with scale invariant loss

$$\rho\,(\frac{\delta - \mu - \eta\sigma}{\sigma})\,, \tag{5}$$

ρ being nonnegative, absolutely continuous, convex with $\rho(0) = 0$.

Remark 1. *Results derived for the canonical form in (4) apply for independent observables $X_1, \ldots, X_n \sim N(\theta, \sigma^2)$, with sample mean $\bar{X}$, and for estimating a quantile $\theta + \beta\sigma$ of order β by δ' under loss $\rho^*(\frac{\delta' - \theta - \beta\sigma}{\sigma})$. This is achieved by setting $X = \sqrt{n}\,\bar{X}, S^2 = \sum_{i=1}^n (X_i - \bar{X})^2, \mu = \sqrt{n}\,\theta, \eta = \sqrt{n}\,\beta, \delta = \sqrt{n}\,\delta'$, and $\rho(z) = \rho^*(z/\sqrt{n})$.*

Remark 2. *In cases, where the parameter space is unrestricted (i.e., $\mu \in \Re, \sigma > 0$), we may limit the analysis without loss of generality to cases where $\eta > 0$. This is so since for the model: $X' \sim N(\mu', \sigma^2)$, $S^2 \sim Gamma(\frac{n-1}{2}, 2\sigma^2)$ independent, a loss $\rho_0(\frac{\delta'(X',S) - \mu' - \eta\sigma}{\sigma})$ incurred by $\delta'(X', S)$ for estimating $\mu' + \eta\sigma; \eta < 0$; matches the loss $\rho(\frac{\delta(X,S) - \mu - \eta\sigma}{\sigma})$ for the canonical form in (4) with $X = -X', \mu = -\mu', \eta = -\eta', \delta = -\delta'$, and $\rho(z) = \rho_0(-z)$ for all $z \in \Re$. However, for positive η and cases with a positivity constraint $\mu \geq 0$, the equivalent problem for negative η is as above but with a negativity constraint $\mu' \leq 0$.*

Following Zidek (1971), our conditional risk analysis proceeds as follows. In (4), set $Y = \frac{X}{S}, V = \frac{S}{\sigma}$, and $\lambda = \frac{\mu}{\sigma}$. Next, consider the class of scale invariant estimators of the form $\delta_\psi(X, S) = S\psi(Y)$, of which the MRE estimator belongs to with $\delta_{mre}(X, S) = S(Y + c_{\rho,n}\eta)$ (see (12) for a representation of $c_{\rho,n}$). Then, decompose the risk of δ_ψ as:

$$R((\mu,\sigma),\delta_\psi) = E[\rho(\frac{S\,\psi(Y)\,-\,\mu\,-\,\eta\sigma}{\sigma})] = E[r(\lambda,\psi(Y))], \tag{6}$$

with

$$r(\lambda,\psi(y)) = E[\rho(V\,\psi(y)\,-\,\lambda\,-\,\eta)\,|Y=y], \tag{7}$$

being the conditional risk of δ_ψ (given $Y=y$) and depending on the parameters (μ,σ) only through their ratio λ. Hence, as synthesized by the next lemma and corollary, dominance or complete class results are potentially available by comparisons of the conditional risks $r(\lambda,\psi(y))$ only.

Lemma 1. *(**a**) For fixed λ, there exists an optimal choice $\psi^*_\lambda(y)$ which minimizes in $\psi(y)$ the conditional risk in (7), and which is given by*

$$E[\,\rho'(W\,\psi^*_\lambda(y)-\lambda-\eta)\,]\;=0\,, \tag{8}$$

with the distribution of W depending on λ and y, with density proportional to

$$w^n\,e^{-(\frac{1+y^2}{2})\,(w-\frac{\lambda y}{1+y^2})^2}\,1_{(0,\infty)}(w)\,. \tag{9}$$

*(**b**) For fixed λ, δ_{ψ_1} dominates δ_{ψ_2} ($\delta_{\psi_1}\neq\delta_{\psi_2}$) whenever for all $y\in\Re$: $\delta_{\psi_2}(y)\geq\delta_{\psi_1}(y)\geq\delta_{\psi^*_\lambda}(y)$ or $\delta_{\psi_2}(y)\leq\delta_{\psi_1}(y)\leq\delta_{\psi^*_\lambda}(y)$, with strict inequality between $\psi_1(y)$ and $\psi_2(y)$ on a set of positive Lebesgue measure.*

Proof. **(a)** First, observe that $r(\lambda,\psi(y))$ is convex in $\psi(y)$ since ρ is convex. Therefore, by differentiation, $\psi^*_\lambda(y)$ will satisfy

$$E[\,V\rho'(V\,\psi^*_\lambda(y)-\lambda-\eta)\,|Y=y]\;=0\,. \tag{10}$$

A simple evaluation yields a conditional density $f_{V|Y=y}$ as in (9) with n replaced by $n-1$. Finally, defining W as a random variable with density proportional to $w f_{V|Y=y}(w)$ in (10) leads to (8) and

(9).

(b) This is a consequence of part (a) and the convexity of $r(\lambda, \psi(y))$ as a function of $\psi(y)$. $\square$

A critical insight is that $\psi^*_\lambda(y)$ may be, as a function of λ, appropriately bounded, and thus lead to complete class results as implied by part **(b)** of the previous lemma. Indeed, in the cases under study here, $\psi^*_\lambda(y)$ will be shown to be upper bounded (e.g., Theorems 1 and 2), and the inadmissibility results will concern estimators δ_ψ with ψ too large on regions of positive Lebesgue measure.

Corollary 1. *Suppose there exists an upper envelope $\bar{\psi}(y)$ on a region D for $\psi^*_\lambda(y)$ such, that for all $y \in D$, $\bar{\psi}(y) \geq \psi^*_\lambda(y)$ for all $(\mu, \sigma) \in \Theta$. Suppose further that δ_ψ is a scale invariant estimator and C is a subset of D such that $\nu(C) > 0$, where $C = \{y \in D : \psi(y) > \bar{\psi}(y)\}$ and ν is Lebesgue measure. Then δ_ψ is inadmissible for estimating $\mu + \eta\sigma$; $(\mu, \sigma) \in \Theta$; and dominated by $\delta_{\psi'}$ with $\psi'(y) = \bar{\psi}(y)\, I_C(y) + \psi(y)\, I_{C'}(y)$, $y \in \Re$.*

Proof. Follows from part **(b)** of Lemma 1 with $\psi_2 \equiv \psi$ and $\psi_1 \equiv \psi'$, since we have for $y \in C$: $\delta_\psi(y) > \delta_{\psi'}(y) = \delta_{\bar{\psi}}(y) \geq \delta_{\psi^*_\lambda}(y)$ for all $(\mu, \sigma) \in \Theta$. $\square$

Remark 3. *In applying Corollary 1 with continuous ψ and $\bar{\psi}$, it suffices to determine a singleton y_0 such that $\psi(y_0) > \bar{\psi}(y_0)$, in which case C can be taken as a small enough neighbourhood of y_0. We also point out that we are not necessarily concerned here with admissible improvements as Corollary 1 $\delta'_\psi s$ will themselves be inadmissible as soon as the envelopes $\bar{\psi}(y)$ do not coincide with* $\sup_{(\mu,\sigma)\in\Theta} \{\psi^*_\lambda(y)\}$ *for some y's, can be seen by a further application of Corollary 1.*

We next collect some useful properties of the distribution of W, as defined in (9).

Lemma 2. *Let $a = \frac{\lambda y}{\sqrt{1+y^2}}$ and $c(\lambda, y, n) = \frac{1}{\sqrt{1+y^2}}\left(\frac{a}{2} + \sqrt{\frac{a^2}{4} + n}\right)$. We have that*

(a) $E(W)$ is a strictly increasing function of n;

(b) $c(\lambda, y, n) < E(W) < c(\lambda, y, n+1)$*;*

(c) Median$(W) \geq c(\lambda, y, n)$*.*

Proof. See Appendix.

Remark 4. *It is easy to verify that the density of* W *is unimodal with mode at* $c(\lambda, y, n)$*. Consequently, the above inequalities may also be interpreted as mean-mode and median-mode inequalities. The former (i.e.,* $E(W) \geq c(\lambda, y, n)$*) is due to Zidek, 1971, a proof of which is integrated below for sake of completeness.*

3. Inadmissibility of the MRE estimator under absolute value invariant error loss

We now turn to a representation and useful property of the MRE estimator under location-scale changes for absolute value invariant loss. The lower bound given in part (b) of the following lemma consists of a preliminary step towards applying the strategy outlined above in Corollary 1.

Lemma 3. **(a)** *For model (4), the MRE estimator of* $\mu + \eta\sigma$ $(\eta > 0)$ *under loss* $|\frac{d-\mu-\eta\sigma}{\sigma}|$ *is given by* $\delta_{\mathrm{MRE}}(X, S) = X + c_{1,n}\eta S$*, where* $c_{1,n}$ *is uniquely defined by the equation*

$$E_{0,1}[S\{1 - 2\Phi(\eta(1 - c_{1,n}S)\}] = 0\,, \tag{11}$$

Φ *being the standard normal cdf.*

(b) *Moreover, we have* $c_{1,n} > \frac{1}{\sqrt{n}}$ *for all* $\eta > 0$*,* $n \geq 2$*.*

Proof. **(a)** It is well understood that equivariant estimators here are of the form $X + c\eta S$ and have constant risk for losses as in (5) (e.g., Rukhin, 1983). It follows that the optimal choice of c

minimizes $E_{0,1}[\rho(X + c\eta S - \eta)]$ in c, and is uniquely given by

$$E_{0,1}[S\rho'(X + c\eta S - \eta)] = 0\,, \tag{12}$$

given that ρ is convex. For $\rho(z) = |z|$ in (5), this becomes $E_{0,1}[SE_{0,1}[\,\mathrm{sgn}(X + c\eta S - \eta)|S\,]] = 0$, which yields (11) given the independence of X and S.

(b) We have from (11), since $\Phi(\cdot)$ is an increasing function and $\eta > 0$, that

$$\begin{aligned} c_{1,n} > \frac{1}{\sqrt{n}} \quad &\Leftrightarrow \quad E_{0,1}[\frac{S}{\sqrt{n}}\{1 - 2\Phi(\eta(1 - \frac{S}{\sqrt{n}})\}] < 0 \\ &\Leftrightarrow \quad \int_0^\infty u^{n-1}\{1 - 2\Phi(\eta(1-u))\}\, e^{-\frac{nu^2}{2}}\, du \,<\, 0 \\ &\Leftrightarrow \quad h(\eta) > 1/2 \text{ for all } \eta > 0, \end{aligned}$$

with $h(\eta) = E[\Phi(\eta(1-U))]$, and U having density proportional to $u^{n-1}e^{-\frac{nu^2}{2}}\,1_{(0,\infty)}(u)$. We proceed by showing that $h(\cdot)$ is strictly increasing on $(0, \infty)$ which will suffice for establishing the result given that $h(0) = 1/2$. A direct computation yields

$$h^{'}(\eta) \propto \int_0^\infty (1-u)\, e^{-\frac{\eta^2(1-u)^2}{2}}\, u^{n-1}\; e^{-\frac{nu^2}{2}}\, du,$$

so that $h^{'}(\eta) > 0$ if and only if

$$\frac{\int_0^\infty u^n \;\; e^{\frac{-(n+\eta^2)}{2}\,(u - \frac{\eta^2}{n+\eta^2})^2}\, du}{\int_0^\infty u^{n-1}\, e^{\frac{-(n+\eta^2)}{2}\,(u-\frac{\eta^2}{n+\eta^2})^2}\, du} < 1\,. \tag{13}$$

Finally, it follows from a re-parametrization of (9) and part (b) of Lemma 2 that the lhs of (13) is bounded above by $c(\frac{\eta^2}{\sqrt{n+\eta^2-1}}, \sqrt{n+\eta^2-1}, n) = 1$, which establishes the result. $\square$

Theorem 1. *For model (4) and for estimating the quantile $\mu + \eta\sigma$ ($\eta > 0$) under loss $|\frac{d-\mu-\eta\sigma}{\sigma}|$, the estimator δ_{mre} is inadmissible and dominated by $\delta_{\psi'}(X,S) = \psi'(\frac{X}{S})S$ with $\psi'(y) = \{y + \min(\frac{1}{y} + \frac{\eta^2 y}{4n}, c_{1,n}\eta)\}1_{(0,\infty)}(y) + \{y + c_{1,n}\eta\}\,1_{(-\infty,0]}(y)$.*

Proof. We apply Corollary 1 with $\psi(y) = y + c_{1,n}\,\eta$, $D = (0,\infty)$, and $\bar{\psi}(y) = (y + \frac{1}{y} + \frac{\eta^2 y}{4n})\,1_{(0,\infty)}(y)$, in which case $C = \{y > 0 : \psi(y) > \bar{\psi}(y)\} = \{y : |y - \frac{2nc_{1,n}}{\eta}| < \frac{2n}{\eta}\sqrt{c_{1,n}^2 - \frac{1}{n}}\} \neq \emptyset$ by virtue of part

(b) of Lemma 3. There remains to establish that $\sup_{(\mu,\sigma)\in\Theta}\{\,\psi^*_\lambda(y)\,\} \leq \bar{\psi}(y)$ for all $y > 0$. From Lemmas 1 and 2, we have for all $y > 0$:

$$\psi^*_\lambda(y) = \frac{\lambda+\eta}{\text{Median}(W)} \leq \frac{\lambda+\eta}{c(\lambda, y, n)}\,. \tag{14}$$

Now write $\frac{1}{c(\lambda,y,n)} = \frac{y}{\sqrt{1+y^2}}(\frac{y+\frac{1}{y}}{n})\,(\sqrt{\frac{a^2}{4}+n}-\frac{a}{2})$, with $a = \frac{\lambda y}{\sqrt{1+y^2}}$ (as in Lemma 2). Setting $b = \frac{\eta y}{\sqrt{1+y^2}}$ and $a_0(b) = \frac{1}{2b}(4n - b^2)$, we obtain (following Zidek, 1971) for all $y > 0$

$$\begin{aligned}
\sup_{(\mu,\sigma)\in\Theta} \psi^*_\lambda(y) &\leq \sup_{\lambda\in\Re} \frac{\lambda+\eta}{c(\lambda, y, n)} \\
&= \sup_{\lambda\geq-\eta} \frac{\lambda+\eta}{c(\lambda, y, n)} \\
&= \frac{1}{n}\,(y+\frac{1}{y})\sup_{a\geq-b}\{(a+b)\,(\sqrt{\frac{a^2}{4}+n}-\frac{a}{2})\} \\
&= \frac{1}{n}\,(y+\frac{1}{y})\{(a_0(b)+b)\,(\sqrt{\frac{a_0^2(b)}{4}+n}-\frac{a_0(b)}{2})\} \\
&= \bar{\psi}(y)\,. \qquad \Box
\end{aligned}$$

Much of our motivation for establishing and disseminating Theorem 1 rests not only with the technical challenges and interesting intermediate results required, but also with the common features with Zidek's result for squared error invariant loss. Expanding on this, we point out that the MRE estimator under squared error invariant loss $(\frac{d-\mu-\eta\sigma}{\sigma})^2$, which may be obtained from (12), is given by

$$X + c_{2,n}\eta S \;\text{ with }\; c_{2,n} = \frac{E_{0,1}(S)}{E_{0,1}(S^2)} = \frac{\Gamma(\frac{n}{2})}{\sqrt{2}\,\Gamma(\frac{n+1}{2})}. \tag{15}$$

Furthermore, one can verify without too much trouble that $c_{2,n}c_{2,n+1} = \frac{1}{n}$, that $c_{2,n}$ decreases in n, and that consequently $\frac{1}{\sqrt{n}} < c_{2,n} < \frac{1}{\sqrt{n-1}}$ for all $n \geq 2$. Consequently, the given lower bounds for $c_{1,n}$ and $c_{2,n}$ coincide even though $c_{1,n}$ depends on η and $c_{2,n}$ does not. And, as established by Zidek (1971), Theorem 1 holds as stated if we replace the L_1 loss $(|\frac{d-\mu-\eta\sigma}{\sigma}|)$ by the L_2 loss $(\frac{d-\mu-\eta\sigma}{\sigma})^2$ and $c_{1,n}$ by $c_{2,n}$. Hence, our proof for L_1 loss is parallel and unified with the L_2 loss proof, notwithstanding the derivation of the key properties in Lemmas 2 and 3; i.e., Median$(W) \geq c(\lambda, y, n)$ and $c_{1,n} > \frac{1}{\sqrt{n}}$ as opposed to their L_2 loss analogs $E(W) > c(\lambda, y, n)$ and $c_{2,n} > \frac{1}{\sqrt{n}}$. Generalizations of the inadmissibility of the MRE estimator to other losses ρ in (5) remains of interest and an open question.

4. Estimating a quantile in presence of a lower bounded mean

4.1. Inadmissibility of a generalized Bayes estimator

We consider now our quantile estimation problem as defined in (4) and (5), but with a lower bound constraint on μ, say $\mu \geq 0$ without loss of generality. We establish the inadmissibility under squared error invariant loss of the generalized Bayes estimator δ_{π_0} of $\mu + \eta\sigma$ with $\eta > 0$, where the prior measure is $\pi_0(\mu, \sigma) = \frac{1}{\sigma} 1_{(0,\infty)}(\mu)\, 1_{(0,\infty)}(\sigma)$. Representing the truncation onto the restricted parameter space of the non-informative prior $\pi(\mu, \sigma) = \frac{1}{\sigma} 1_{(-\infty,\infty)}(\mu)\, 1_{(0,\infty)}(\sigma)$, the choice or study of π_0 is particularly interesting. On one hand, the Bayes estimator δ_π coincides with δ_{mre} and is thus inadmissible under both losses L_1 and L_2 following the results of Section 3 and those of Zidek (1971). On the other hand, truncations such as π_0 have been studied for their efficiency before. For instance, Katz (1961) considered estimating a non negative normal mean μ with known variance under loss $(d - \mu)^2$, for which the Bayes estimator with respect to the truncation $1_{(0,\infty)}(\mu)$ is both minimax and admissible. Further results for other location models and other losses, such as those obtained by Farrell (1965) and Marchand and Strawderman (2005) among others, go in the same direction, while Bayesian HPD credible intervals based on such truncations of non-informative priors have been shown to have satisfactory frequentist coverage properties (e.g., Marchand and Strawderman, 2006).

The next lemma, whose proof is relegated to the Appendix, pertains to the Bayes estimator δ_{π_0}.

Lemma 4. *Let* $\beta_\rho(w,z) = E[V \int_{-\infty}^{Vw} \rho'(u-\eta+V(z-w))\phi(u)du\,]$ *for* $z,w \in \Re$, *where* ϕ *stands for the standard normal pdf and* $V =^d \frac{S}{\sigma}$ *in (4).*

(a) For model (4) and loss (5), the Bayes estimator δ_{π_0} *is scale invariant, and given by* $\delta_{\pi_0}(X,S) = S\psi_{\pi_0}(\frac{X}{S})$, *with* $\beta_\rho(y,\psi_{\pi_0}(y)) = 0$ *for all* $y \in \Re$;

(b) For scaled invariant squared error loss, δ_{π_0} *may be expressed as in part (a) with*

$$\psi_{\pi_0}(y) = y + \frac{A_n(y) + \eta c_{2,n+1} B_n(y)}{B_{n+1}(y)}, \tag{16}$$

where $A_n(y) = \frac{1}{n}(1+y^2)^{-\frac{n}{2}}$, $B_n(y) = \int_{-\infty}^{y}(1+x^2)^{-(\frac{n+1}{2})}dx$; $y \in \Re, n \geq 2$; *and* $c_{2,n+1} = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{2}\Gamma(\frac{n+2}{2})}$.

(c) Furthermore, we have $2\,c_{2,n+1}B_n(y) \geq (c_{2,n} + c_{2,n+1})\,B_{n+1}(y)$ *for all* $n \geq 2, y > 0$.

We now establish the inadmissibility of the Bayes estimator δ_{π_0} by making use of the above properties and Corollary 1.

Theorem 2. *For model (4) and for estimating the quantile* $\mu + \eta\sigma$ $(\eta > 0)$ *under the constraint* $\mu \geq 0$ *and loss* $\frac{(d-\mu-\eta\sigma)^2}{\sigma^2}$, *the Bayes estimator* δ_{π_0} *given in Lemma 4 is inadmissible and dominated by* $\delta_{\psi''}(X,S) = \psi''(\frac{X}{S})S$, *with* $\psi''(y) = \psi_{\pi_0}(y)\,I_{(-\infty,0]}(y) + min(\psi_{\pi_0}(y), \bar{\psi}(y))\,I_{(0,\infty)}(y)$, *with* $\bar{\psi}(y) = y + \frac{1}{y} + \frac{\eta^2 y}{4n}; y > 0$.

Proof. With the knowledge that $\sup_{\mu\geq 0,\sigma>0} \psi^*_\lambda(y) \leq \bar{\psi}(y)$ for all $y > 0$ (i.e., Zidek, 1971, or as pointed out above in the last paragraph of Section 3), we follow Corollary 1 and Remark 3, where it suffices to show that the set $\{y > 0 : \psi_{\pi_0}(y) > \bar{\psi}(y)\}$ has positive Lebesgue measure, or again that there exists a positive y_0 such that $\psi_{\pi_0}(y_0) > \bar{\psi}(y_0)$. We take $y_0 = \frac{2nc_{2,n}}{\eta}$ in which case $\bar{\psi}(y_0) = y_0 + \frac{1}{y_0} + \frac{\eta^2 y_0}{4n} = y_0 + \frac{\eta}{2n\, c_{2,n}} + \frac{\eta\, c_{2,n}}{2} = y_0 + \frac{\eta}{2}(c_{2,n} + c_{2,n+1})$, since $nc_{2,n}c_{2,n+1} = 1$. Hence, we obtain from (16)

$$\begin{aligned}
\psi_{\pi_0}(y_0) - \bar{\psi}(y_0) &= \{y_0 + \frac{A_n(y_0) + \eta c_{2,n+1} B_n(y_0)}{B_{n+1}(y_0)}\} - \{y_0 + \frac{\eta}{2}(c_{2,n} + c_{2,n+1})\} \\
&= \frac{1}{B_{n+1}(y_0)}\{A_n(y_0) + \eta c_{2,n+1} B_n(y_0) - \frac{\eta}{2}(c_{2,n} + c_{2,n+1})\, B_{n+1}(y_0)\} \\
&> 0\,,
\end{aligned}$$

by virtue of part (c) of Lemma 4 and since $A_n(\cdot)$ and $B_{n+1}(\cdot)$ are positive valued functions. □

Remark 5. *The inadmissibility of the estimators δ_{π_0} (Theorem 2) and δ_{MRE} (Zidek, 1971), are obtained as an application of Corollary 1 by showing the estimators expand "too much". Moreover, since δ_{π_0} is Bayes with respect to the truncated version $\pi_0(\cdot,\cdot)$ of $\pi(\cdot,\cdot)$ onto the restricted parameter space, one might anticipate that δ_{π_0} expands further on δ_{mre}, which would provide an easy route to establishing Theorem 2. However, this is not necessarily the case. Indeed, for scale invariant squared error loss, it follows from (16) and (15) that $sgn(\psi_{\pi_0}(y) - \psi_{mre}(y)) = sgn(h(y))$, with $h(y) = A_n(y) + \eta c_{2,n+1} B_n(y) - \eta c_{2,n} B_{n+1}(y)$. Furthermore, one verifies that*

$$A_n'(y) = -y B_{n+1}'(y) = -\frac{y}{\sqrt{1+y^2}} B_n'(y),$$

from which we infer that

$$h'(y) = \sqrt{1+y^2}\, B_{n+1}'(y)\{\eta c_{2,n+1} - \frac{\eta c_{2,n}}{\sqrt{1+y^2}} - \frac{y}{\sqrt{1+y^2}}\}.$$

Finally, for $\eta > \frac{1}{c_{2,n+1}}$, we see that $h'(y)$ is positive for large enough y since $B_{n+1}'(\cdot)$ is positive and $\lim_{y\to\infty}\{\eta c_{2,n+1} - \frac{\eta c_{2,n}}{\sqrt{1+y^2}} - \frac{y}{\sqrt{1+y^2}}\} = \eta c_{2,n+1} - 1 > 0$, which implies that $\psi_{\pi_0}(y) < \psi_{\text{MRE}}(y)$ for large enough y given that $\lim_{y\to\infty} h(y) = 0$. [2]

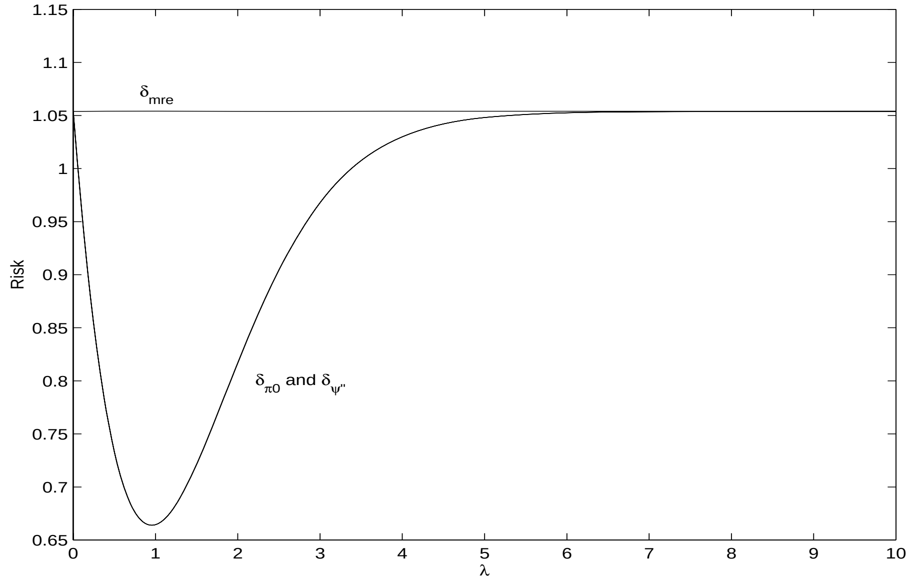
[2] However, it can be verified from the above expressions that $\psi_{\pi_0}(y) \geq \psi_{\text{MRE}}(y)$ for all $y \in \Re$ whenever $\eta \leq \frac{1}{c_{2,n+1}}$.

4.2. Further remarks and numerical evaluations

In the presence of the constraint $\mu \geq 0$, a minimum risk equivariant estimator $\delta_{mre}(X,S) = X + \eta c_\rho S$ of $\mu + \eta\sigma$ is clearly inefficient when $\eta \geq 0$ and improved upon by its truncation onto $[0,\infty)$ for any loss in (5) since $\mu + \eta\sigma \geq 0$, while $P_{\mu,\sigma}(\delta_{mre}(X,S) < 0) > 0$ for all $\mu \geq 0$, $\sigma > 0$. However, as shown recently by Marchand and Strawderman (2010), δ_{mre} remains minimax even in the presence of the constraint $\mu \geq 0$, and its constant risk equals the minimax risk for quite general ρ in (5) subject to risk finiteness. Therefore such minimum risk equivariant estimators remain useful benchmarks, and the determination of dominating estimators, which remains to be studied, is worthwhile and will necessarily produce minimax estimators. Another motivation for the search of efficient estimators in the presence of a lower bound on the mean resides in further applications for two sample additional information problems as presented and expanded upon in the next section.

Although plausible and supported by some numerical evidence, part of which is illustrated in Figure 1, we do not know if δ_{π_0} is such a minimax dominating estimator, despite being inadmissible itself for $\eta > 0$ under squared error ρ, except in the case $\eta = 0$ and scale invariant squared error ρ where Kubokawa (2004) obtained a class of dominating (minimax) estimators which includes δ_{π_0}. We conclude this section with Figure 1, which represents the risk functions of the δ_{mre}, the generalized Bayes estimator δ_{π_0} and Lemma 2's estimator $\delta_{\psi''}$ for $n = 10, \eta = 1$, as a function of $\lambda = \frac{\mu}{\sigma} \geq 0$. With other choices of $n, \eta > 0$ leading to similar results, it is very interesting to see the important gains in risk provided by δ_{π_0} in comparison to δ_{mre}, and the minuscule gains in risk for $\delta_{\psi''}$ as opposed to δ_{π_0}.

Figure 1 Risks (n=10,η=1.0) as a function of λ = μ/σ ≥ 0.



5. Estimation of quantiles in the presence of additional information on the means

We describe here a correspondence between: **(i)** a two-sample problem with additional information present on the ordering of the means, and **(ii)** the quantile estimation context of Section 4 with a lower bounded mean. The end result will be that any dominance or admissibility result under squared error loss in **(ii)** translates to a companion result in **(i)** and vice-versa.

- Start with a canonical form as in (4) with

$$X_1 \sim N(\mu_1, \sigma^2),\ S_1^2 \sim \text{ Gamma}(\frac{n-1}{2}, 2\sigma^2) \text{ independent, } (n \geq 2),$$

  with the objective of estimating efficiently $\mu_1 + \eta\sigma$, $\eta > 0$, under loss $(\frac{d-\mu_1-\eta\sigma}{\sigma})^2$, and where we already know that the plausible δ_{mre} is an inadmissible estimator.

- Suppose now that a second, independently generated, sample is available with summaries

$$X_2 \sim N(\mu_2, \sigma^2),\ S_2^2 \sim \text{ Gamma}(\frac{m-1}{2}, 2\sigma^2) \text{ independent, } (m \geq 2),$$

and suppose further that the means μ_1 and μ_2 are ordered in such a way that

$$\mu_1 \geq \mu_2 \quad \text{(additional information).} \tag{17}$$

- Clearly, given the homogeneity of the variances[3], more degrees of freedom are available and $X_1 + \eta c_{2,m+n}\sqrt{S_1^2 + S_2^2}$ seems preferable to $X_1 + c_{2,n}S_1$.[4] Indeed, the former dominates the latter as the risk of δ_{mre}, given by $1 + \eta^2(1 - \frac{(\Gamma(\frac{n}{2}))^2}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n+1}{2})})$, can be shown to decrease with n.

- But, one may strive to do much better given the additional information (17) by making use of a variant of the so-called rotation technique, introduced by Blumenthal, Cohen, Sackrowitz in the late 60's, revisited by van Eeden and Zidek in a series of more recent papers, and as further described (with references) by Marchand and Strawderman (2004). The key feature of the technique is a decomposition of the problem in **(i)** into two separate and additive subproblems, one of which corresponds to the problem in **(ii)**. To pursue, set

$$Y_1 = \frac{X_1 - X_2}{2},\ Y_2 = \frac{X_1 + X_2}{2},\ W = \frac{S_1^2 + S_2^2}{2},$$

$$\theta_1 = \frac{\mu_1 - \mu_2}{2},\ \theta_2 = \frac{\mu_1 + \mu_2}{2},\ \text{ and }\ \tau = \frac{\sigma}{\sqrt{2}}.$$

Observe that Y_1, Y_2, and W are independent with $Y_1 \sim N(\theta_1, \tau^2)$, $Y_2 \sim N(\theta_2, \tau^2)$, and $W \sim \text{Gamma}(\frac{n+m-2}{2}, 2\tau^2)$.

We then have the following.

[3]A similar development can be presented in the case where the ratio of variances is known.

[4]Admittedly, the ad hoc substitution $\frac{S_1^2+S_2^2}{2}$ is not necessarily an efficient linear combination estimate of σ^2 (clearly, under squared error loss, it is suboptimal unless $m = n$), but we continue with this nevertheless plausible choice for illustrative and application purposes, as indeed linear combinations $aS_1^2 + (1-a)S_2^2$ for $a \notin \{0, 1/2, 1\}$ are not Gamma distributed.

Lemma 5. *Consider the above data* (Y_1, Y_2, W) *for estimating the quantile* $\mu_1 + \eta\sigma$ *with the information* $\mu_1 \geq \mu_2$*, and loss* $(\frac{d-\mu_1-\eta\sigma}{\sigma})^2$*. Consider also estimators of the form* $\delta_\phi(Y_1, Y_2, W) = Y_2 + \phi(Y_1, W)$*. Then the risk of such an estimator is given by:*

(a) $R((\mu_1, \mu_2, \sigma), \delta_\phi) = \frac{1}{\sigma^2}\{\tau^2 + E[(\phi(Y_1, W) - \theta_1 - \sqrt{2}\,\eta\,\tau)^2]\}$*;*

(b) $\delta_{\phi_1}(Y_1, Y_2, W)$ *dominates* $\delta_{\phi_2}(Y_1, Y_2, W)$ *if and only if* $\phi_1(Y_1, W)$ *dominates* $\phi_2(Y_1, W)$ *as an estimator of the quantile* $\theta_1 + \eta^*\tau$*, with* $\eta^* = \sqrt{2}\,\eta$ *under the constraint* $\theta_1 \geq 0$*, based on the data* (Y_1, W) *as in (4).*

Proof. Part (b) is a consequence of part (a). For part (a), we have

$$\begin{aligned}\sigma^2 R((\mu_1, \mu_2, \sigma), \delta_\phi) &= E[(Y_2 + \phi(Y_1, W) - \mu_1 - \eta\sigma)^2] \\ &= E[\{(Y_2 - \theta_2) + (\phi(Y_1, W) - \theta_1 - \eta\sqrt{2}\,\tau)\}^2] \\ &= \tau^2 \;+\; E[(\phi(Y_1, W) - \theta_1 - \eta\sqrt{2}\,\tau)^2],\end{aligned}$$

given the independence of Y_2 and (Y_1, W), and since $E(Y_2 - \theta_2) = 0$ and $E[Y_2 - \theta_2)^2] = \tau^2$. □

Example 1. *The MRE estimator of* $\mu_1 + \eta\sigma$ *is given by* $X_1 + \eta c_{2,m+n}\sqrt{2W} = \delta_{\phi_{mre}}(Y_1, Y_2, W)$ *with* $\phi_{mre}(Y_1, W) = Y_1 + c_{2,m+n}\eta^*\sqrt{W}$*. The above lemma tells us that any dominating estimator* $\phi_1(Y, W)$ *of* $\phi_{mre}(Y_1, W)$ *for estimating the quantile* $\theta_1 + \eta^*\tau$*, such as its truncation* $\max(0, \phi_{mre}(Y_1, W))$ *for* $\eta \geq 0$*, leads to a corresponding dominating estimator* $\delta_{\phi_1}(Y_1, Y_2, W)$*, such as* $Y_2 + \max(0, \phi_{mre}(Y_1, W))$ *for* $\eta \geq 0$*. Similarly, the estimator* $Y_2 + \delta_{\pi_0}(Y_1, W)$*, where* δ_{π_0} *is Section 4's Bayes estimator of the quantile of order* η^**, is inadmissible for* $\eta > 0$ *and can be improved upon by making use of Lemma 5 and dominating estimators of* δ_{π_0}*.*

Example 2. *The above decomposition also applies for the case of the median (or mean) with* $\eta = 0$*. In this case* X_1 *is the MRE estimator of* μ_1 *(under scale invariant squared error loss), and is of* *course admissible in absence of the second sample. With the additional information $\mu_1 \geq \mu_2$, X_1 is inadmissible and Example 1's class of dominating estimators includes $Y_2 + \delta_{\pi_0}(Y_1, W)$, where δ_{π_0} is given in (16), by virtue of Kubokawa's (2004) finding.*

6. Appendix

Proof of Lemma 2

(a) The result is immediate since the family of densities in (9) possess a monotone increasing likelihood ratio in W, with parameter n.

(b) Set $Z =^d \sqrt{1+y^2}\, W$, so that Z has density proportional to $z^n e^{-\frac{z^2}{2}+az}\, 1_{(0,\infty)}(z)$. Now write $E_n(Z) = \frac{I_{n+1}(a)}{I_n(a)}$, with $I_n(a) = \int_0^\infty z^n e^{-\frac{z^2}{2}+az} dz$, and integrate by parts to obtain the recurrence

$$I_n(a) = \frac{1}{n+1} I_{n+2}(a) - \frac{a}{n+1} I_{n+1}(a),$$

or again,

$$E_{n+1}(Z) = \frac{n+1}{E_n(Z)} + a\,, \tag{18}$$

which is valid for all $n \geq 0$. Pursue by applying twice the result in (a) yielding the inequalities: $E_n^2(Z) - aE_n(Z) - (n+1) < 0$, and $E_{n+1}^2(Z) - aE_{n+1}(Z) - (n+1) > 0$. Finally, since $E_n(Z) > \frac{a}{2}$,[5] the above inequalities imply that:

$$\frac{a}{2} + \sqrt{\frac{a^2}{4} + n} < E_n(Z) < \frac{a}{2} + \sqrt{\frac{a^2}{4} + (n+1)}\,,$$

which is equivalent to, and establishes part (b).

[5] For $y \leq 0$, this is obvious, while for $y > 0$, it is easy to see that $E_n(Z) \geq a > \frac{a}{2}$.

(c) As in Remark 4, the density f_Z of Z has mode at $M = \frac{a}{2} + \sqrt{\frac{a^2}{4} + n}$, and we seek to establish that Median$(Z) \geq M$. A sufficient condition for the above inequality to hold is:

$$r(z) = \frac{f_Z(M-z)}{f_Z(M+z)} \leq 1, \quad \text{for all } \; z \in [0, M). \tag{19}$$

Setting $T(z) = \log r(z) = n \log \frac{M-z}{M+z} + 2(M-a)z$, it is easy to verify that $T(\cdot)$ is concave on $(0, m)$ with $T(0) = 0$, and $T'(0^+) = \frac{-2n}{M} + 2(M-a) = 0$, which establishes that $T(z) \leq 0$ for all $z \in [0, m)$, equivalently (19), and hence the desired result. $\square$

Proof of Lemma 4

Proof. **(a)** Under prior π_0 and loss $\rho(\frac{d-\mu-\eta\sigma}{\sigma})$, $\delta_{\pi_0}(x,s)$ minimizes in δ for all (x,s) the posterior expected loss $E[\rho(\frac{\delta-\mu-\eta\sigma}{\sigma})|(X,S) = (x,s)]$, equal to

$$\int_0^\infty \int_0^\infty \rho(\frac{\delta-\mu-\eta\sigma}{\sigma}) \, \frac{1}{\sigma} \, \phi(\frac{x-\mu}{\sigma}) \, \frac{1}{\sigma} h(\frac{s}{\sigma}) \, \frac{du \, d\sigma}{\sigma},$$

h being the density of V. With the change of variables $(\mu, \sigma) \to (u = \frac{x-\mu}{\sigma}, v = \frac{s}{\sigma})$, the above becomes proportional to

$$\int_0^\infty \int_{-\infty}^{v\frac{x}{s}} \rho(v(\frac{\delta - x}{s}) + u - \eta) \, \phi(u) \, h(v) \, du \, dv \, . \tag{20}$$

Searching to seize the form of δ as a function of (x,s) and as a minimizer of (20), observe that $\frac{1}{s}(\delta(x,s) - x)$ depends on (x,s) only through $y = \frac{x}{s}$, so that δ_{π_0} is indeed scale invariant. The result follows by differentiation of (20) in δ and since ρ is convex.

(b) Solving $\beta_\rho(y, \psi_{\pi_0}(y)) = 0$ for $\rho(y) = y^2$ yields directly

$$\psi_{\pi_0}(y) = y + \frac{E[V\phi(Vy)] + \eta E[V\Phi(Vy)]}{E[V^2\Phi(Vy)]}$$

by collecting terms and since $\int_{-\infty}^t u\phi(u)du = -\phi(t)$ for all $t \in \Re$. The result follows by making use of identities for the terms $E[V\phi(Vy)]$ and $E[V^k\Phi(Vy)]$, given and proven below in Lemma 6, as well as the definitions of $A_n(\cdot)$, $B_n(\cdot)$, and $c_{2,n+1}$. $\square$

(c) First, we have $B_n(y) = B_n(0) + \int_0^y (1+x^2)^{-(\frac{n+1}{2})}\,dx \geq B_n(0) + \int_0^y (1+x^2)^{-(\frac{n+2}{2})}\,dx$, with $B_n(0) = \int_{-\infty}^0 (1+x^2)^{-(\frac{n+1}{2})}\,dx = \sqrt{\frac{\pi}{2}}\, c_{2,n}$. From this, we obtain

$$B_n(y) - B_{n+1}(y) \geq B_n(0) - B_{n+1}(0) = \sqrt{\frac{\pi}{2}}\,(c_{2,n} - c_{2,n+1})\,. \tag{21}$$

As well, notice that $B_{n+1}(y)$ increases in y for $y > 0$ with

$$B_{n+1}(y) \leq \int_{-\infty}^{\infty} (1+x^2)^{-(\frac{n+2}{2})}\,dx \;=\; 2\sqrt{\frac{\pi}{2}}\,c_{2,n+1}. \tag{22}$$

Finally, from (21) and (22), we obtain for all $n \geq 2, y > 0$: $2\,c_{2,n+1}(B_n(y) - B_{n+1}(y)) \geq (c_{2,n} - c_{2,n+1})\,B_{n+1}(y)$ yielding the result. □

Lemma 6. *Let ϕ and Φ represent the pdf and cdf (resp.) of a standard normal distribution and let $V^2 \sim Gamma(\frac{n-1}{2}, 2)$; $n \geq 2$. We have for all $k \geq 0$, $t \in \Re$,*

(a) $E[V^k \phi(Vt)] = \frac{2^{\frac{k-1}{2}}\,\Gamma(\frac{n+k-1}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}\,(1+t^2)^{-(\frac{n+k-1}{2})}$;

(b) $E[V^k \Phi(Vt)] = \frac{\Gamma(\frac{n+k}{2})\,2^{\frac{k}{2}}}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}\,\int_{-\infty}^t (1+x^2)^{-(\frac{n+k}{2})}\,dx$, *for all* $k \geq 0$.

Proof. With the density of V given by $\frac{v^{n-2}\,e^{-v^2/2}}{\Gamma(\frac{n-1}{2})2^{(n-3)/2}} I_{(0,\infty)}(v)$, part (a) follows directly with the identity $\int_0^\infty v^\alpha\, e^{-(v^2/2\beta)}\,dv = \Gamma(\frac{\alpha+1}{2})\,2^{\frac{\alpha-1}{2}}\,\beta^{\frac{\alpha+1}{2}}$; $\alpha \geq 0, \beta > 0$. For part (b), observe that $\frac{\partial}{\partial t} E[V^k \Phi(Vt)] = E[V^{k+1}\phi(Vt)]$, which implies that $E[V^k \Phi(Vt)] = \int_{-\infty}^t E[V^{k+1}\phi(Vx)]\,dx + c$. Since $E(V^k\Phi(Vt)) \to 0$ as $t \to -\infty$, we obtain $c = 0$ and the stated result. □

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