On the bayesianity of maximum likelihood estimators of restricted location parameters under absolute value error loss

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ABSTRACT

We investigate the potential bayesianity of maximum likelihood estimators (mle), under absolute value error loss, for estimating the location parameter θ of symmetric and unimodal density functions in the presence of **(i)** a lower (or upper) bounded constraint, and **(ii)** an interval constraint, for θ. With these problems being expressed in terms of integral equations, we establish for logconcave densities: the generalized bayesianity of the mle in **(i)**; and the proper bayesianity and admissibility of the mle in **(ii)** which extends the normal model result of Iwasa and Moritani. In **(i)**, a key feature concerns a correspondence with a Riemann-Hilbert problem, while in **(ii)** we use Fredholm's technique and a contraction mapping argument. We demonstrate that logconcavity is a critical condition with sufficient conditions for non-Bayesianity and, accordingly, with a class of counterexamples. Note that the bayesianity of the mle under absolute value loss in the restricted location parameter case is in marked counterdistinction to that under quadratic loss, where, typically, a generalized Bayes estimator must be a smooth function. Finally, various other remarks, illustrations and numerical evaluations are provided.

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1. Introduction

Statistical inference for restricted parameter spaces leads to a surprising number of challenging problems (see Marchand and Strawderman, 2004; or van Eeden, 2006 for recent reviews). A simple case concerns the case of a positive normal mean θ, and its estimation based on an observable $X \sim N(\theta, 1)$. It has long been known that the plausible maximum likelihood solution given by $\delta_{\text{mle}}(x) = \max(0, x)$ is not necessarily adequate as: **(i)** it is not a sufficient statistic, and **(ii)** it is inadmissible under squared error loss $(d - \theta)^2$ (Sacks, 1963). Observe that an understanding of **(ii)** does not necessitate the derivation of dominating estimators, as those given by Shao and Strawderman (1996) much later than Sacks, but rather stems from the requirement that admissible estimators here be Generalized Bayes. The non-bayesianity under squared error loss follows from the absence of a solution in π, supported on $[0, \infty)$, of the functional equation

$$\delta_{\text{mle}}(x) = E_\pi(\theta|x)\,,\ \text{for all } x \in \mathbb{R}. \tag{1}$$

Furthermore, it is instructive to recall why (1) does not admit a solution in π; with cases $x \in (-\infty, 0]$ forcing π to be degenerate at 0, and cases where $x \in (0, \infty)$ (i.e., $\delta_\pi(x) = x > 0$) not satisfiable with such a degenerate π.

Now, consider the same situation but with absolute value error loss, given by

$$L(\theta, d) = |d - \theta|\,, \tag{2}$$

and which leads to the functional equation in π; π being a prior supported on $[0, \infty)$;

$$\delta_{\text{mle}}(x) = \text{Median}_\pi(\theta|x)\,,\ \text{for all } x \in \mathbb{R}. \tag{3}$$

Here, in contrast, the above argument fails and we have no indication whatsoever that solutions of (3) are non-existent, although candidate solutions in π will necessarily have to have a point mass on 0. Moreover, if there were a solution to (3), it would vividly illustrate the opposition between the losses as far as the decision theoretic properties of $\delta_{\text{mle}}(X)$ are concerned. This paper deals with such a contrast, and the interesting functional integral equations that arise in connection with (3). For location models with symmetric, unimodal, positive, and logconcave densities, which includes the normal case above, we establish that (3) possesses a solution in π. Furthermore, we provide an analogous development for the doubly-bounded case (i.e., $\theta \in [0, m]$ above), again for symmetric, unimodal, and logconcave densities, showing that $\delta_{\text{mle}}(X)$ is unique (proper) Bayes, and hence also admissible. The latter finding generalizes the normal case discovery of Iwasa and Moritani (1997). Additionally, various other considerations are addressed, such as: **(a)** sufficient conditions for non-Bayesianity along with examples, **(b)** cases with nuisance parameters, and **(c)** applications to certain types of truncated models.

Section 2 expands on functional equation (3) and reviews key features of the powerful Riemann-Hilbert technique for solving integral equations. Further technical elements are also given in Section 3 and 4. In Section 3, we separate the lower-bounded and doubly-bounded cases. For the lower-bounded case and for densities which are unimodal, symmetric, positive, and logconcave, we arrive at a correspondence with a Riemann-Hilbert problem (Theorem 1) which allows a solution to (3) in π. Along with additional arguments (Lemma 9) aimed at showing that the solution is feasible (i.e., real-valued and nonnegative), we conclude that the $\delta_{\text{mle}}(X)$ is a Generalized Bayes estimator under absolute value error loss (Corollary 1). For the doubly-bounded case, similar developments are given under the same model assumptions, but we rely on Fredholm's technique, (as did Iwasa and Moritani), and a contraction mapping argument, (which Iwasa and Moritani did not use and which simplifies the proof that the solution is nonnegative and hence can be a valid prior). Here the bayesianity of $\delta_{\text{mle}}(X)$ will imply its admissibility, thus extending the normal model results of Iwasa and Moritani.

Finally in Section 4, we illustrate the results with analytical and numerical evaluations of π in (3) for double-exponential and normal models. The numerical evaluations take advantage of the fact that the

Riemann-Hilbert approach leads to efficient and convenient methods for generating approximate solutions.

2. Definitions and preliminary results

We work throughout with the model:

$$X \sim f_0(x-\theta), \text{ with } f_0 \text{ a known unimodal, positive, absolutely continuous, and symmetric density about } 0. \quad (4)$$

We investigate the potential bayesianity of $\delta_{\text{mle}}(X)$ as an estimator of θ for cases where: **(i)** θ is lower (or upper)-bounded; and **(ii)** θ is restricted to an interval. Without loss of generality since we can always translate and change signs, it suffices to consider only the cases: **(i)** $\theta \geq 0$; and **(ii)** $\theta \in [0, m]$, m known; in which cases $\delta_{mle}(x) = \max(0, x)$ and $\delta_{\text{mle}}(x) = \min(x, m) I_{(0,\infty)}(x)$ respectively. The positive answers provided in this work arise for cases where f_0 is logconcave, or equivalently for families of densities in (4) which possess an (increasing) monotone likelihood ratio (mlr) in X (i.e., $\frac{f_0(x-\theta_2)}{f_0(x-\theta_1)}$ increasing in x for all (θ_1, θ_2) such that $\theta_2 > \theta_1$). Common examples include normal, doule-exponential or Laplace, logistic, hyperbolic secant, and exponential power densities $f_0(y) \propto e^{-|y|^\beta} I_{(-\infty,\infty)}(y)$ with $\beta \geq 1$, among others. Although the logconcavity assumption is called upon in Theorem 1 (via Lemma 8) and Theorem 3, a primary utility of the assumption transpires in Lemma 1 in showing that the Bayes estimator is monotone, (as is the case for $\delta_{\text{mle}}(X)$).

Our first task is to translate (3) to a useful integral equation for priors that are mixtures of a point mass at 0 (necessarily) and an absolutely continuous part with respect to Lebesgue measure on $(0, \infty)$ in **(i)**; and a two-point symmetric distribution on $\{0, m\}$ (necessarily) and a symmetric absolutely continuous part with respect to Lebesgue measure on $(0, m)$ in **(ii)**. The study or restriction to absolutely continuous parts suffices for our purposes.

Lemma 1. *For models as in (4) with logconcave densities, $\delta_{mle}(X)$ is a Bayes estimator of θ with respect to a prior π of the above form if and only if*

(a)

$$f_0(x) + \int_0^x g(\theta) f_0(x-\theta) d\theta = \int_x^\infty g(\theta) f_0(x-\theta) d\theta, \quad \textit{for } x \geq 0, \tag{5}$$

in **(i)** *with* $\pi(\theta) \propto I_{\{0\}}(\theta) + g(\theta) I_{(0,\infty)}(\theta)$*;*

(b)

$$f_0(x) + \int_0^x g(\theta) f_0(x-\theta) d\theta = f_0(x-m) + \int_x^m g(\theta) f_0(x-\theta) d\theta, \quad \textit{for } x \in [0,m], \tag{6}$$

in **(ii)** *with* $\pi(\theta) \propto I_{\{0,m\}}(\theta) + g(\theta) I_{(0,m)}(\theta)$*.*

Proof. First observe that the mlr property of the densities $f_0(x-\theta)$ implies, for any prior π (for which the posterior is defined), an increasing mlr property for the posteriors $\pi(\theta|x)$, with x viewed as the parameter. Consequently, since these posterior medians will be nondecreasing with the mlr property, it will suffice for both cases **(i)** and **(ii)** to study (3) for $x \geq 0$ and $x \in [0,m]$ respectively. Now, in **(ii)** for $x \in [0,m]$, it must be the case that $\frac{1}{2} = P_\pi(\theta \leq x|x)$, in other words

$$\frac{1}{2} = \frac{f_0(x) + \int_0^x g(\theta) f_0(x-\theta) d\theta}{f_0(x) + f_0(x-m) + \int_0^m g(\theta) f_0(x-\theta) d\theta},$$

which is equivalent to (6), and establishes part (b). Finally, part (a) follows along the same lines or by taking $m = +\infty$ in (6).

We now turn to useful elements required to study solutions of integral equations which can be written as

$$\ell(x) = \int_0^m q(\theta) k(x-\theta)\, d\theta, \text{ for } x \in (0,m), \tag{7}$$

where ℓ and k are given, q is to be found, and m can be positive or $+\infty$. Such equations, which are said to be of Wiener-Hopf form, include both (5) and (6) by setting $q \equiv g$, either $\ell(x) = f_0(x)$ or $\ell(x) = f_0(x) - f_0(x-m)$, and $k(y) = -f_0(y)\,\mathrm{sgn}(y)$; $x, y \in \mathbb{R}$. There are several potential techniques to solve such equations. For the case $m < +\infty$, as in Iwasa and Moritani (1997), Fredholm's technique can be applied. Such an application arises in Section 3.2 and no further mention of this case will occur in this section. For the case $m = +\infty$ however, the situation is different with Fredholm's technique not applicable and a different method required. Among other choices, the Riemann-Hilbert method seems to be advantageous, since the technique can be used to establish the existence and number of solutions simply by determining the so-called index (Definition 1) associated with the problem, which is usually easy to compute. Although, our immediate concern will be to assess whether there exists at least one solution to (3), we also make use of the Riemann-Hilbert framework to obtain efficient computation of approximate solutions. Some further definitions and lemmas are required in order to reformulate integral equations like (7) (with $m = \infty$) as Riemann-Hilbert problems. The Riemann-Hilbert problem, presented in Definition 3, is the function-theoretical problem of finding analytic functions on $\mathbb{C} \setminus \mathbb{R}$ (see Definition 2) having a prescribed jump discontinuity on the real line $\mathbb{R}$. Often the problem is restricted by requiring boundedness (or exponential growth at a given rate) at infinity in all directions.

Definition 1. *Suppose q is a complex-valued smooth function defined on a smooth oriented curve Γ, such that $q(\Gamma)$ is closed and compact, then the index is defined to be the winding number of $q(\Gamma)$ about the origin.*

We refer to Gakhov (1990) or Payandeh (2007) for several examples and properties associated with the above definition of index. We will require the following (e.g., Gakhov, 1990).

Lemma 2. *If q is a function $q : \mathbb{R} \longrightarrow \mathbb{C}$ with real part positive, or with the imaginary part positive, then the index over $\mathbb{R}$ is whenever defined equal to zero.*

In terms of function theory, we may note that

$$\operatorname{Ind} \Gamma_q = \frac{1}{2\pi i} \oint_{q(\Gamma)} \frac{dz}{z} \,,$$

where it is understood that poles on Γ are handled by the usual method of indenting the contour.

Definition 2. *A scalar function $\Phi : \mathbb{C} \longrightarrow \mathbb{C}$ is said to be* sectionally analytic *if Φ is analytic in $\mathbb{C} \setminus \mathbb{R}$.*

A large supply of sectionally analytic functions having a prescribed jump discontinuity, and having limiting value 0 along vertical rays in the complex plane, are given by the Sokhotskyi-Plemelj formulas.[2]. It is usual to denote by Φ^+ the function on $\mathbb{R}$ obtained from the radial limit on the upper half-plane (noted D^+ throughout) of a sectionally analytic function Φ, and similarly to denote by Φ^- the function obtained from the radial limit on the lower half-plane (noted D^-). [3] We now proceed with a formulation of the Riemann-Hilbert problem.

Definition 3. *The* Riemann-Hilbert problem with index ν *consists in finding a sectionally analytic function* Φ *such that the upper and lower radial limits* $\Phi^\pm$ *satisfy*

$$\Phi^+(\omega) = r(\omega)\Phi^-(\omega) - s(\omega), \quad \textit{for } \omega \in \mathbb{R}, \tag{9}$$

where r *and* s *are given bounded and continuous functions on* $\mathbb{R}$ *such that:*

1. r *and* s *satisfy a Hölder condition on* $\mathbb{R}$ *(i.e.,* $|\varphi(t_2) - \varphi(t_1)| \leq a|t_2 - t_1|^\lambda$*, for any real valued* t_1*,* t_2*, for some* $a > 0$*,* $\lambda \in (0, 1]$*, and for* $\varphi \equiv r, \varphi \equiv s$ *), and* $r(\omega)$ *and* $s(\omega)$ *go to zero when* $\omega \to \infty$ *faster than some negative power of* ω*,*

2. r *does not vanish on* $\mathbb{R}$*, has index* ν*, and* $\lim_{\omega \longrightarrow \pm\infty} s(\omega) = 0$*.*

The condition that r does not vanish is overly restrictive, but in the cases we are interested in, the above conditions can be, as suggested by Gakhov (1990), met by dividing by a suitable polynomial in ω in order

[2]The Sokhotskyi-Plemelj integral for a function f, on $\mathbb{R}$, is defined by the principal value integral, $⨍$, as follows

$$\phi_f(\lambda) = \frac{1}{2\pi i} ⨍_{\mathbb{R}} \frac{f(\omega)}{\omega - \lambda} d\omega, \quad \text{for } \lambda \in \mathbb{C}. \tag{8}$$

In the boundary-value literature, it is well known that $\phi_f^\pm(\omega) = \lim_{\lambda \to \omega + i0^\pm} \phi_f(\lambda) = \pm f(\omega)/2 + H(f, \omega)/(2i)$, where $H(f)$ stands for the Hilbert transform of f and $\omega \in \mathbb{R}$.

[3]Since a sectionally analytic function is continuous in the open upper half-plane, (and the open lower half-plane) there is no compelling reason to consider limits more general than radial limits. One could replace the radial limits by non-tangential limits if desired.

to remove finitely many isolated zeros (see Theorem 1). Also, it would be enough that the functions go to a constant at infinity, but the solution to the problem behaves somewhat better in our case of functions going to zero at infinity.

The above mentioned Sokhotskyi-Plemelj formulas (which play a role in the proof of Lemma 4) lead to an existence proof for solutions to the Riemann-Hilbert problem. The number of bounded solutions is controlled by the index ν, and various situations arise. We will be concerned with the specific case $\nu = 0$ and the next result gives conditions for which unique solutions will arise. It will be significant for us that the Cauchy-type integrals appearing in the Sokhotskyi-Plemelj formula preserve the Hölder property, because this insures that the solution is Hölder-continuous. Further related technical results are relegated to Sections 3 and 4, and we refer the reader to Gakhov (1990) for proofs and related developments for cases where $\nu \neq 0$.

Lemma 3. *For a Riemann-Hilbert problem with index* 0*, there exists among solutions converging to* 0 *at* $+\infty$ *an essentially unique solution to (9). The solution is continuous in the Hölder sense, and goes to zero at infinity with the same exponent as* $r(\cdot)$ *and* $s(\cdot)$*.*

The first part of the above lemma is found in Gakhov (1990), page 106, and the remark on continuity follows from the form of the solution plus the theorem on page 38, *ibid.* Finally, the remark on going to zero follows from section 4.6 of Gakhov (and the fact that $r(\cdot)$ and $s(\cdot)$ go to zero, rather than a constant, at infinity.)

Finally, we make reference to the Paley-Wiener theorem (e.g., Dym and Mckean, 1972, pp. 158), which in one formulation can be stated as follows.

Lemma 4. *Suppose* $f \in L^2(\mathbb{R})$*. The following are equivalent:*

(i) The real function f *vanishes on* $\mathbb{R}^-$*;*

(ii) The Fourier transform $\hat{f}$ of f is holomorphic on the upper half-plane and the L_2-norms of the functions $x \mapsto \hat{f}(x+iy_0)$ are continuous and uniformly bounded for all $y_0 \geq 0$.

3. Main results

In this section, we study the potential bayesianity of δ_{mle} for models in (4) with logconcave densities as set up in equations (3), (5), and (6). For lower bounded cases, we obtain positive answers in all generality by developing a key correspondence with a Riemann-Hilbert problem. For doubly bounded cases, we use Fredholm's technique and a contraction mapping arguments to also obtain affirmative answers, thus generalizing the normal case result by Iwasa and Moritani.

3.1. Lower bounded case

We begin with the following useful technical lemmas before pursuing with the main results. The following is a form of the Riemann-Lebesgue lemma.

Lemma 5. *(Pinkus & Zafrany 1997) Suppose $f : \mathbb{R} \longrightarrow \mathbb{C}$ is piecewise continuous and absolutely integrable. Then its Fourier transform $\hat{f}$ is continuous on $\mathbb{R}$ and $\lim_{\omega \to \pm\infty} \hat{f}(\omega) = 0$.*

Lemma 6. *Suppose t and u are functions such that $\int_0^\infty x t(x) dx < \infty$, t is positive, and u satisfies the Hölder condition with $\lambda = 1$ (i.e., Lipschitz). Set $v(\omega) = \int_0^\infty t(x) u(x\omega) dx,\ \omega \in \mathbb{R}$. Then $v(\cdot)$ also satisfies the Hölder condition with $\lambda = 1$ (i.e., Lipschitz).*

Proof. It suffices to show that $\left|\frac{v(\omega_1)-v(\omega_2)}{\omega_1-\omega_2}\right| < M$, for all ω_1, ω_2 $(\omega_1 \neq \omega_2)$ and for some $M < \infty$. By the representation of v, and since u satisfies the Hölder condition with $\lambda = 1$, we have

$$\begin{aligned}\left|\frac{v(\omega_1) - v(\omega_2)}{\omega_1 - \omega_2}\right| &\leq \int_0^\infty t(x) \left|\frac{u(x\omega_1) - u(x\omega_2)}{\omega_1 - \omega_2}\right| dx \\ &\leq a \int_0^\infty x\, t(x)\, dx \quad \text{(for some } a > 0) \\ &< \Delta,\end{aligned}$$

for any Δ such that $a \int_0^\infty xt(x)dx < \Delta$.

Lemma 7. *Suppose L is a real-valued function such that $x \int_0^x L(t)dt \in L^1(\mathbb{R}^+)$. Then $v(y) = \frac{1}{y} \int_0^\infty L(t)e^{-iyt}dt;\ y \in \mathbb{R}$; satisfies the Lipschitz condition.*

Proof. Since differentiable functions with a bounded derivative satisfy the Lipschitz condition, it suffices to show that $v'(\cdot)$ is bounded. Expressing e^{-iyt} as $iy \int_t^\infty e^{-ixy}\, dx$, and $v(y)$ as $i \int_0^\infty e^{-ixy} \int_0^x L(t)dtdx$, we obtain $v'(y) = \int_0^\infty xe^{-ixy} \int_0^x L(t)dtdx$ so that the norm of $v'(\cdot)$ is bounded by $\int_0^\infty x \int_0^x L(t)dtdx$. The result follows by assumption on L.

Hereafter, we will denote $\bar{F}_0$ and F_0 as the survival and cumulative distribution functions (respectively) associated with f_0 (i.e., $F_0 \equiv 1 - \bar{F}_0$, $\bar{F}_0(x) = \int_{(x,\infty)} f_0(y)dy$ for $x \in \mathbb{R}$). As presented with the next result, decreasing logconcave functions, which have tails decreasing more rapidly than an exponential density, will possess the useful property of finite moments.

Lemma 8. *In (4) with logconcave f_0 on $(0, \infty)$, we have that:*

(a) $\bar{F}_0$ is logconcave on $(0, \infty)$;

(b) the functions $x^k f_0(x)$ and $x^k \bar{F}_0(x)$; $k \geq 0$ belong to $L^1(\mathbb{R}^+)$.

Proof. The results are known. See for instance An (1998).

In the next theorem, aimed at identifying solutions for g in (5), we will search for solutions g such that the limit at $+\infty$ exists and is finite (say equal to c) and decompose g as $g^* + c$ in such a way that $g^*(\theta) \to 0$ as $\theta \to +\infty$, (this is so in order to make use of Lemma 3).

Theorem 1. *Suppose f_0 is as in (4) and logconcave. Consider equation (7) with $m = \infty$, $q \equiv g$, $\ell(x) = f_0(x)$, and $k(y) = -f_0(y)\ sgn(y)$; $x, y \in \mathbb{R}$; i.e.,*

$$f_0(x) = \int_0^\infty g(\theta)k(x-\theta)\,d\theta\,,\ \textit{for}\ x \geq 0, \tag{10}$$

which we wish to solve in g. Among functions g whose limit exist and is finite at $+\infty$ (say equal to c), there exists an essentially unique continuous solution g_0 given by

$$g_0(\theta) = g^*(\theta) + c\,, \quad \textit{with}\ g^*(\theta) = \mathfrak{F}^{-1}(\Phi^-(\cdot);\theta), \tag{11}$$

for: **(i)** $c = \{\, 2\int_0^\infty \bar{F}_0(x)dx\,\}^{-1}$, **(ii)** $\mathfrak{F}^{-1}(\Phi^-(\cdot);\theta)$ *the inverse Fourier transform of Φ^- evaluated at θ, and* **(iii)** Φ^- *the (essentially) unique solution of a Riemann-Hilbert problem, as in (9), with:*

$$\begin{aligned} r(\omega) &= \frac{1}{\omega}\int_{-\infty}^\infty k(x)e^{-i\omega x}dx, \\ s(\omega) &= \frac{1}{\omega}\int_{-\infty}^\infty f^*(x)e^{-i\omega x}dx, \end{aligned}$$

and $f^*(x) = \{f_0(x) - c\bar{F}_0(x)\}I_{[0,\infty)}(x)$, (i.e., at ω, $\omega r(\omega)$ and $\omega s(\omega)$ are up to a multiplicative constant respectively the Fourier transforms of k and f^*).

Proof. We first establish an explicit correspondence with a Riemann-Hilbert problem. We then prove that the latter Riemann-Hilbert problem has index 0 which will imply, via Lemma 3, the existence of an essentially unique solution to (10). Finally, the established correspondence will lead to (11).

Observe that both f^* and k are piecewise continuous and integrable with

$$\int_{-\infty}^\infty k(x)\,dx \;=\; 0, \tag{12}$$

and, with the choice of c in **(i)**,

$$\int_{-\infty}^\infty f^*(x)\,dx \;=\; \int_0^\infty f^*(x)\,dx \;=\; 0. \tag{13}$$

Setting $h(x) = \{\int_0^\infty g^*(\theta)k(x-\theta)d\theta\}\ I_{(-\infty,0)}(x)$ (in order to extend the domain of (10) to $\mathbb{R}$), equation (10) with $g \equiv g^* + c$ becomes equivalent to:

$$\begin{aligned} & \int_0^\infty [g^*(\theta) + c]\, k(x-\theta)d\theta = f_0(x),\ x \geq 0 \\ \Leftrightarrow & \int_0^\infty g^*(\theta)k(x-\theta)d\theta = f_0(x) - c\bar{F}_0(x),\ x \geq 0 \\ \Leftrightarrow & \int_0^\infty g^*(\theta)k(x-\theta)d\theta = f^*(x) + h(x),\ x \in \mathbb{R}, \end{aligned} \tag{14}$$

where f^*, k are given (for c as in **(i)**), and g^*, h are to be determined. Fourier transforming and division by ω leads to:

$$\widehat{g^*}(\omega)\frac{\widehat{k}(\omega)}{\omega} = \frac{\widehat{f^*}(\omega)}{\omega} + \frac{\widehat{h}(\omega)}{\omega}, \quad \omega \in \mathbb{R}, \tag{15}$$

which can be relabelled as

$$\Phi^-(\omega)\frac{\widehat{k}(\omega)}{\omega} = \frac{\widehat{f^*}(\omega)}{\omega} + \Phi^+(\omega), \quad \omega \in \mathbb{R}, \tag{16}$$

where $\widehat{g}$, $\widehat{k}$, $\widehat{f^*}$ and $\widehat{h}$ represent the Fourier transforms of g, k, f^*, and h respectively. Now, clearly (16) and (9) are of the same form with $r(\omega) = \frac{\widehat{k}(\omega)}{\omega}$, and $s(\omega) = \frac{\widehat{f^*}(\omega)}{\omega}$; $\omega \in \mathbb{R}$. Hence, if there exists a solution Φ^- to this problem, g_0 as in (11) will solve (10). We show that such a solution Φ^- exists by establishing that (16) satisfies the conditions of Definition 3 and Lemma 3 (index 0) provided c is as given in **(i)**. (This choice of c is forced upon us to avoid generalized functions in the solution g_0.)

To show that r satisfies the Hölder condition (1) of Definition 3, write

$$r(\omega) = \int_0^\infty f_0(x)\frac{e^{i\omega x} - e^{-i\omega x}}{\omega}\, dx \tag{17}$$

$$= 2i\int_0^\infty f_0(x)\frac{sin(\omega x)}{\omega}\, dx\ ; \omega \in \mathbb{R}; \tag{18}$$

from which we infer the result by making use of Lemma 6 with the choices $u(\omega) = \frac{\sin(\omega)}{\omega}$, $t(x) = xf_0(x)$ for r; and since $\int_0^\infty x\{x\, f_0(x)\}\, dx < \infty$, given the logconcavity of f_0 and Lemma 8.

To show that s satisfies a Hölder condition, we write $s(\omega) = \int_0^\infty f^*(x)\frac{e^{-i\omega x}}{\omega}\,dx$, and we use Lemma 7 with $L(t) = f^*(t)$ requiring

$$x\int_0^x f^*(t)dt \in L^1(\mathbb{R}^+). \tag{19}$$

To establish (19), we show that: **(a)** $\int_0^x f^*(t)\,dt \leq 0$, for all $x \geq 0$, and **(b)** $|\int_0^x f^*(t)\,dt| \leq \overline{F}_0(x)$, for all $x \geq 0$, which will suffice since $x\overline{F}_0(x) \in L^1(\mathbb{R}^+)$ by Lemma 8. For **(b)**, note that $\int_0^\infty f^*(t)dt = 0$ from (13), so that $|\int_0^x f^*(t)dt| = \int_x^\infty [f_0(t) - c\overline{F}_0(t)]dt \leq \overline{F}_0(x)$. For **(a)**, observe that f^* must change signs at least once (or be 0 everywhere) on $(0,\infty)$ since $\int_0^\infty f^*(t)dt = 0$. Furthermore, the logconcavity of f_0 on $(0,\infty) \Rightarrow$ the logconcavity of $\overline{F}_0$ on $(0,\infty)$(Lemma 8) $\Rightarrow -\dfrac{\overline{F}_0}{f_0}$ increasing on $(0,\infty) \Rightarrow \dfrac{f^*}{f_0}$ increasing on $(0,\infty) \Rightarrow f^*$ changes signs once on $(0,\infty)$ from $-$ to $+$ (or is 0 everywhere) $\Rightarrow$ **(a)**.

To show that our Riemann-Hilbert problem has index 0, it is sufficient to show that the imaginary part of $r(\omega)$ is positive for all $\omega \in \mathbb{R}$ (Lemma 2). Since $r(\cdot)$ is an even function, it suffices to consider $\omega \geq 0$ only. Substituting $y = \omega x$ in (18) and partitioning, we obtain

$$\begin{aligned} r(\omega) &= 2i\int_0^\infty f_0(\frac{y}{\omega})\frac{sin(y)}{\omega^2}dy \\ &= 2i\sum_{j=0}^\infty \int_{j\pi}^{(j+1)\pi} f_0(\frac{y}{\omega})\frac{sin(y)}{\omega^2}dy \\ &= 2i\sum_{j=0}^\infty \int_{2j\pi}^{(2j+1)\pi} \left(f_0(\frac{y}{\omega}) - f_0(\frac{y+\pi}{\omega})\right)\frac{sin(y)}{\omega^2}\,dy\,. \end{aligned}$$

Since the sin function takes nonnegative values on all these above domains of integration and since f_0 is unimodal, we infer that all the above integrands are positive. Therefore, $r(\cdot)$ is a positive imaginary function and the corresponding Riemann-Hilbert problem has index 0.

Finally, further required conditions of Definition 3 are verified as follows :

- r does not vanish on $\mathbb{R}$ as seen just previously;
- the functions given by $\omega r(\omega)$ $(= \hat{k}(\omega))$ and $\omega s(\omega)$ $(= \hat{f}^*(\omega))$ are continuous on $\mathbb{R}$ with $\lim_{\omega\longrightarrow\pm\infty} \omega r(\omega) =$

$0 = \lim_{\omega \longrightarrow \pm\infty} \omega s(\omega)$ in view of Lemma 5 and the piecewise continuous and integrable behaviours of k and f^*. Hence both r and s go to zero at infinity faster than some negative power of ω, in particular ω^{-1};

- boundedness and continuity of r and s follow from the above and the previously established Hölder conditions.

Hence, the solution Φ^- is a Hölder-continuous function that goes to zero as $1/\omega$ for large ω. Thus the function is continuous and L^1, and the inverse Fourier transform of such a function is continuous by Lemma 5. (It is routine to verify that this lemma holds for the inverse Fourier transform as well as the forward transform). This completes the proof. □

The above Theorem yields the solution g_0 of (5) in g within the class of continuous functions having the properties stated. We however must investigate whether g_0 is real-valued and positive to render the solution statistically relevant.

Lemma 9. *Under the conditions of Theorem 1, the solution g_0 in (11) is real-valued and nonnegative a.e.*

Proof. See Appendix.

Corollary 1. *For models in (4) with logconcave f_0 and $\theta \geq 0$, δ_{mle} is a Generalized Bayes estimator of θ under absolute value error loss as in (2). Furthermore, the corresponding prior π is given by $\pi(\theta) \propto 1_{\{0\}}(\theta) + g_0(\theta) I_{(0,\infty)}(\theta)$, with g_0 given in (11).*

Remark 1. *(Admissibility and minimaxity)*

Although we now know that δ_{mle} is Generalized Bayes under loss (2) for models in (4) with f_0 logconcave and a lower bound constraint, we do not know whether or not δ_{mle} is admissible under loss (2). Brown (1979) conjectures that a necessary feature for admissibility in our problem is that $\pi(\theta)$ behaves like $|\theta|^b$

as $|\theta| \to \infty$, with $0 \le b \le 1$. Our prior is flat at $\theta \to \infty$ and has $b = 0$ thus passing Brown's conjectured test, but the question of admissibility remains unsolved. With regards to minimaxity however, we recall that δ_{mle} is minimax under loss (2) for models in (4) (e.g., Farrell, 1964; Marchand and Strawderman, 2005).

In the development above, the assumption of logconcavity of f_0 is critical (see first paragraph of Section 2) in establishing the bayesianity of δ_{mle}. The next result isolates a departure from logconcavity, associated with the tail behaviour of f_0, which suffices for the non-bayesianity of δ_{mle} and leads to various interesting "counterexamples".

Theorem 2. *Suppose for models in (4) that f_0 is such that for all $\theta > 0$: $\lim_{x\to\infty} \frac{f_0(x-\theta)}{f_0(x)} = 1$. Then under loss (2) and for $\theta \in [0,\infty)$, δ_{mle} is* **not** *a Generalized Bayes estimator of θ.*

Proof. First, observe that for any $\theta > 0$, we have $\lim_{x\to-\infty} \frac{f_0(x-\theta)}{f_0(x)} = 1$ as well, given the assumed symmetry and tail behaviour of f_0. For π to be a potential solution of (3), we require:

$$P_\pi(\theta = 0|x) \ge \frac{1}{2}, \text{ for all } x \le 0; \tag{20}$$

$$\text{and } P_\pi(\theta \ge x|x) \ge \frac{1}{2}, \text{ for all } x > 0. \tag{21}$$

Limit π to be a prior which is a mixture of a point mass at 0, with a density g on $(0,\infty)$ with respect to a $\sigma-$finite measure μ on $(0,\infty)$; as otherwise $P_\pi(\theta = 0|x) = 0$ for all $x \in \mathbb{R}$. Set

$$\eta = \int_{(0,\infty)} g(\theta)d\mu(\theta).$$

The proof consists in separating the cases: **(i)** $\eta > 1$, and **(ii)** $\eta \le 1$ and showing respectively that (20) and (21) cannot hold. Notice that **(i)** must be the applicable case for any improper π.

- **(i)** For $\eta > 1$, there must exist positive and finite values M_1, M_2, and ϵ such that $\int_{(M_1,M_2)} g(\theta)d\mu(\theta) = 1+\varepsilon$. It would then follow that

$$P_\pi(\theta = 0|x) = \frac{f_0(x)}{f_0(x) + \int_{(0,\infty)} g(\theta)f_0(x-\theta)d\mu(\theta)},$$

$$P_\pi(\theta = 0|x) \leq \frac{1}{1 + \int_{(M_1,M_2)} g(\theta) \frac{f_0(x-\theta)}{f_0(x)} d\mu(\theta)},$$

and, by dominated convergence as $\frac{f_0(x-\theta)}{f_0(x)} \leq 1$ for all x negative and $\theta \geq 0$,

$$\begin{aligned} \lim_{x\to-\infty} P_\pi(\theta = 0|x) &\leq \frac{1}{1 + \int_{(M_1,M_2)} g(\theta) \{ \lim_{x\to-\infty} \frac{f_0(x-\theta)}{f_0(x)} \} \, d\mu(\theta)} \\ &\leq \frac{1}{2+\varepsilon} < \frac{1}{2}, \end{aligned}$$

in contradiction with (20).

(ii) For $\eta \leq 1$, suppose in order to arrive at a contradiction that (21) holds, or equivalently

$$B(x) \leq A(x), \text{ for all } x > 0, \tag{22}$$

where $B(x) = f_0(x) + \int_{(0,x)} g(\theta) f_0(x-\theta) d\mu(\theta)$, and $A(x) = \int_{[x,\infty)} g(\theta) f_0(x-\theta) d\mu(\theta)$. Now, evaluating $\int_{(0,\infty)} B(x)dx$ and $\int_{(0,\infty)} A(x)dx$, we obtain with a change in the order of integration: $\int_{(0,\infty)} B(x)dx = \frac{\eta}{2} + \frac{1}{2} > \frac{\eta}{2} - \int_{(0,\infty)} \bar{F}_0(\theta)\, g(\theta) d\mu(\theta) = \int_{(0,\infty)} A(x)dx$, which is not consistent with (22), establishes a contradiction, and completes the proof of the Theorem.

Remark 2. *Two interesting examples where the conditions of Theorem 2 are satisfied and where we can infer that* δ_{mle} *is* **not** *a Bayes estimator under loss (2) are:* **(i)** *Student distributions with* $\nu \geq 1$ *degrees of freedom and scale parameter* $\sigma > 0$*, including Cauchy with* $\nu = 1$*, where* $f_0(x) \propto (1 + \frac{x^2}{\nu\sigma^2})^{-(\nu+1)/2}$*; and* **(ii)** *Exponential power densities with scale parameter* σ*, shape parameter* β*;* $\beta \in (0,1)$*; where* $f_0(x) \propto \exp\{-|\frac{x}{\sigma}|^\beta\}$.

3.2. Doubly-bounded case

Here, we work with logconcave densities in (4) with $\theta \in [0, m]$. The maximum likelihood estimator is given by $\delta_{\text{mle}}(x) = x I_{[0,\, m]}(x) + m I_{(m,\, \infty)}(x)$ and will be Bayes, with respect to a prior density of the form

$\pi(\theta) = I_{\{0,m\}}(\theta) + g(\theta)I_{(0,m)}(\theta)$; where g is an unknown, continuous function; if and only if (6) holds. The next results generalize the normal case result of Iwasa and Moritani (1997). We will require the following preliminary, and more general, result.

Lemma 10. *A solution g_0 of $h(x)+\int_0^x g(\theta)h(x-\theta)d\theta = h(x-m)+\int_x^m g(\theta)h(x-\theta)d\theta$; $x \in (0,m)$; is both real-valued and positive (a.e.) on $(0,m)$, where m is finite. We require that h be symmetric, monotone decreasing and positive for positive x, h' exists (a.e.), and that $\|h'(x)\|_{L^1} < 2h(0)$.*

Proof. We may first deduce from Fredholm's theory (see for instance Tricomi, 1985) the existence of a unique solution of the above problem. The solution g_0 is also a solution of the related equation given by differentiation:

$$g_0(x) = \rho(x) + \int_0^m g_0(\theta)k^*(x-\theta)\,d\theta\,,\ x \in (0,m),$$

where $\rho(x) = \frac{h'(x-m)-h'(x)}{2h(0)}$ and $k^*(x) = |h'(x)|/(2h(0))$. We may regard the above equation as the fixed point problem

$$g = L(g),$$

where $L : L^\infty([0,m]) \longrightarrow L^\infty([0,m])$ is the affine linear operator such that

$$L : g \mapsto \rho(x) + \int_0^m g(\theta)k^*(x-\theta)\,d\theta.$$

If we can only show that it is a contraction mapping (*i.e.,* the distance, in norm, between two functions decreases after L is applied), then L would have a unique fixed point in $L^\infty([0,m])$, by the contraction mapping theorem. Furthermore, since by the hypotheses, ρ and k^* are positive functions, L maps the positive cone of L^∞ to the positive cone. In particular, it is still a contraction mapping when restricted to the positive cone, and therefore there exists a (unique) fixed point of L within the positive cone, implying that the original unrestricted operator L actually has a positive function as its unique fixed point. It remains to show that the contraction property holds, but from the hypothesis on the L^1 norm of h we have $d = \|k^*(\theta)\|_{L^1(-\infty,\infty)} < 1$, so that $\|L(g_1-g_2)\|_{L^\infty} \leq d\|g_1-g_2\|_{L^\infty} < |g_1-g_2\|_{L^\infty}$.

We thus see that a solution of the differentiated problem is necessarily positive, and thus that the solution of the original problem is positive. □

Theorem 3. *For models as in (4) with logconcave f_0, equation (6) has an (essentially) unique, nonnegative, and bounded solution in g.*

Proof. The result follows from Lemma 10. Indeed equation (6) is equivalent to the stated equation of Lemma 10 with $h(x) = h_0(x) = f_0(x) I_{(-2m,2m)}(x)$, and we check that $\|h_0'\|_{L^1}$ is less than $2h(0)$, which follows from positivity and unimodality. □

The main result of this section is now immediate.

Corollary 2. *For models in (4) with logconcave f_0 and $\theta \in [0, m]$, δ_{mle} is a Bayes and admissible estimator of θ under loss (2).*

Proof. The proof follows from Lemma 1, Theorem 3, and the well known fact that unique Bayes rules with finite Bayes risks are admissible (e.g., Berger, 1985, theorem 8, page 253).

Here is a further application for a related truncated model.

Corollary 3. *Consider the truncated model $Y =^d X|X \in A$ where X has density as in (4) with f_0 logconcave, $\theta \in [0, m]$, and A is an (otherwise arbitrary) subset of $[0, m]$. Then, for such a model, the estimator $\delta_0(y) = y$ is an admissible estimator of θ under loss (2).*

Proof. In order to arrive at a contradiction, suppose there exists a dominating $\delta'(Y)$, i.e.,

$$\int_A |\delta'(y) - \theta|\, f_\theta(y) dy \leq \int_A |\delta_0(y) - \theta|\, f_\theta(y) dy\,,$$

for all $\theta \in [0, m]$ with strict inequality for a θ_0. Then it would follow, under (4) with $\theta \in [0, m]$, that the estimator given by $\delta^*(x) = \delta_{\text{mle}}(x) 1_{A'}(x) + \delta'(x) 1_A(x)$ would dominate δ_{mle} under loss (2), thus contradicting the results of Corollary 2.

We conclude this section with further results of interest.

Remark 3. *Corollary 2's admissibility of the* δ_{mle} *under loss (2) persists in the presence of an unknown scale parameter* σ *for models* $X \sim \frac{1}{\sigma} f_1(\frac{x-\theta}{\sigma})$*, with a logconcave* f_0 *satisfying the conditions in (4). This is so, since for any fixed* $\sigma = \sigma_0$*, if* δ_{mle} *were inadmissible, we would have a contradiction with the findings of Corollary 2 applied to the density* $f_0(x) = \frac{1}{\sigma_0} f_1(\frac{x}{\sigma_0})$ *in (4).*

Remark 4. *As shown in Theorem 4 immediately below, the non-logconcave densities of Theorem 2 also provide examples for which* δ_{mle} *is not Bayes for* θ *under loss (2) and* $\theta \in [0, m]$*. This is established in a similar fashion as in Theorem 2, and furthermore implies the inadmissibility of* δ_{mle} *(apply Theorems 10 and 12, p.545-546 of Berger, 1985) for such* f_0*'s.*

Theorem 4. *Suppose for models in (4) that* f_0 *is such that for all* $\theta > 0$*:* $\lim_{x\to\infty} \dfrac{f_0(x-\theta)}{f_0(x)} = 1$*. Then under loss (2) and for* $\theta \in [0, m]$*,* δ_{mle} *is* **not** *a Generalized Bayes estimator of* θ*.*

Proof. Proceed first as in Theorem 2 by limiting π to be of the form $\pi(\theta) = I_{\{0,m\}}(\theta) + g(\theta) I_{(0,m)}(\theta)$; with g being a density on $(0, m)$ with respect to a $\sigma-$finite measure μ on $(0, m)$;. Set

$$\eta = \int_{(0,m)} g(\theta) d\mu(\theta).$$

Now, as in part (i) of the proof of Theorem 2 and (20) still required here, we have whenever $\eta > 0$

$$\begin{aligned} \lim_{x\to-\infty} P_\pi(\theta = 0|x) &= \lim_{x\to-\infty} \frac{f_0(x)}{f_0(x) + f_0(x-m) + \int_{(0,m)} g(\theta) f_0(x-\theta)\, d\mu(\theta)} \\ &= \frac{1}{2+\eta} < \frac{1}{2}, \end{aligned}$$

in contradiction with (20) (whenever $\eta > 0$). Finally, $\eta = 0$ implies $g(\cdot) = 0$ a.e. μ on $(0, m)$, in which case the posterior median (i.e., of $\theta|x$) cannot equal x for all $x \in (0, m)$ unless $f_0(x) = f_0(x - m)$ for all $x \in (0, m)$, which in turn is not possible given the assumptions on f_0 in (4).

4. Illustrations and numerical evaluations

The results of Section 3, namely Corollaries 1 and 2, apply to a large class of densities in (4) with logconcave f_0. Among the models for which these results apply, we mention normal, double-exponential or Laplace, exponential power as in Remark 2 with $\beta \geq 1$, logistic, hyperbolic secant among others. The results of Section 3 tell us that there exists a solution π to the functional equation $\delta_{\text{mle}} \equiv \delta_\pi$. With evaluations obtainable from (11) in the lower bounded case, we investigate here such applications for the double-exponential and normal cases. In the double-exponential case, an explicit solution (Example 1), as well as a characterization (Remark 5, Theorem 5), are also available but we also illustrate the use of (11). In the normal case, we give numerical developments aimed at evaluating (11). Further issues and examples relative to numerical evaluations are addressed in Kucerovsky and Payandeh (2007) and Payandeh (2007).

We give first a general result (Carlemann's method) which gives a useful setup from which we can extract solutions of a Riemann-Hilbert problem of nonnegative integer index ν, and namely index 0 cases.

Lemma 11. *Suppose that it is required to solve a Riemann-Hilbert problem as in Definition 3 with non-negative integer index ν. If we can find functions r^+, r^-, s^+, and s^- satisfying for all $\omega \in \mathbb{R}$:*

1. $r(\omega) = r^+(\omega)r^-(\omega)$ *with r^+ analytic and bounded in D^+, and r^- analytic and bounded in D^-,*
2. $s(\omega)/r^+(\omega) = s^+(\omega) + s^-(\omega)$ *with s^+ analytic and bounded in D^+, and s^- analytic and bounded in D^-,*

then the (bounded) solutions of the given problem are:

$$\begin{aligned}
\Phi^+(\omega) &= -r^+(\omega)[s^+(\omega) + P_\nu(\omega)/\omega^\nu] \\
\Phi^-(\omega) &= [s^-(\omega) + P_\nu(\omega)/\omega^\nu]/r^-(\omega)\,,
\end{aligned}$$

where $P_\nu(\omega)$ is a polynomial of degree ν, if ν is positive, and $P_0(\omega) = 0$.

Example 1. *(Double Exponential)*

For a double-exponential model with scale parameter σ *(*$\sigma > 0$*) and density* $f_0(x) = \frac{1}{2\sigma}\, e^{-\frac{|x|}{\sigma}}$ *in (4), it is easy to verify by substitution that* $g \equiv \frac{1}{\sigma}$ *is a solution to both (5) and (6). On the other hand for the lower bounded case, Theorem 1 points to a Riemann-Hilbert equivalence with computations of* c*,* r *(in (17)),* s *and* f^* *with the above* f_0 *yielding*

$$c = \frac{1}{\sigma},\ f^*(x) = \frac{1-c\sigma}{2\sigma} e^{-\frac{x}{\sigma}} = 0,\ (x \geq 0),\ r(\omega) = \frac{i\sigma}{1+\sigma^2\omega^2},\ \textit{and}\ s(\omega) = \frac{1-c\sigma}{2\omega(1+i\omega\sigma)} = 0\,(\omega \in \mathbb{R})\,.$$

Now, applying the decompositions of Lemma 11, we may set for $\omega \in \mathbb{R}$*:* $r^+(\omega) = 1\,;r^-(\omega) = \frac{i\sigma}{1+\sigma^2\omega^2}\,;s^+(\omega) = s^-(\omega) = 0$*. Therefore, we infer that* $g^*(\theta)$ *is the inverse Fourier transform at* θ *of* $\Phi^- \equiv 0$*. Hence,* $g^* \equiv \mathfrak{F}^{-1}(0) \equiv 0$*, and* $g_0(\theta) = 0 + c = \frac{1}{\sigma}$ *for* $\theta \geq 0$*.*

Remark 5. *The dual problem of finding* g *in (5) or (6) is to determine* f_0 *from* g*. The double-exponential model provides an interesting example, and a link with the constant hazard rate property of the exponential distribution. As a complement to Example 1, the following says that if the continuous part* g *of the prior* c *is constant, then the model is necessarily double-exponential.*

Theorem 5. *In (5) or (6),* $g \equiv c$ *(with* $c > 0$*) implies that* f_0 *is a double-exponential density with scale parameter* $1/c$ *(or hazard rate* c*).*

Proof. Consider (6) where $g \equiv c$, along with the symmetric of f_0 imply:

$$\begin{aligned}
f_0(x) + c\int_0^x f_0(x-\theta)d\theta &= f_0(x-m) + c\int_x^m f_0(x-\theta)d\theta,\ x \in [0,m], \\
\Leftrightarrow f_0(x) + cF_0(x) &= f_0(x-m) + cF_0(m-x),\ x \in [0,m], \\
\Rightarrow \qquad & f_0 + cF_0 \text{ constant on } [0,m] \\
\Rightarrow \qquad & f_0' + cf_0 \equiv 0 \text{ on } [0,m] \\
\Rightarrow \qquad & f_0(x) = ce^{-cx} \text{ for } x \geq 0.
\end{aligned}$$

Finally, a similar analysis or setting directly $g^* \equiv 0$ in (14) establishes the result for the lower-bounded case (5).

Remark 6. *As seen in Example 1, it is sometimes possible to solve a Riemann-Hilbert problem by inspection, in particular when s and r are rational functions. Indeed, as prescribed by Lemma 11, if r is rational, then r^+ and r^- as above may be found by writing the denominator of r as a product of two polynomials, one having zeros only in D^+ and the other having zeros only in D^-. Then s/r^+ is again a rational function, and we can expand this rational function as a partial fraction, whereupon s^+ and s^- can be read off from the expansion. However, to obtain a completely explicit solution, we must find the zeros of the relevant polynomials (which can be done by standard numerical methods). In the same vein, and as suggested by Abrahams (2000) who proposed a Padé approximant to solve a homogeneous Riemann-Hilbert (i.e., $s \equiv 0$ in (9)) problem with zero index, one can seek in more general situations to approximate r and s by rational functions and proceed as above in order to obtain an approximation for g_0.*

Example 2. *(Normal) For a standard normal density $f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ in (3), Theorem 1's corresponding Riemann-Hilbert problem is obtained as follows. First, r and s can be expressed in general, (using (17) for r, and with little bit of computation for s), as:*

$$\begin{aligned} r(\omega) &= \frac{1}{\omega}(\mathfrak{F}(f_0^+; -\omega) - \mathfrak{F}(f_0^+; \omega)), \\ s(\omega) &= \frac{1}{\omega^2}(\mathfrak{F}(f_0^+; \omega)(\omega - ci) + \frac{ci}{2}), \end{aligned}$$

where f_0^+ is the positive part of f_0 and $\mathfrak{F}$ stands for the Fourier transform. Routine evaluations with the standard normal density f_0 yield

$$c = \sqrt{\frac{\pi}{2}}, \quad \text{and} \quad \mathfrak{F}(f_0^+; \omega) = \frac{1}{2} e^{-\omega^2/2} - \frac{i}{\sqrt{\pi}} D(\omega/\sqrt{2}),$$

where $D(\cdot)$ is the Dawson function given by $D(y) = e^{-y^2} \int_0^y e^{t^2} dt$. Substituting and collecting terms leads *to:*

$$\begin{aligned} r(\omega) &= \frac{-i}{\omega}\frac{2}{\sqrt{\pi}}D(\omega/\sqrt{2}); \\ \text{and} \;\; s(\omega) &= \frac{1}{2\omega^2}\left[(\omega - ci)e^{-\omega^2/2} + ci + (c - i\omega)\frac{2}{\sqrt{\pi}}D(\omega/\sqrt{2})\right]. \end{aligned}$$

Our specification of the corresponding Riemann-Hilbert problem is thus explicit. However, to obtain approximate solutions for Φ^- and g^*, we used accurate rational approximations for $D(\cdot)$ and f_0 given by Lether (1997) and MAPLE 10 respectively permitting the implementation of Lemma 7. Proceeding numerically with an inverse Fourier transform gave us an approximate solution for g_0 which is represented in Figure 1. Finally, one way to assess the accuracy is to compare the left-hand and right-hand sides of (10) with g being replaced by our approximated g_0. These differences in absolute value as a function of x ($x > 0$), are shown in Figure 2 and suggest a reasonably accurate approximation.

5. Concluding remarks

The bayesianity of the maximum likelihood estimators under loss (2) and for models in (4) extends (implies) to the multiparameter case when estimating $\theta = (\theta_1, \ldots, \theta_p)$:

(i) under loss $L(\theta, d) = \sum_{i=1}^{p} |d_i - \theta_i|$;

(ii) for rectangular constraints of the form $\theta_i \in A_i$, with $A_i = [a_i, b_i]$ or $A_i = [a_i, \infty)$ for $i = 1, \ldots p$;

(iii) and observables $X_1, \ldots X_p$ independently generated with $X_i - \theta_i \sim f_i$, f_i being positive, continuous, logconcave and symmetric densities (as in (4)) for $i = 1, \ldots, p$.

Indeed, in order to justify the above, first observe that

$$\hat{\theta}_{\text{mle}} = (\hat{\theta}_{1,\text{mle}}, \ldots, \hat{\theta}_{p,\text{mle}}),$$

where; for $i = 1, \ldots, p$; $\hat{\theta}_{i,\text{mle}}$ is the maximum likelihood estimator of θ_i with $\theta \in A_i$ and based on X_i only. Also, the bayesianity is established with the choice of prior $\pi(\theta_1, \ldots, \theta_p) = \Pi_{i=1}^{p} \pi_i(\theta_i)$, with $\hat{\theta}_{i,\text{mle}} \equiv \eta_{\pi_i}$; as given in the above sections; since the posterior loss is additive with independent components of the posterior $\theta | x_1, \ldots, x_p$, i.e.,

$$\arg\min_{\theta} E[\sum_{i=1}^{p} |d_i - \theta_i| \mid x_1, \ldots, x_p] = \arg\min_{\theta} \sum_{i=1}^{p} E[|d_i - \theta_i| \mid x_i] = (\eta_{\pi_1}, \ldots, \eta_{\pi_p}).$$

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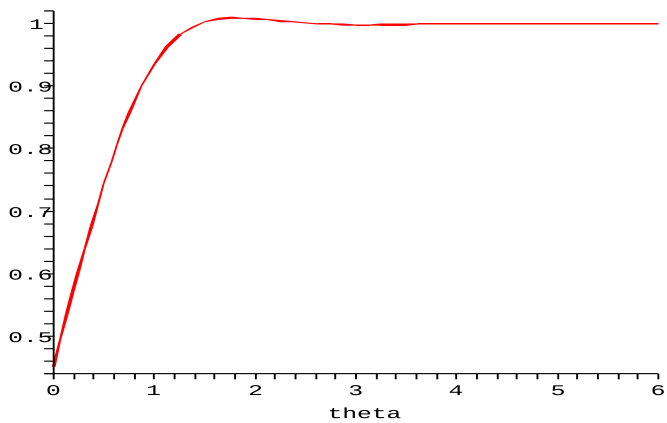

Figure 1: Normal model approximation for g_0

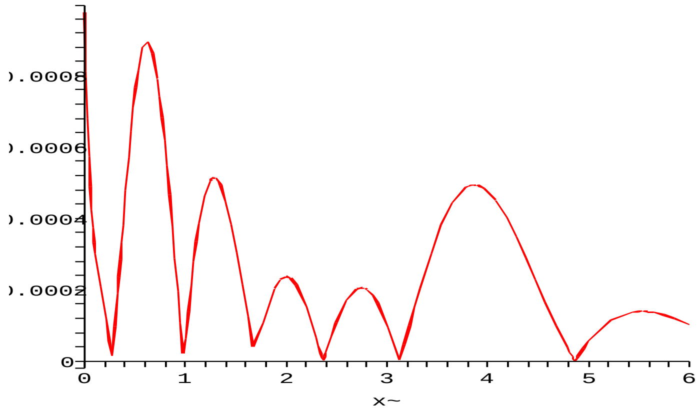

Figure 2: Error of approximated solution

6. Appendix

PROOF OF LEMMA 9

To show that g_0 must take only real values, and in order to arrive at a contradiction, assume that a solution of equation (10), say, g_0 is a complex-valued function. Therefore, g_0 can be written as $g_0(\theta) = Re(g_0(\theta)) + Im(g_0(\theta))$, where Re and Im stand for real and imaginary parts of g_0, respectively. By the uniqueness in the sense of Theorem 1, the real and imaginary parts must therefore be linearly dependent, from which it follows that g_0 is a scalar multiple of a real function, and must in fact therefore be real.

To show nonnegativity, suppose in order to arrive at a contradiction that $\mu(S^-) > 0$, where $S^- = \{\theta \geq 0 : g_0(\theta) < 0\}$ and μ is Lebesgue measure. We will require the following intermediate observations.

1. (boundedness and continuity of g_0) By Theorem 1, g_0 is a continuous function that goes to a constant at infinity. It is thus bounded.

2. (the set S^-) By the continuity of g_0, the set S^- is an open set, and has finite Lebesgue measure since $\lim_{\theta\to\infty} g_0(\theta) = c > 0$; i.e., there exists $\bar{\theta}$ such that $g_0(\theta) \geq 0$ for all $\theta \geq \bar{\theta}$, which implies indeed

that $\mu(S^-) \leq \bar{\theta} < \infty$. Since the intervals form a basis for the open sets, from this and some measure theory it follows that we can find intervals I_j having the property that for all $\epsilon > 0$, there exists $N \geq 1$ such that $\bigcup_{j=1}^N I_j \subseteq S^- \subseteq \bigcup_{j=1}^\infty I_j$, with $\mu(S^- - \bigcup_{j=1}^N I_j) \leq \epsilon$.

Assume now (without loss of generality) that $I_j = [a_j, b_j]$, with $0 \leq a_j \leq b_j \leq a_{j+1} \leq b_{j+1}$, for all $j \geq 1$. Consider now, for a given $\epsilon > 0$ (and N) in 2. above $\Delta = \sum_{i=1}^N \{ f_0(a_i) - f_0(b_i) \}$. On one hand, we would have in cases where $\mu(S^-) > 0$, since f_0 is unimodal,

$$\Delta \geq f_0(a_1) - f_0(b_1) > 0\,. \tag{23}$$

On the other hand, from (10), we have

$$\begin{aligned}
\Delta &= \sum_{i=1}^N \int_0^\infty g_0(\theta)[k(a_i - \theta) - k(b_i - \theta)]\,d\theta \\
&= \sum_{i=1}^N \{ \int_0^{a_i} g_0(\theta)[f_0(b_i - \theta) - f_0(a_i - \theta)]\,d\theta + \int_{b_i}^\infty g_0(\theta)[f_0(a_i - \theta) - f_0(b_i - \theta)]\,d\theta \\
&+ \int_{a_i}^{b_i} g_0(\theta)[f_0(a_i - \theta) + f_0(b_i - \theta)]\,d\theta \} \\
&= A + B + C \text{ (say)}.
\end{aligned} \tag{24}$$

We pursue by showing that ϵ and a covering $I_1, \ldots I_N$ can be selected in such a way (see (27) below) that $\Delta < f_0(a_1) - f_0(b_1)$, with Δ as in (24), thus contradicting (23) and implying $\mu(S^-) = 0$, as desired.

Now denote the complement set of S^- as S^+, denote for sake of simplicity $L_i(\theta)$ as the difference $f_0(b_i - \theta) - f_0(a_i - \theta)$, set $b_0 = 0$, and partition the sets of the form (b_i, ∞) and $(0, a_i)$ as:

$$(b_i, \infty) = \bigcup_{j=i}^{\infty} (b_j, a_{j+1}) \cup \bigcup_{j=i+1}^{N} [a_j, b_j)$$

and

$$(0, a_i) = \bigcup_{j=1}^{i-1} [a_j, b_j) \cup \bigcup_{j=0}^{i-1} [b_j, a_{j+1})\,, \text{ for } i \in \{1, \ldots, N\},$$

to obtain

$$
\begin{aligned}
A &= \sum_{i=1}^{N} \int_0^{a_i} g_0(\theta)\, L_i(\theta)\, d\theta \\
&= \sum_{i=1}^{N} \sum_{j=0}^{i-1} \int_{b_j}^{a_{j+1}} g_0(\theta)\, L_i(\theta)\, d\theta + \sum_{i=2}^{N} \sum_{j=1}^{i-1} \int_{a_j}^{b_j} g_0(\theta)\, L_i(\theta)\, d\theta \\
&= \sum_{j=0}^{N-1} \int_{b_j}^{a_{j+1}} g_0(\theta) \{ \sum_{i=j+1}^{N} L_i(\theta)\} \{1_{\{S^- - \cup_{j=1}^N I_j\}}(\theta) + 1_{S^+}(\theta)\} d\theta + \sum_{j=1}^{N-1} \int_{a_j}^{b_j} g_0(\theta) \{ \sum_{i=j+1}^{N} L_i(\theta)\}\, d\theta \\
&\leq M \sum_{j=0}^{N-1} \int_{b_j}^{a_{j+1}} \sum_{i=j+1}^{N} \{-L_i(\theta)\}\, 1_{\{S^- - \cup_{j=1}^N I_j\}}(\theta)\, d\theta + \sum_{j=1}^{N-1} \int_{a_j}^{b_j} g_0(\theta) \{ \sum_{i=j+1}^{N} L_i(\theta)\}\, d\theta \\
&< \epsilon M f_0(0) - \sum_{j=1}^{N-1} \int_{a_j}^{b_j} g_0(\theta) f_0(b_j - \theta)\, d\theta \,, \qquad (25)
\end{aligned}
$$

with the equalities above following by decomposition and change of variables; with the first inequality following by unimodality of f_0, the nonnegativity of g_0 on the set S^+, and the inequality $|g_0| \leq M$; with the second inequality a consequence of part (a) Lemma 12 (stated and proven below) used twice, the negativity of g_0 on $\cup_{j=1}^{N-1}(a_j, b_j)$, and the inequality $\mu(S^- - \cup_{j=1}^N I_j) \leq \epsilon$.

Similarly,

$$
\begin{aligned}
B &= \sum_{i=1}^{N} \int_0^{a_i} g_0(\theta) \{-L_i(\theta)\}\, d\theta \\
&= \sum_{i=1}^{N} \sum_{j=i}^{\infty} \int_{b_j}^{a_{j+1}} g_0(\theta) \{-L_i(\theta)\}\, d\theta + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \int_{a_j}^{b_j} g_0(\theta) \{-L_i(\theta)\}\, d\theta \\
&= \sum_{j=1}^{\infty} \int_{b_j}^{a_{j+1}} g_0(\theta) \{ \sum_{i=1}^{j\wedge N} \{-L_i(\theta)\}\} \{1_{\{S^- - \cup_{j=1}^N I_j\}}(\theta) + 1_{S^+}(\theta)\} d\theta + \sum_{j=2}^{N} \int_{a_j}^{b_j} g_0(\theta) \{ \sum_{i=1}^{j-1} \{-L_i(\theta)\}\}\, d\theta \\
&\leq M \sum_{j=1}^{\infty} \int_{b_j}^{a_{j+1}} \sum_{i=1}^{j\wedge N} (L_i(\theta))\, 1_{\{S^- - \cup_{j=1}^N I_j\}}(\theta)\, d\theta + \sum_{j=2}^{N} \int_{a_j}^{b_j} g_0(\theta) \{ \sum_{i=1}^{j-1} \{-L_i(\theta)\}\}\, d\theta \\
&< \epsilon M f_0(0) - \sum_{j=2}^{N} \int_{a_j}^{b_j} g_0(\theta) f_0(a_j - \theta)\, d\theta \,, \qquad (26)
\end{aligned}
$$

with also the last inequality a consequence of part (b) of Lemma 12 (used twice).

Assembling the above inequalities, making use still of the negativity of g_0 on $\cup_{j=1}^N(a_j, b_j)$, we would have from (24), (25), and (26), if indeed $\mu(S^-)$ were positive,

$$\begin{aligned}\Delta &= A+B+C \\ &< 2\epsilon M f_0(0) + (-\sum_{j=1}^{N-1}\int_{a_j}^{b_j} g_0(\theta) f_0(b_j-\theta)\,d\theta) + \\ &\quad (-\sum_{j=2}^{N}\int_{a_j}^{b_j} g_0(\theta) f_0(a_j-\theta)\,d\theta) + \sum_{j=1}^{N}\int_{a_j}^{b_j} g_0(\theta)[f_0(a_j-\theta)+f_0(b_j-\theta)]\,d\theta \\ &= 2\epsilon M f_0(0) + \int_{a_1}^{b_1} g_0(\theta)\, f_0(a_1-\theta)\,d\theta + \int_{a_N}^{b_N} g_0(\theta)\, f_0(b_N-\theta)\,d\theta \le 2\epsilon M f_0(0).\end{aligned}$$

Finally, the contradiction is arrived at by selecting ϵ and $I_1,\ldots I_N$, in such a way that $2\epsilon M f_0(0) \le f_0(a_1) - f_0(b_1)$, for instance with

$$\epsilon = \frac{f_0(a_1)-f_0(b_1)}{2M f_0(0)}\,. \qquad \square \tag{27}$$

Lemma 12. *(a) For fixed $N \ge 2$ and $j \in \{1,\ldots,N-1\}$, and $\theta \in (a_j, b_j)$, we have $\sum_{i=j+1}^{N}[f_0(b_i-\theta) - f_0(a_i-\theta)] \ge -f_0(b_j-\theta) \ge -f_0(0)$;*

(b) For fixed $j \ge 2$ and $\theta \in (a_j, b_j)$, we have $\sum_{i=1}^{j-1}[f_0(a_i-\theta) - f_0(b_i-\theta)] \ge -f_0(a_j-\theta) \ge f_0(0)$.

Proof. We exploit the properties in (4), namely unimodality (i.e., $f_0(\cdot)$ is decreasing on $(0,\infty)$, and increasing on $(-\infty,0)$), and telescoping arguments applicable by rearranging terms (in turn permissible since the given sums are finite). For part (a), we have

$$\sum_{i=j+1}^{N} L_i(\theta) = f_0(b_N-\theta) - f_0(a_{j+1}-\theta) + \sum_{i=j+1}^{N-1}[f_0(b_i-\theta) - f_0(a_{i+1}-\theta)] \ge -f_0(a_{j+1}-\theta) \ge -f_0(b_j-\theta) \ge f_0(0),$$

since $\theta \in (a_j, b_j)$. Similarly, for part (b), for fixed $j \ge 2$ and $\theta \in (a_j, b_j)$, we have

$$\sum_{i=1}^{j-1}\{-L_i(\theta)\} = f_0(a_1-\theta) + \{\sum_{i=1}^{j-1}[f_0(a_{i+1}-\theta) - f_0(b_i-\theta)] - f_0(a_j-\theta)\} \ge f_0(a_1-\theta) - f_0(a_j-\theta) \ge -f_0(a_j-\theta) \ge f_0(0).$$

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