FROM TRISECTIONS IN MODULE CATEGORIES TO QUASI-DIRECTED COMPONENTS

E. R. ALVARES, I. ASSEM, F. U. COELHO, M. I. PEÑA, AND S. TREPODE

ABSTRACT. In this paper, we define and study a special type of trisections in a module category, namely the compact trisections which characterise quasi-directed components. We apply this notion to the study of laura algebras and we use it to define a class of algebras with predictable Auslander-Reiten components.

Let Λ be an artin algebra, modΛ be the category of finitely generated right Λ-modules and indΛ be a full subcategory of modΛ consisting of exactly one representative from each isomorphism class of indecomposable Λ-modules. In this paper, we are interested in the study of a particular type of components of the Auslander-Reiten quiver $\Gamma(\text{mod}\Lambda)$ of Λ, namely the so-called quasi-directed components which appeared naturally in the study of laura algebras [3, 19]. They are components containing at most finitely many modules lying on oriented cycles and which moreover are generalised standard (in the sense of [23]). These components and their connection to laura algebras have been studied in [27, 14]. Our objective in this paper is to give simple new categorical characterisations of quasi-directed components.

For this purpose, our main tool is the notion of trisection, introduced in [13]: a trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of indΛ is a triple of full disjoint subcategories of indΛ such that ind$\Lambda = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and $\text{Hom}_\Lambda(\mathcal{B}, \mathcal{A}) = \text{Hom}_\Lambda(\mathcal{C}, \mathcal{B}) = \text{Hom}_\Lambda(\mathcal{C}, \mathcal{A}) = 0$. Here, we define and study a special type of trisections, namely the compact trisections which characterise quasi-directed components. A second approach is motivated by the observation that a trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in indΛ is really determined by the central subcategory $\mathcal{B}$. Along this line, and in the spirit of [19], we define the core of a component Γ to be a full subcategory of indΛ which is the convex envelope in indΛ of a left and of a right section of Γ (in the sense of [1]). We prove here that the presence of a core characterises quasi-directed components. Moreover, these conditions yield a description of the module category of Λ which then resembles closely that of laura algebra. Namely, our main result is as follows.

THEOREM. *The following conditions are equivalent for an artin algebra Λ:*

(a) Γ*(modΛ) admits a separating quasi-directed component Γ.*

(b) *indΛ admits a compact trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.*
(c) *Γ(modΛ) admits a separating convex component Γ having a left section Σ_l and a right section Σ_r whose convex envelope Δ is a core in Γ.*

If these conditions are satisfied, then Γ has only finitely many τ_Λ-orbits and is convex in Γ(modΛ). Moreover, if $\Gamma' \neq \Gamma$ is a component of Γ(modΛ), then either $\Gamma' \subset \mathcal{A}$ or $\Gamma' \subset \mathcal{C}$. Moreover,

(a) *$Hom_\Lambda(\Gamma', \Gamma) \neq 0$ implies $\Gamma' \subset \mathcal{A}$;*
(b) *$Hom_\Lambda(\Gamma, \Gamma') \neq 0$ implies $\Gamma' \subset \mathcal{C}$.*

In particular, if Γ is non-semiregular, then $\Lambda/\mathrm{Ann}(\Gamma)$ is a laura algebra having Γ as a unique quasi-directed faithful component while, if Γ is semiregular, then $\Lambda/\mathrm{Ann}(\Gamma)$ is a tilted algebra having Γ as a connecting component.

Since quasi-directed components are intimately connected to laura algebras, we get, as corollaries, several new characterisations of classes of laura algebras by means of trisections and cores. We end this paper with the construction of a generalisation of the class of laura algebras, which we call quasi-laura, and the nth iterates of this construction, which we call n-quasi-laura.

1. Preliminaries

1.1. **Notation.** Throughout this paper, all our algebras are basic and connected artin algebras. For an algebra Λ, we denote by modΛ its category of finitely generated right Λ-modules and by indΛ a full subcategory of modΛ consisting of exactly one representative from each isomorphism class of indecomposable modules. When we speak about a Λ-module (or an indecomposable Λ-module), we always mean implicitly that it belongs to modΛ (or to indΛ, respectively). Also, all subcategories of modΛ are full and so are identified with their object classes.

A subcategory $\mathcal{C}$ of indΛ is called *finite* if it has only finitely many objects, and it is *cofinite* if its complement $\mathcal{C}^c = $ indΛ $\setminus \mathcal{C}$ is finite. We sometimes write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. We denote by add$\mathcal{C}$ the subcategory of modΛ with objects the finite direct sums of summands of modules in $\mathcal{C}$ and, if M is a module, we abbreviate add$\{M\}$ as add$M$. Similarly, we denote by ind$M$ the subcategory of indΛ having as objects the indecomposable summands of M. We denote the projective (or injective) dimension of a module M as pd$M$ (or id$M$, respectively). The global dimension of Λ is denoted by gl.dimΛ. For a point x in the quiver of Λ, we denote by P_x (or I_x, or S_x) the indecomposable projective (or injective, or simple, respectively) module corresponding to this point. Finally, if $\mathcal{C}, \mathcal{D}$ are two full subcategories of modΛ, we write $Hom_\Lambda(\mathcal{C}, \mathcal{D}) = 0$ to express that $Hom_\Lambda(X, Y) = 0$ for any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

For an algebra Λ, we denote by Γ(modΛ) its Auslander-Reiten quiver and by $\tau_\Lambda = $ DTr, $\tau_\Lambda^{-1} = $ TrD its Auslander-Reiten translations. For further definitions or facts needed on modΛ or Γ(modΛ), we refer the reader to [7, 8].

1.2. **Paths.** Let Λ be an algebra. Given $M, N \in \operatorname{ind}\Lambda$, a *path* from M to N in $\operatorname{ind}\Lambda$ (denoted by $M \rightsquigarrow N$) is a sequence of non-zero morphisms

$$(*) \qquad\qquad M = X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_{t-1} \xrightarrow{f_t} X_t = N$$

$(t \geq 1)$, where $X_i \in \operatorname{ind}\Lambda$ for all i. We then say that M is a *predecessor* of N and N is a *successor* of M. A path from M to M involving at least one non-isomorphism is a *cycle*. A module $M \in \operatorname{ind}\Lambda$ which lies on no cycle is *directed.* If each f_i in $(*)$ is irreducible, we say that $(*)$ is a path of irreducible morphisms. A path $(*)$ of irreducible morphisms is *sectional* if $\tau_\Lambda X_{i+1} \neq X_{i-1}$ for all i with $0 < i < t$. A *refinement* of $(*)$ is a path in $\operatorname{ind}\Lambda$

$$M = X_0' \longrightarrow X_1' \longrightarrow \cdots \longrightarrow X_{s-1}' \longrightarrow X_s' = N$$

such that there is an order-preserving injection $\sigma \colon \{1, \cdots, t-1\} \rightarrow \{1, \cdots, s-1\}$ satisfying $X_i = X_{\sigma i}'$ for all i with $0 < i < t$.

Let $\mathcal{C}$ be a subcategory of $\operatorname{ind}\Lambda$. Then, $\mathcal{C}$ is called *directed* (or *almost directed*) if all (or all but finitely many, respectively) objects of $\mathcal{C}$ are directed. Also, $\mathcal{C}$ is *closed under predecessors* if, whenever $M \rightsquigarrow N$ is a path in $\operatorname{ind}\mathcal{C}$, with $N \in \mathcal{C}$, then $M \in \mathcal{C}$. An example is the left part $\mathcal{L}_\Lambda$ of $\operatorname{mod}\Lambda$, defined in [13], which is the full subcategory of $\operatorname{ind}\Lambda$ with object class

$$\mathcal{L}_\Lambda \;=\; \{M \in \operatorname{ind}\Lambda : \operatorname{pd}L \leq 1 \text{ whenever there is a path } L \rightsquigarrow M\}$$

Another example is, for a given module M, the full subcategory $\operatorname{Pred}M$ of the predecessors of M: this is the full subcategory of $\operatorname{ind}\Lambda$ consisting of the modules X such that there exist an indecomposable summand Y of M and a path $X \rightsquigarrow Y$.

We define dually subcategories closed under successors, examples of which are the right part $\mathcal{R}_\Lambda$ of $\operatorname{mod}\Lambda$, and the subcategory $\operatorname{Succ}M$ of $\operatorname{ind}\Lambda$ consisting of the successors of M.

Given $X, Y \in \operatorname{ind}\Lambda$, we denote by $\operatorname{rad}_\Lambda(X, Y)$ the group of the morphisms $f \colon X \longrightarrow Y$ which are not isomorphisms and by $\operatorname{rad}_\Lambda^\infty(X, Y)$ the intersection of all powers $\operatorname{rad}_\Lambda^i(X, Y)$, $i \geq 1$, of $\operatorname{rad}_\Lambda(X, Y)$. The following result has been proven in different versions (see, for instance, [21](2.1) or [22](2.1)). The one below is [27](1.1).

LEMMA. *Let Λ be an algebra and $X, Y \in \operatorname{ind}\Lambda$ be such that $\operatorname{rad}_\Lambda^\infty(X, Y) \neq 0$. Then, for each $t \geq 0$, there exists a path in $\operatorname{ind}\Lambda$*

$$X = X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \xrightarrow{f_t} X_t \xrightarrow{g_t} Y$$

with the X_i pairwise non-isomorphic, $g_t f_t \cdots f_1 \neq 0$, each f_i is a composite of irreducible morphisms and $g_t \in \operatorname{rad}_\Lambda^\infty(X_t, Y)$.

1.3. **Covariant and contravariant finiteness.** Let $\mathcal{X}$ be an additive subcategory of $\operatorname{mod}\Lambda$. For a Λ-module M, a *right $\mathcal{X}$-approximation of M* is a morphism $f_M \colon X_M \longrightarrow M$ with $X_M \in \mathcal{X}$ such that any morphism $f \colon X \longrightarrow M$ with $X \in \mathcal{X}$ factors through f_M. The subcategory $\mathcal{X}$ is *contravariantly finite* if any

Λ-module has a right $\mathcal{X}$-approximation. We define dually *left $\mathcal{X}$-approximations* and *covariantly finite subcategories*. Finally, $\mathcal{X}$ is *functorially finite* if it is both contravariantly and covariantly finite. Observe that finite and cofinite subcategories are functorially finite (see [9]).

1.4. If $\mathcal{X}$ is an additive subcategory of modΛ, closed under extensions, then a module $M \in \mathcal{X}$ is called *Ext-projective* (or *Ext-injective*) in $\mathcal{X}$ if $\operatorname{Ext}^1_\Lambda(M, -)|_{\mathcal{X}} = 0$ (or $\operatorname{Ext}^1_\Lambda(-, M)|_{\mathcal{X}} = 0$, respectively), see [10]. It is shown in [10](3.3)(3.7) that, if $\mathcal{X}$ is a torsion (or torsion-free) class, then an indecomposable module $M \in \mathcal{X}$ is Ext-projective in $\mathcal{X}$ if and only if $\tau_\Lambda M$ is torsion-free (or, $M \in \mathcal{X}$ is Ext-injective in $\mathcal{X}$ if and only if $\tau_\Lambda^{-1} M$ is torsion).

1.5. We recall from [13] that an algebra Λ is called *quasi-tilted* if gl.dim$\Lambda \leq 2$ and ind$A = \mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda$. The module category of a quasi-tilted algebra has been described in [11, 15, 16]. Several generalisations of this notion have been introduced over the years (see [4] for a survey). We recall the definition of laura algebras [3]: an algebra Λ is *laura* if $\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda$ is cofinite in indΛ.

Quasi-tilted algebras are clearly laura, and a laura algebra which is not quasi-tilted is called *strict*. The module category of a strict laura algebra has been described in [3, 19]. In particular, a strict laura algebra has a unique faithful, non-semiregular Auslander-Reiten component which is quasi-directed (that is, it is generalised standard and almost directed). This component is called its connecting component, in analogy with the connecting components of tilted algebras.

2. Trisections

2.1. Following [13], we define a trisection of indΛ.

DEFINITION. Let Λ be an artin algebra. A *trisection* of indΛ is a triple of disjoint full subcategories $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of indΛ such that:

  (a) ind$\Lambda = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, and
  (b) $\operatorname{Hom}_\Lambda(\mathcal{C}, \mathcal{B}) = \operatorname{Hom}_\Lambda(\mathcal{C}, \mathcal{A}) = \operatorname{Hom}_\Lambda(\mathcal{B}, \mathcal{A}) = 0$.

Notice that this definition of trisection is different from the one given in [18].

If any of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ is empty, then the definition above reduces to that of a split torsion pair.

If $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a trisection of indΛ, then the subcategory $\mathcal{A}$ is closed under predecessors and, dually, $\mathcal{C}$ is closed under successors. Also, $\mathcal{B}$ is *convex* in indΛ, that is, whenever there exists a path

$$X = X_0 \to X_1 \to \cdots \to X_t = Y$$

in indΛ, with $X, Y \in \mathcal{B}$, then $X_i \in \mathcal{B}$ for each i.

2.2. We need another definition due to Ringel [20](p. 120). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three disjoint full subcategories of indΛ, then we say that $\mathcal{B}$ *separates* $\mathcal{A}$ from $\mathcal{C}$ if $\mathcal{B} \neq \emptyset$ and any morphism $X \to Y$ with $X \in \mathcal{A}$ and $Y \in \mathcal{C}$ factors through add$\mathcal{B}$. If $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a trisection and $\mathcal{B}$ separates $\mathcal{A}$ from $\mathcal{C}$, then we say that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a *separated trisection*.

LEMMA. *Let* $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *be a separated trisection of indΛ and $f : M \to N$ be an irreducible morphism with M, N indecomposable.*

   (a) *If* $M \in \mathcal{A}, N \notin \mathcal{A}$ *then* $N \in \mathcal{B}$.
   (b) *If* $M \notin \mathcal{C}, N \in \mathcal{C}$ *then* $M \in \mathcal{B}$.

Proof. We prove only (a) since the proof of (b) is dual.
(a) Assume $M \in \mathcal{A}$ and $N \in \mathcal{C}$, then $f : M \to N$ must factor through add$\mathcal{B}$. But then $M \in \mathcal{B}$ or else $N \in \mathcal{B}$, a contradiction in either case. $\square$

2.3. EXAMPLES. (a) Let Λ be a representation-infinite hereditary algebra, then indΛ admits a separated trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, where $\mathcal{A}$ consists of the postprojective modules, $\mathcal{C}$ of the preinjective, and $\mathcal{B}$ of the regular.

(b) The motivating example is from the study of quasi-tilted algebras: it is shown in [13] (II,1.6) and (II, 1.7) that any quasi-tilted algebra Λ admits a trisection $(\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda, \mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda, \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)$. Moreover, by [11] and [15](4.3), this trisection is separated.

(c) Let Λ be the radical square zero algebra given by the quiver

FIGURE 1.

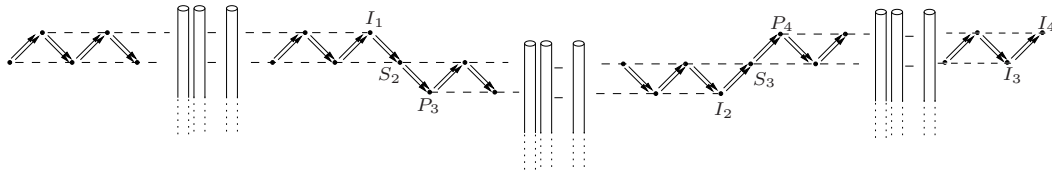Then the Auslander-Reiten quiver $\Gamma(mod\ \Lambda)$ of Λ is of the form

FIGURE 2.

Then indΛ admits a trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ where $\mathcal{A}$ consists of all predecessors of S_2, $\mathcal{C}$ of all successors of S_3 and $\mathcal{B}$ consists of the remaining indecomposables. Trivially, $\mathcal{B}$ separates $\mathcal{A}$ from $\mathcal{C}$.

2.4. LEMMA. *The following statements are equivalent for a trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $\mathrm{ind}\Lambda$.*

    (a) *$pdM \leq 1$ for all $M \in \mathcal{A}$.*
    (b) *$\mathcal{A} \subset \mathcal{L}_\Lambda$.*
    (c) *Each indecomposable injective $I \in \mathcal{A}$ satisfies the following: any path $I \rightsquigarrow X$ with $X \in \mathcal{A}$ is refinable to a path of irreducible morphisms and any such path is sectional.*

Proof. Since $\mathcal{A}$ is closed under predecessors and $pdM \leq 1$ for each $M \in \mathcal{L}_\Lambda$, the equivalence of (a) and (b) is trivial. The equivalence between (b) and (c) follows easily from [3](1.6). $\square$

2.5. We need the following lemma, essentially due to Smalø [26].

LEMMA. *Let $\mathcal{A}$ be a full subcategory of $\mathrm{ind}\Lambda$ closed under predecessors. Then $\mathrm{add}\mathcal{A}$ is contravariantly finite if and only if $\mathrm{add}\mathcal{A}^c$ is covariantly finite.*

2.6. The main result of this section is the following theorem.

THEOREM. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three disjoint full subcategories of $\mathrm{ind}\Lambda$ such that $\mathrm{ind}\Lambda = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and assume that $\mathcal{B}$ separates $\mathcal{A}$ from $\mathcal{C}$. Then $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a separated trisection with $\mathrm{add}\mathcal{B}$ functorially finite if and only if the following two conditions hold:*

    (a) *$\mathrm{add}\mathcal{A}$ is contravariantly finite and closed under predecessors, and*
    (b) *$\mathrm{add}\mathcal{C}$ is covariantly finite and closed under successors.*

Proof. Necessity. We claim that $\mathrm{add}(\mathcal{A} \cup \mathcal{B})$ is contravariantly finite. Let $X \in \mathrm{ind}\Lambda$. If $X \in \mathcal{A} \cup \mathcal{B}$, then the identity $1_X : X \to X$ is a right $\mathrm{add}(\mathcal{A} \cup \mathcal{B})$-approximation. If $X \notin \mathcal{A} \cup \mathcal{B}$, then $X \in \mathcal{C}$. Since $\mathrm{add}\mathcal{B}$ is contravariantly finite, then X has a right $\mathrm{add}\mathcal{B}$-approximation $f_\mathcal{B} : X_\mathcal{B} \to X$ with $X_\mathcal{B} \in \mathrm{add}\mathcal{B} \subset \mathrm{add}(\mathcal{A} \cup \mathcal{B})$. Let $g : Y \to X$ be a morphism, with $Y \in \mathrm{add}(\mathcal{A} \cup \mathcal{B})$. We may assume that Y is indecomposable. If $Y \in \mathcal{B}$, then g factors through $f_\mathcal{B}$. Suppose that $Y \in \mathcal{A}$. Since $\mathcal{B}$ separates $\mathcal{A}$ from $\mathcal{C}$, there exists $Z \in \mathrm{add}\mathcal{B}$ and morphisms $g_2 : Y \to Z$ and $g_1 : Z \to X$ such that $g = g_1 g_2$. Since g_1 factors through $f_\mathcal{B}$, we have established our claim. Applying now (2.5) yields that $\mathrm{add}\mathcal{C}$ is covariantly finite. Dually, $\mathrm{add}\,\mathcal{A}$ is contravariantly finite. The other statements in (a),(b) are evident.

Sufficiency. Since $\mathcal{A}$ is closed under predecessors, $\mathcal{C}$ is closed under successors and the subcategories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are disjoint, we easily infer that

$$\mathrm{Hom}_\Lambda(\mathcal{C}, \mathcal{B}) = \mathrm{Hom}_\Lambda(\mathcal{C}, \mathcal{A}) = \mathrm{Hom}_\Lambda(\mathcal{B}, \mathcal{A}) = 0$$

and so $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a trisection of $\mathrm{ind}\,\Lambda$. It remains to show that $\mathrm{add}\mathcal{B}$ is functorially finite.

It follows from the hypothesis that $\mathcal{A} \cup \mathcal{B}$ is closed under predecessors. Since $\mathrm{add}\mathcal{C}$ is covariantly finite, then $\mathrm{add}(\mathcal{A} \cup \mathcal{B})$ is contravariantly finite by Lemma (2.5). Let $X \in \mathrm{ind}\Lambda$, then there exists a right $\mathrm{add}(\mathcal{A} \cup \mathcal{B})$-approximation $f_{\mathcal{A} \cup \mathcal{B}} : X_{\mathcal{A} \cup \mathcal{B}} \to X$

with $X_{\mathcal{A}\cup\mathcal{B}} \in \mathrm{add}(\mathcal{A}\cup\mathcal{B})$. Write $X_{\mathcal{A}\cup\mathcal{B}} = X_{\mathcal{A}} \oplus X_{\mathcal{B}}$ with $X_{\mathcal{A}} \in \mathrm{add}\mathcal{A}$ and $X_{\mathcal{B}} \in \mathrm{add}\mathcal{B}$. Let $\iota : X_{\mathcal{B}} \to X_{\mathcal{A}\cup\mathcal{B}}$ be the natural inclusion and define $f_{\mathcal{B}} : X_{\mathcal{B}} \to X$ to be the composition $f_{\mathcal{A}\cup\mathcal{B}} \circ \iota$. Let $g : Y \to X$ be a morphism with $Y \in \mathrm{add}\mathcal{B}$. Since $Y \in \mathrm{add}(\mathcal{A}\cup\mathcal{B})$, then g factors through $f_{\mathcal{A}\cup\mathcal{B}}$ that is, there exists $h : Y \to X_{\mathcal{A}\cup\mathcal{B}}$ such that $g = f_{\mathcal{A}\cup\mathcal{B}} \circ h$.

Since $Y \in \mathrm{add}\mathcal{B}$, then $\mathrm{Hom}_{\Lambda}(Y, X_{\mathcal{A}}) = 0$. Therefore $\mathrm{Im}h \subset X_{\mathcal{B}}$ and we have the required factorisation by $f_{\mathcal{B}}$. This shows that $\mathrm{add}\mathcal{B}$ is contravariantly finite. Dually, $\mathrm{add}\mathcal{B}$ is covariantly finite. Hence it is functorially finite. $\qquad\square$

Note that the conditions of the theorem are satisfied by the algebra Λ of Example (2.3)(c), or by any tilted algebra with a non-semiregular connecting component.

3. Compact Trisections

3.1. Let Λ be an artin algebra. We recall that a subcategory $\mathcal{B}$ of $\mathrm{ind}\Lambda$ is called *connected* if, for any $X, Y \in \mathcal{B}$, there exists a walk $X = X_0 - X_1 - \cdots - X_t = Y$ in $\mathrm{ind}\Lambda$ with all $X_i \in \mathcal{B}$. Here, $M - N$ indicates that there exists either a morphism $M \to N$ or a morphism $N \to M$.

LEMMA. *Let Λ be an artin algebra, and $\mathcal{B}$ be a non-empty finite, connected and convex subcategory of $\mathrm{ind}\Lambda$. Then there is a unique component $\Gamma^{\mathcal{B}}$ of $\Gamma(\mathrm{mod}A)$ such that $\Gamma^{\mathcal{B}}$ intersects $\mathcal{B}$.*

Proof. If Λ is representation-finite, there is nothing to prove. Assume Λ to be representation-infinite and suppose there are two modules $M, N \in \mathcal{B}$ and a component $\Gamma \subset \Gamma(\mathrm{mod}A)$ such that $M \notin \Gamma$ and $N \in \Gamma$. Since $\mathcal{B}$ is connected, then we can suppose, without loss of generality, that there exists a morphism $f : M \longrightarrow N$ or $f : N \longrightarrow M$ in $\mathrm{mod}\Lambda$. Assume the former. Since $M \notin \Gamma$ and $N \in \Gamma$, then $f \in \mathrm{rad}_{\Lambda}^{\infty}(M, N)$. By (1.2), for each $t \geq 1$, there exists a path

$$M = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_t} M_t \xrightarrow{g_t} N$$

in $\mathrm{ind}\Lambda$ such that the M_i are pairwise non-isomorphic, $g_t f_t \cdots f_1 \neq 0$, each f_i is a composite of irreducible morphisms and $g_t \in \mathrm{rad}^{\infty}(M_t, N)$. Now, by convexity, we infer that $M_i \in \mathcal{B}$, for each $i = 1, \cdots, t$, thus getting a contradiction to the finiteness of $\mathcal{B}$. A similar proof holds for a morphism $f : N \longrightarrow M$. $\qquad\square$

3.2. LEMMA. *Let Λ be an artin algebra, and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a separated trisection of $\mathrm{ind}\Lambda$, with $\mathcal{B}$ finite and connected. Let Γ be a component of $\Gamma(\mathrm{mod}\Lambda)$.*

(a) *If $\Gamma \neq \Gamma^{\mathcal{B}}$, then either $\Gamma \subset \mathcal{A}$ or $\Gamma \subset \mathcal{C}$.*
(b) *If $\Gamma \cap \mathcal{A} \neq \emptyset$ and $\Gamma \cap \mathcal{C} \neq \emptyset$, then $\Gamma = \Gamma^{\mathcal{B}}$.*

Proof. (a) Since $\Gamma \neq \Gamma^{\mathcal{B}}$, clearly any indecomposable module in Γ lies either in $\mathcal{A}$ or in $\mathcal{C}$. There remains to show that, if $M, N \in \Gamma$, then both lie in $\mathcal{A}$ or both lie in $\mathcal{C}$. If this is not the case, then we can assume that we have an irreducible morphism $M \to N$ or $N \to M$. By definition of trisection, we can supppose that

we have an irreducible morphism $M \to N$ with $M \in \mathcal{A}$ and $N \in \mathcal{C}$. But this contradicts (2.2).

(b) Let M be a module in $\mathcal{A}$, and N be a module in $\mathcal{C}$, both lying in Γ. Then there exists a walk $M = M_0 - M_1 - \cdots - M_t = N$ (with $t \geq 1$) in Γ. Let j be the least integer such that $M_j \in \mathcal{A}$ but $M_{j+1} \notin \mathcal{A}$. Since $\mathcal{A}$ is closed under predecessors, then we have an irreducible morphism $M_j \to M_{j+1}$. But then (2.2) yields $M_{j+1} \in \mathcal{B}$. Therefore $\Gamma = \Gamma^{\mathcal{B}}$ because of (3.1). $\qquad\square$

3.3. DEFINITION. A trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is called *compact* if it is separated and $\mathcal{B}$ is finite, connected and contains all the projectives and all the injectives in $\Gamma^{\mathcal{B}}$.

REMARK. Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a compact trisection. Since $\mathcal{B}$ is finite, then it is functorially finite and so the results of Section 2 apply trivially. The converse is however not true since the trisection shown in Example 2.3(c) is not compact, but it is functorially finite.

EXAMPLES. (a) Let Λ be an artin algebra. Then, Λ is representation-finite if and only if $(\emptyset, \mathrm{ind}\Lambda, \emptyset)$ is the unique compact trisection of $\mathrm{ind}\Lambda$.

(b) In [21](2.7)p.215, Ringel considered the gluing of two tame hereditary algebras to form a tame one-relation algebra (see also [17]). Since these are tilted algebras, one may define a compact trisection by letting $\mathcal{B}$ be the separating subclass consisting of modules lying on a given complete slice.

(c) Let Λ be the radical square zero algebra given by the quiver

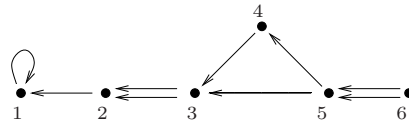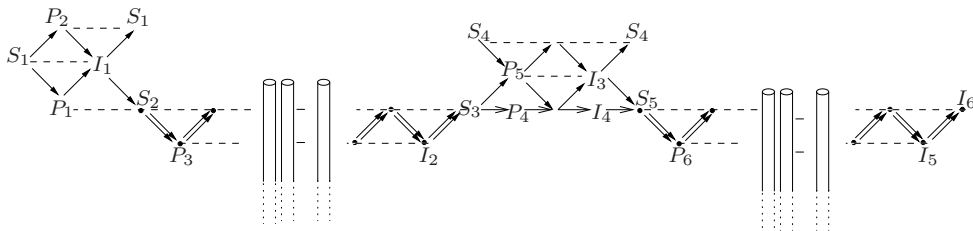
FIGURE 3.

Then $\Gamma(mod\ \Lambda)$ is of the form

FIGURE 4.

where we identify the two copies of the simple module S_1 and the two copies of the simple S_4. Here, we let $\mathcal{A}$ consist of all predecessors of τS_3 and $\mathcal{C}$ of all successors of $\tau^- S_5$.

3.4. Let Λ be an artin algebra, and let Γ be a component of $\Gamma(\text{mod}\Lambda)$. We recall from [1] that a full subquiver Σ of Γ is called a *right* (or *left*) *section* provided:

(1) Σ is acyclic,
(2) Σ is convex in Γ, and
(3) for each $Y \in \Gamma$ such that there exists a path from Σ to Y (or from Y to Σ, respectively), there exists a unique $s \geq 0$ (or $s \leq 0$, respectively) such that $\tau_\Lambda^s Y \in \Sigma$.

LEMMA. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a compact trisection in* $\text{ind}\Lambda$. *Then*

(a) *The class Σ_r of the indecomposable Ext-projectives in $\Gamma^\mathcal{B} \cap \mathcal{C}$ forms a right section in $\Gamma^\mathcal{B}$, and*
(b) *The class Σ_l of the indecomposable Ext-injectives in $\Gamma^\mathcal{B} \cap \mathcal{A}$ forms a left section in $\Gamma^\mathcal{B}$.*

Proof. We prove only (a), since the proof of (b) is dual.
(a) The acyclicity of Σ_r follows from the dual of [1](5.3), and its convexity in $\Gamma^\mathcal{B}$ follows from [6](2.2), because $\Gamma^\mathcal{B} \cap \mathcal{C}$ contains no projective modules. Let now $X \in \Gamma^\mathcal{B}$ be a successor of Σ_r and consider a walk $Y = Y_0 - Y_1 - \cdots - Y_t = X$, with $Y \notin \mathcal{C}$ and $Y_i \in \mathcal{C}$ for all i such that $1 \leq i \leq t$. Since all the projective modules of $\Gamma^\mathcal{B}$ are in $\mathcal{B}$, and applying τ_Λ if necessary, the above walk gives rise to a path from $\tau_\Lambda^r X$, for some $r \geq 0$, to a module not in $\mathcal{C}$. In particular, $\tau_\Lambda^r X \notin \mathcal{C}$. Hence there exists $s \geq 0$ such that $\tau_\Lambda^s X \in \mathcal{C}$ but $\tau_\Lambda^{s+1} X \notin \mathcal{C}$, implying that $\tau_\Lambda^s X$ is an Ext-projective of $\Gamma^\mathcal{B} \cap \mathcal{C}$. Hence $\tau_\Lambda^s X \in \Sigma_r$, as required. $\square$

3.5. COROLLARY. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a compact trisection in* $\text{ind}\Lambda$. *Then all modules in $\Gamma^\mathcal{B} \setminus \mathcal{B}$ are directed in $\Gamma^\mathcal{B}$.*

Proof. If $X \in \Gamma^\mathcal{B} \setminus \mathcal{B}$ is not directed in $\Gamma^\mathcal{B}$, then there exists a cycle $X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t = X$ between indecomposable Λ-modules on $\Gamma^\mathcal{B}$. Assume first that $X \in \mathcal{C}$. The same argument as in (3.4) yields an $r \geq 0$ such that the path

$$\tau^r X = \tau^r X_0 \longrightarrow \tau^r X_1 \longrightarrow \cdots \longrightarrow \tau^r X_t = \tau^r X$$

intersects Σ_r, contradicting its acyclicity and its convexity. Similarly if $X \in \mathcal{A}$. $\square$

3.6. COROLLARY. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a compact trisection in* $\text{ind}\Lambda$ *and let $X \in \Gamma^\mathcal{B}$.*

(a) *If $X \in \mathcal{C}$, then X has only finitely many predecessors in $\mathcal{C}$.*
(b) *If $X \in \mathcal{A}$, then X has only finitely many successors in $\mathcal{A}$.*

Proof. We only prove (a) since the proof of (b) is dual.
(a) It follows from (3.4) and (3.5) that X has only finitely many predecessors lying in $\Gamma^\mathcal{B} \cap \mathcal{C}$. We now observe that the predecessors of X in $\mathcal{C}$ are exactly those

lying in $\Gamma^{\mathcal{B}}$. Indeed, if $f \colon Y \longrightarrow X$ is a non-zero morphism with $Y \in \mathcal{C} \setminus \Gamma^{\mathcal{B}}$, then $f \in \operatorname{rad}_{\Lambda}^{\infty}(Y, X)$. But then (1.2) yields infinitely many predecessors of X lying in $\Gamma^{\mathcal{B}} \cap \mathcal{C}$, a contradiction. $\qquad\square$

3.7. Lemma. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a compact trisection in $\operatorname{ind}\Lambda$ and let $M \notin \Gamma^{\mathcal{B}}$ be an indecomposable module.*

  (a) *If $\operatorname{Hom}_{\Lambda}(M, \Gamma^{\mathcal{B}}) \neq 0$, then $M \in \mathcal{A}$.*
  (b) *If $\operatorname{Hom}_{\Lambda}(\Gamma^{\mathcal{B}}, M) \neq 0$, then $M \in \mathcal{C}$.*
  (c) *Either $\operatorname{Hom}_{\Lambda}(\Gamma^{\mathcal{B}}, M) = 0$ or $\operatorname{Hom}_{\Lambda}(M, \Gamma^{\mathcal{B}}) = 0$.*

Proof. (a) Suppose there is a non-zero morphism $f \colon M \to N$ with $N \in \Gamma^{\mathcal{B}}$. Clearly, $f \in \operatorname{rad}_{\Lambda}^{\infty}(M, N)$. So, by the dual of (1.2), there exists, for each $t \geq 1$, a path in $\operatorname{ind}\Lambda$

$$M \xrightarrow{g_t} M_t \xrightarrow{f_t} M_{t-1} \to \cdots \to M_1 \xrightarrow{f_1} M_0 = N$$

where the M_i are pairwise non-isomorphic, $f_1 \cdots f_t g_t \neq 0$, each f_i is a composition of irreducible morphisms and $g_t \in \operatorname{rad}_{\Lambda}^{\infty}(M, M_t)$. If $N \in \mathcal{A}$, then $M \in \mathcal{A}$ because $\mathcal{A}$ is closed under predecessors. If $N \in \mathcal{B}$, then the finiteness of $\mathcal{B}$ yields an $i_0 \geq 0$ such that $i \geq i_0$ implies $M_i \in \mathcal{A}$. Again, $M \in \mathcal{A}$. If $N \in \mathcal{C}$ then, by (3.6), N has only a finite number of predecessors in $\mathcal{C}$. Also, $\mathcal{B}$ is finite. Therefore, there exists $i_0 \geq 0$ such that $i \geq i_0$ implies $M_i \in \mathcal{A}$, and thus $M \in \mathcal{A}$.

(b) The proof is dual.

(c) It follows from (a) and (b). $\qquad\square$

3.8. Corollary. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a compact trisection in $\operatorname{ind}\Lambda$, then $\Gamma^{\mathcal{B}}$ is convex in $\operatorname{ind}\Lambda$, and quasi-directed.*

Proof. Let $M = M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{t-1} \longrightarrow M_t = N$ be a path in $\operatorname{ind}\Lambda$, with $t \geq 2$, $M, N \in \Gamma^{\mathcal{B}}$ and $M_i \notin \Gamma^{\mathcal{B}}$ for all i such that $1 \leq i < t$. If both M, N are in $\mathcal{B}$, we are done. Thus we may assume that M or N do not lie in $\mathcal{B}$. By (3.7), $M_1 \in \mathcal{C}$ and $M_{t-1} \in \mathcal{A}$. But $\mathcal{A}$ is closed under predecessors, and $\mathcal{C}$ is closed under successors. Hence a contradiction. This proves the convexity.

Because of (3.5), any module in $\Gamma^{\mathcal{B}} \cap \mathcal{A}$ or in $\Gamma^{\mathcal{B}} \cap \mathcal{C}$ is directed in $\Gamma^{\mathcal{B}}$, hence in $\operatorname{ind}\Lambda$ (because of convexity). So, if a module of $\Gamma^{\mathcal{B}}$ is not directed, then it must lie in $\mathcal{B}$. Since $\mathcal{B}$ is finite, $\Gamma^{\mathcal{B}}$ is almost directed. By [14](4.2.3), $\Gamma^{\mathcal{B}}$ is quasi-directed (and, in particular, generalised standard). $\qquad\square$

4. Cores

4.1. The following definition is inspired by the notion of multisection of [19].

Definition. Let Λ be an artin algebra and Γ be a component of $\Gamma(\operatorname{mod}\Lambda)$. A full subcategory Δ of Γ is a *core* of Γ (and Γ is said to *have a core*) if:

  (a) Δ is convex in $\operatorname{ind}\Lambda$.
  (b) Δ intersects each τ_{Λ}-orbit in Γ, and only finitely many times.

(c) Δ is almost directed.

EXAMPLES. (a) A complete slice in the connecting component of a tilted algebra is a core in this component.

(b) Let Λ be a representation-infinite strict laura algebra, and Γ be a connecting component of $\Gamma(\text{mod}\Lambda)$. Let Σ_l be a left section, and Σ_r be a right section of Γ. Then the convex hull $\Delta = \mathcal{C}(\Sigma_l \cup \Sigma_r)$ of Σ_l and Σ_r (that is, the full subcategory consisting of all the modules $M \in \Gamma$ such that there is a path $M' \rightsquigarrow M \rightsquigarrow M''$, with $M', M'' \in \Sigma_l \cup \Sigma_r$) is a core in Γ, which contains all the non-directed modules of Γ. Indeed, by [1](2.2), all modules which are either proper predecessors of Σ_l (or proper successors of Σ_r) in Γ are directed. Hence all non-directed modules in Γ lie in Δ. Since there are only finitely many of them (because Λ is laura), Δ is almost directed. It follows from the definition of Δ that it is convex in $\text{ind}\Lambda$. Let $X \in \Gamma$ be a proper predecessor of Σ_l (or a proper successor of Σ_r), then clearly its τ_Λ-orbit intersects Σ_l (or Σ_r, respectively) exactly once. If $X \in \Gamma$ is neither, then $X \in \Delta$ so its τ_Λ-orbit intersects Δ. It can only intersect it finitely many times because, by the description of the connecting component of a laura algebra [3], Δ is finite.

4.2. Before stating, and proving, the main result of this section, we need one last definition.

DEFINITION. Let Λ be an artin algebra, a component Γ of $\Gamma(\text{mod}\Lambda)$ is *separating* if the components distinct from Γ belong to one of two classes $\overleftarrow{\Gamma}$ and $\overrightarrow{\Gamma}$ such that, if $\Gamma' \in \overleftarrow{\Gamma}$ and $\Gamma'' \in \overrightarrow{\Gamma}$, then

   (i) $\text{Hom}_\Lambda(\Gamma, \Gamma') = 0$, $\text{Hom}_\Lambda(\Gamma'', \Gamma) = 0$, $\text{Hom}_\Lambda(\Gamma'', \Gamma') = 0$.
   (ii) Any morphism $M' \to M''$ with $M' \in \Gamma'$ and $M'' \in \Gamma''$ factors through $\text{add}\Gamma$.

As an easy example, a connecting component of a tilted algebra is separating.

THEOREM. *The following conditions are equivalent for an artin algebra Λ:*

   (a) *$\Gamma(\text{mod}\Lambda)$ admits a separating quasi-directed component Γ.*
   (b) *$\text{ind}\Lambda$ admits a compact trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.*
   (c) *$\Gamma(\text{mod}\Lambda)$ admits a separating convex component Γ having a left section Σ_l and a right section Σ_r whose convex envelope Δ is a core in Γ.*

If these conditions are satisfied, and $\Gamma' \neq \Gamma$ is a component of $\Gamma(\text{mod}\Lambda)$, then either $\Gamma' \subset \mathcal{A}$ or $\Gamma' \subset \mathcal{C}$. Moreover,

   (i) *$Hom_\Lambda(\Gamma', \Gamma) \neq 0$ implies $\Gamma' \subset \mathcal{A}$;*
   (ii) *$Hom_\Lambda(\Gamma, \Gamma') \neq 0$ implies $\Gamma' \subset \mathcal{C}$.*

Proof. (a) $\Rightarrow$ (b). Assume that Λ is an algebra having a separating quasi-directed component Γ. We may assume that Λ is representation-infinite. In particular, the left stable or the right stable part of Γ is infinite or both are. Let $\{_i\Sigma\}$ be

a family of sections in the connected components of the left stable part $_l\Gamma$ and let $\{\Sigma'_j\}$ be a family of sections in the connected components of the right stable part Γ_r. Next, we define $\mathcal{B}$ to consist of all the indecomposable modules which are neither proper predecessors of the $_i\Sigma$, nor proper successors of the Σ'_j (that is, lying between the $_i\Sigma$ and the Σ'_j). We define $\Gamma \cap \mathcal{A}$ (or $\Gamma \cap \mathcal{C}$) to consist of all the indecomposables in Γ which are proper predecessors of the $_i\Sigma$ (or proper successors of the Σ'_j, respectively). Also, if $\Gamma' \neq \Gamma$ is a component of $\Gamma(\mathrm{mod}\Lambda)$, we let $\Gamma' \subset \mathcal{A}$ if $\Gamma' \in \overleftarrow{\Gamma}$ and $\Gamma' \subset \mathcal{C}$ if $\Gamma' \in \overrightarrow{\Gamma}$. By [27](2.4.1 and 2.4.2) we have that $\mathcal{B}$ is finite. It is then easily seen that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a compact trisection of $\mathrm{ind}\Lambda$.

(b) $\Rightarrow$ (c). By (3.8), the component $\Gamma = \Gamma^{\mathcal{B}}$ is convex (and even quasi-directed). Let $\overleftarrow{\Gamma}^{\mathcal{B}}$ and $\overrightarrow{\Gamma}^{\mathcal{B}}$ consist of the components of $\Gamma(\mathrm{mod}\Lambda)$ distinct from $\Gamma^{\mathcal{B}}$, and lying in $\mathcal{A}$ and $\mathcal{C}$, respectively (this definition makes sense because of (3.2)). It is then evident that $\Gamma^{\mathcal{B}}$ is separating.
By (3.4), the class Σ_l (or Σ_r) of the Ext-injectives in $\Gamma \cap \mathcal{A}$ (or of the Ext-projectives in $\Gamma \cap \mathcal{C}$) is a left section (or a right section, respectively). Replacing each of Σ_l and Σ_r, if necessary, by their Auslander-Reiten translates, we may assume that no projective and no injective module in $\Gamma^{\mathcal{B}}$ is a predecessor of Σ_l, or a successor of Σ_r. Since the convex hull Δ of Σ_l and Σ_r is clearly connected and convex in $\mathrm{ind}\Lambda$ (because it is convex in the convex component $\Gamma^{\mathcal{B}}$), we have to show that Δ intersects each τ_Λ-orbit in $\Gamma^{\mathcal{B}}$ finitely many times, and that it is almost directed. Note first that, since $\mathcal{B}$ is a finite subcategory of the quasi-directed component $\Gamma^{\mathcal{B}}$, then $\mathcal{B}$ has finitely many τ_Λ-orbits.
We claim that $\Delta = \Sigma_l \cup \mathcal{B} \cup \Sigma_r$. This yields the result because: (i) each of $\Sigma_l, \mathcal{B}$ and Σ_r is finite; (ii) Σ_l crosses each τ_Λ-orbit of $\Gamma \cap \mathcal{A}$; (iii) Σ_r crosses each τ_Λ-orbit of $\Gamma \cap \mathcal{C}$; (iv) all the non-directing modules of Γ lie in $\mathcal{B}$ by (3.5). Observe first that $\mathcal{B} \subset \Delta$. Indeed, if $X \in \mathcal{B} \setminus \Delta$, then X precedes Σ_l or succedes Σ_r. In the first case, X precedes an Ext-injective in $\Gamma \cap \mathcal{A}$. Therefore $X \in \mathcal{A}$. Similarly, in the second case, $X \in \mathcal{C}$ and we are done. So $\Sigma_l \cup \mathcal{B} \cup \Sigma_r \subset \Delta$.
Now, if $X \in \Delta \setminus (\Sigma_l \cup \Sigma_r)$, then there exists a path $E' \rightsquigarrow X \rightsquigarrow U'$ of irreducible morphisms, with $E' \in \Sigma_l$ and $U' \in \Sigma_r$. We may assume that the immediate successor of E' on this path in not is Σ_l. By (2.2), it lies in $\mathcal{B}$. Similarly for the immediate predecessor of U'. But this implies that $X \in \mathcal{B}$ and the claim is proven.

(c) $\Rightarrow$ (a) Let $\mathcal{A}$ consist of the predecessors of Σ_l, $\mathcal{B} = \Delta \setminus (\Sigma_l \cup \Sigma_r)$ and $\mathcal{C}$ consist of the successors of Σ_r. Clearly, $\Gamma^{\mathcal{B}} = \Gamma$. By hypothesis, Γ is convex and almost directed. By [14](4.2.3), it is quasi-directed. This proves (a)
The last statement follows from (3.2). $\square$

As a consequence, we have a description of the Auslander-Reiten components of $\Gamma(\mathrm{mod}\Lambda)$ where Λ admits a compact trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. There is, first, the

separating quasi-directed convex component $\Gamma^{\mathcal{B}}$, and the other components belong either to $\mathcal{A}$ or to $\mathcal{C}$ (according as to whether they precede, or succeed, $\Gamma^{\mathcal{B}}$). Let $A = \text{End}(\bigoplus_{P_x \in \mathcal{A}} P_x)$ be the support algebra of $\mathcal{A}$ (see [1]), then any component of $\Gamma(\text{mod}\Lambda)$ which lies in $\mathcal{A}$ is also a component of $\Gamma(\text{mod}A)$. Dually, a component of $\Gamma(\text{mod}\Lambda)$ which lies in $\mathcal{C}$ is also an Auslander-Reiten component of the support algebra of $\mathcal{C}$.

4.3. COROLLARY. *Let Λ be an algebra admitting a compact trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Let $\Lambda_{\mathcal{B}} = \text{End } P$, where P is the direct sum of all indecomposable projective Λ-modules P_x such that P_x and I_x both lie in $\mathcal{B}$. Then*

(a) *$\text{ind}\Lambda_{\mathcal{B}} \subset \mathcal{B}$ and so $\Lambda_{\mathcal{B}}$ is representation-finite.*
(b) *$\Lambda_{\mathcal{B}}$ is a full convex subcategory of Λ.*

Proof. (a) Let M be an indecomposable $\Lambda_{\mathcal{B}}$-module and $e_x \in \Lambda_{\mathcal{B}}$ be a primitive idempotent such that $Me_x \neq 0$. Then there exist non-zero morphisms $P_x \to M \to I_x$. Since $P_x, I_x \in \mathcal{B}$, the convexity of $\mathcal{B}$ gives $M \in \mathcal{B}$. Hence $\text{ind}\Lambda_{\mathcal{B}} \subset \mathcal{B}$.

(b) Let $x = x_0 \to x_1 \to \cdots \to x_t = y$ be a path in the quiver of Λ, with x, y in the quiver of $\Lambda_{\mathcal{B}}$. Then we have a path of projectives

$$P_y = P_{x_t} \to \cdots \to P_{x_1} \to P_{x_0} = P_x.$$

Since $P_y, P_x \in \mathcal{B}$, then $P_{x_i} \in \mathcal{B}$ for every i. Similarly, $I_{x_i} \in \mathcal{B}$ for every i. Therefore x_i belongs to the quiver of $\Lambda_{\mathcal{B}}$. $\square$

4.4. Let Λ admit a compact trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, the component $\Gamma^{\mathcal{B}}$ is not in general non-semiregular, in contrast with the case of the strict laura algebras. Let indeed Λ be the endomorphism algebra of a regular tilting module over the path algebra of a (wild) quiver Q, and $\mathcal{B}$ consist of a complete slice, then $\Gamma^{\mathcal{B}}$ is of the form $\mathbb{Z}Q$.

PROPOSITION. *Let Λ be an algebra admitting a compact trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, then we have one of two cases:*

(a) *either $\Gamma^{\mathcal{B}}$ is non-semiregular, in which case the algebra $\Lambda/\text{Ann } \Gamma^{\mathcal{B}}$ is a laura algebra;*
(b) *or $\Gamma^{\mathcal{B}}$ is the connecting component of the algebra $\Lambda/\text{Ann } \Gamma^{\mathcal{B}}$ which is tilted.*

Proof. Let $\Lambda' = \Lambda/\text{Ann}\Gamma^{\mathcal{B}}$. Then $\Gamma^{\mathcal{B}}$ is a component of $\Gamma(\text{mod}\Lambda')$. It is faithful, because of the definition of Λ', and generalised standard, by [23](Lemma 2). Therefore, it is quasi-directed as a component of $\Gamma(\text{mod}\Lambda')$. If it is not non-semiregular, then either it is regular, or it is semiregular. In the first case, it follows from [24] that it is of the form $\mathbb{Z}Q$ (for some wild quiver Q), and is the connecting component of $\Gamma(\text{mod}\Lambda')$. In the second case, it follows from [27](Theorem 1), that it is also the connecting component of $\Gamma(\text{mod}\Lambda')$ (in particular, it can be embedded in a quiver of the form $\mathbb{Z}Q$, for some quiver Q). The

conclusion then follows from the observation that, if $\Gamma^{\mathcal{B}}$ is non-semiregular, then, by [27](Theorem 2), Λ' is a laura algebra. $\qquad\square$

Again in contrast with the case of laura algebras, the component $\Gamma^{\mathcal{B}}$ is generally not faithful. For an example, we refer to Section 7.

5. Laura Algebras and Trisections

5.1. In this section, we are interested in a particular trisection of the module category. Let Λ be an artin algebra, we let E be the direct sum of a complet set of representatives of the Ext-injectives in $\text{add}(\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda)$, and U be the direct sum of a complete set of representatives of the Ext-projectives in $\text{add}\,(\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)$. We set

$$\mathcal{L}'_\Lambda = (\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda) \setminus \text{ind}\,E,$$
$$\mathcal{R}'_\Lambda = (\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda) \setminus \text{ind}\,U,$$
$$\mathcal{B} = \text{ind}\Lambda \setminus (\mathcal{L}'_\Lambda \cup \mathcal{R}'_\Lambda).$$

Lemma. *The triple $(\mathcal{L}'_\Lambda, \mathcal{B}, \mathcal{R}'_\Lambda)$ is a trisection in $\text{ind}\Lambda$.*

Proof. Clearly, $\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda$ is closed under predecessors. Let $Y \in \mathcal{L}'_\Lambda$ and $X \rightsquigarrow Y$ be a path in $\text{ind}\Lambda$. Since $Y \in \mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda$, then $X \in \mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda$. Assume thus that $X \in \text{ind}E$. Since Y belongs to $\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda$, we have $Y \in \text{ind}E$ as well, because of [1](6.3). This contradiction shows that $X \notin \text{ind}E$, that is, $X \in \mathcal{L}'_\Lambda$. Similarly, $\mathcal{R}'_\Lambda$ is closed under successors. The statement now follows easily. $\qquad\square$

5.2. **Lemma.** *Let Λ be a laura algebra, then the trisection $(\mathcal{L}'_\Lambda, \mathcal{B}, \mathcal{R}'_\Lambda)$ is separated. If moreover Λ is strict, then $\mathcal{B}$ is finite.*

Proof. If Λ is quasi-tilted, then the trisection $(\mathcal{L}'_\Lambda, \mathcal{B}, \mathcal{R}'_\Lambda)$ is clearly separated. We may thus assume that Λ is strict laura. In this case, it follows from [3](4.6) that $\mathcal{B}$ is finite and that $\text{add}\mathcal{L}'_\Lambda$ is contravariantly finite while $\text{add}\mathcal{R}'_\Lambda$ is covariantly finite. Let $f\colon X \to Y$ be a non-zero morphism with $X \in \mathcal{L}'_\Lambda$ and $Y \in \mathcal{R}'_\Lambda$. By [1](8.4), there exists $E_1 \in \text{add}E$ such that f factors through E_1. Since $E_1 \in \text{add}\mathcal{B}$, then the trisection is separated. $\qquad\square$

5.3. We say that a path in $\text{ind}\Lambda$ is *finite* provided every morphism in this path belongs to a finite power of the radical of $\text{mod}\Lambda$.

Lemma. *Let Λ be an artin algebra, and $\mathcal{B}$ be an almost directed full convex subcategory of $\text{ind}\Lambda$. Then any path in $\mathcal{B}$ is finite.*

Proof. Let $X = X_0 \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_t} X_t = Y$ be a path in $\mathcal{B}$. If it is not finite, then there exists i with $1 \le i \le t$ such that $f_i \in \text{rad}^\infty_\Lambda(X_{i-1}, X_i)$. Then, by [14](4.2.2), there exist infinitely many non-isomorphic non-directed modules lying on a path from X_{i-1} to X_i. Since $\mathcal{B}$ is convex in $\text{ind}\Lambda$, all these modules lie in $\mathcal{B}$, contradicting the hypothesis that $\mathcal{B}$ is almost directed. $\qquad\square$

5.4. Before stating, and proving, our main result of this section, we recall from [3](2.4) that an algebra Λ is laura if and only if any path in indΛ from an indecomposable projective to an indecomposable injective is finite.

THEOREM. *Let Λ be a representation-infinite artin algebra. The following conditions are equivalent.*

(a) *Λ is a strict laura algebra, or a tilted algebra with a non-semiregular connecting component.*
(b) *$(\mathcal{L}'_\Lambda, \mathcal{B}, \mathcal{R}'_\Lambda)$ is a compact trisection.*
(c) *$(\mathcal{L}'_\Lambda, \mathcal{B}, \mathcal{R}'_\Lambda)$ is a separated trisection with add$\mathcal{B}$ functorially finite and $\mathcal{B}$ almost directed.*

Proof. (a) $\Rightarrow$ (b). Assume that Λ is strict laura. By (5.2), $(\mathcal{L}'_\Lambda, \mathcal{B}, \mathcal{R}'_\Lambda)$ is a separated trisection with $\mathcal{B}$ finite. Since Λ is representation-infinite, then the trisection is compact. The proof is similar if Λ is tilted with a non-semiregular connecting component.

(b) $\Rightarrow$ (c) This is trivial.

(c) $\Rightarrow$ (a). Assume that $(\mathcal{L}'_\Lambda, \mathcal{B}, \mathcal{R}'_\Lambda)$ satisfies the stated conditions. If all indecomposable injective modules lie in $\mathcal{R}_\Lambda$, or if all indecomposable projective modules lie in $\mathcal{L}_\Lambda$, then Λ is quasi-tilted, by [13](II-1.14). Since $\mathcal{B}$ is almost directed, it is actually tilted and we are done. By [27](3.8), its connecting component is non-semiregular.

We may thus assume that Λ is not quasi-tilted. Let $(*)\colon I \rightsquigarrow P$ be a path in ind Λ from an injective I to a projective P. We prove now that $(*)$ is finite. If $I, P \in \mathcal{B}$, then this follows from (5.3). We may thus suppose that I or P does not lie in $\mathcal{B}$.

We claim that, if $I \in \mathcal{L}'_\Lambda$, then any path in ind$\Lambda$ from I to an indecomposable not in $\mathcal{R}'_\Lambda$ is finite. Indeed, let

$$I = X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \xrightarrow{f_t} X_t = Y$$

be a path in indΛ with $Y \notin \mathcal{R}'_\Lambda$. Using that $\mathcal{L}'_\Lambda \subset \mathcal{L}_\Lambda$ and it is closed under predecessors, it follows from [1](6.1) that I has only finitely many successors in $\mathcal{L}'_\Lambda$, all of which are Ext-injectives in $\mathcal{L}'_\Lambda$. So the claim is proven if $Y \in \mathcal{L}'_\Lambda$. Suppose $Y \in \mathcal{B}$. Then there exists i with $0 \le i < t$ such that $X_i \in \mathcal{L}'_\Lambda$ and $X_{i+1} \notin \mathcal{L}'_\Lambda$. By [1](8.4), the morphism $f_i : X_i \to X_{i+1}$ factors through an indecomposable Ext-injective in $\mathcal{L}'_\Lambda$. We may thus suppose that X_i is Ext-injective in $\mathcal{L}'_\Lambda$. The morphism f_i factors through the left minimal almost split morphism $h : X_i \to \bigoplus_j Z_j$. Clearly, we may assume that there exists j_0 such that $Z_{j_0} \notin \mathcal{L}'_\Lambda$ and $\mathrm{Hom}_\Lambda(Z_{j_0}, X_{i+1}) \neq 0$. So we get a path

$$I = X_0 \rightsquigarrow X_i \xrightarrow{g} Z_{j_0} \longrightarrow X_{i+1} \rightsquigarrow X_t = Y$$

in ind Λ, with g irreducible. The subpaths $X_0 \rightsquigarrow X_i$ and $Z_{j_0} \rightsquigarrow Y$ are finite, by [1] (6.1) and (5.3), respectively (using that $Z_{j_0} \in \mathcal{B}$ by (2.2)). So the above path is finite. Since it is obtained by factoring f_i through h, then the original path is finite. This establishes our claim.

Dually, if $P \in \mathcal{R}'_\Lambda$, then any path from an indecomposable module not in $\mathcal{L}'_\Lambda$ to P is finite.

If $I \in \mathcal{L}'_\Lambda$ and $P \notin \mathcal{R}'_\Lambda$ or if $I \notin \mathcal{L}'_\Lambda$ and $P \in \mathcal{R}'_\Lambda$, then the required result follows from the claim and its dual. It remains to consider the case where $I \in \mathcal{L}'_\Lambda$ and $P \in \mathcal{R}'_\Lambda$. Since $\mathcal{B}$ is separating, we can assume the path $(*)$ to be of the form

$$ I = X_0 \to \cdots \to X_{i-1} \to X_i \to \cdots \to X_j \to X_{j+1} \to \cdots \to X_t = P $$

with $X_{i-1} \in \mathcal{L}'_\Lambda$, $X_i \in \mathcal{B}$, $X_j \in \mathcal{B}$ and $X_{j+1} \in \mathcal{R}'_\Lambda$. Applying (5.3) together with the claim and its dual, we get that $(*)$ is finite. Hence Λ is a laura algebra. It is strict because we assumed it is not quasi-tilted. $\qquad\square$

5.5. We recall that an algebra Λ is called *strict weakly shod* [12] if it is strict laura and its connecting component is directed.

COROLLARY. *Let Λ be a representation-infinite artin algebra. The following statements are equivalent:*

  (a) *Λ is strict weakly shod or tilted with a non-semiregular connecting component.*
  (b) *$(\mathcal{L}'_\Lambda, \mathcal{B}, \mathcal{R}'_\Lambda)$ is a compact trisection with $\mathcal{B}$ directed.*
  (c) *$(\mathcal{L}'_\Lambda, \mathcal{B}, \mathcal{R}'_\Lambda)$ is a separated trisection with add$\mathcal{B}$ functorially finite and $\mathcal{B}$ directed.*

6. LAURA ALGEBRAS AND CORES

6.1. LEMMA. *Let Λ be an artin algebra, and Γ be a component of $\Gamma(mod\Lambda)$.*

  (a) *If Γ contains an injective, then every module of $\Gamma \cap \mathcal{L}_\Lambda$ is directed.*
  (b) *If Γ contains a projective, then every module of $\Gamma \cap \mathcal{R}_\Lambda$ is directed.*

Proof. This follows from [1](6.5). $\qquad\square$

6.2. Let Λ be an artin algebra. As before, we denote by E (or by U) the direct sum of all indecomposable Ext-injectives in $\mathcal{L}_\Lambda$ (or Ext-projectives in $\mathcal{R}_\Lambda$, respectively). Also, we let F stand for the direct sum of all indecomposable projectives not in $\mathcal{L}_\Lambda$ and Q stand for the direct sum of all indecomposable injectives not in $\mathcal{R}_\Lambda$. Finally, Δ will denote the convex hull in indΛ of the modules in ind$(E \oplus F \oplus U \oplus Q)$.

LEMMA. *Let Λ be a connected artin algebra such that*

$$ ind\,\Lambda = (\ PredE \setminus\ ind\,E) \cup \Delta \cup (\ Succ\,U \setminus\ ind\,U). $$

  (a) *If $\mathcal{L}_\Lambda \neq \emptyset$, then $E \neq 0$ or $U \neq 0$.*
  (b) *If $\mathcal{R}_\Lambda \neq \emptyset$, then $E \neq 0$ or $U \neq 0$.*

Proof. We only prove (a), since the proof of (b) is dual.

(a) Assume $\mathcal{L}_\Lambda \neq \emptyset$, but $E = 0$ and $U = 0$. By hypothesis, ind$\Lambda = \Delta$, and Δ is the convex hull of ind$(F \oplus Q)$. Assume $X \in \mathcal{L}_\Lambda$. Since $\mathcal{L}_\Lambda \subset \Delta$, there exist $X', X'' \in$ ind$(F \oplus Q)$ such that we have paths $X' \rightsquigarrow X \rightsquigarrow X''$. Suppose $X' \in$ ind$F$. Then X' is a projective not in $\mathcal{L}_\Lambda$. However $X \in \mathcal{L}_\Lambda$ which is closed under predecessors, hence a contradiction. Therefore, $X' \in$ ind$Q$, that is, X' is an injective not in $\mathcal{R}_\Lambda$. However, $X \in \mathcal{L}_\Lambda$ implies $X' \in \mathcal{L}_\Lambda$ and since X' is injective, we get $X' \in$ ind$E$, by [1](6.1), a contradiction. $\qquad\square$

6.3. The main result of this section is the following theorem. Recall that an Ext-injective M is of the first kind if there is an injective module I and a path $I \rightsquigarrow M$. Otherwise it is called an Ext-injective of second kind (see [1] or [5]).

THEOREM. *Let Λ be a representation-infinite artin algebra which is not quasi-tilted. Then Λ is strict laura if and only if*

(a) *$\mathcal{L}_\Lambda \neq \emptyset$ or $\mathcal{R}_\Lambda \neq \emptyset$.*
(b) *There is a component Γ of $\Gamma(mod\Lambda)$ which contains a core Δ of Γ and ind$\Lambda = (PredE \setminus indE) \cup \Delta \cup (Succ\, U \setminus ind\, U)$.*

Proof. Necessity. Since Λ is laura and representation infinite, then (a) is satisfied. Let Γ be the connecting component of $\Gamma(mod\Lambda)$. Then, since Λ is strict, $Q \neq 0$ and ind$Q \subset \Gamma$. Because Γ is a convex component (see [27]), the convex hull Δ of ind$(E \oplus F \oplus U \oplus Q)$ is contained in Γ. By [2], we have $\mathcal{L}_\Lambda = $ Pred$E$ and $\mathcal{R}_\Lambda = $ Succ U. Moreover, we can decompose E in the form $E = E_1 \oplus E_2$, where E_1 (or E_2) is the direct sum of the indecomposable Ext-injectives of the first kind (or of the second kind, respectively) and every indecomposable module not in $\mathcal{L}_\Lambda$ is generated by the tilting module $V = \tau_\Lambda^{-1} E_2 \oplus E_1 \oplus F$. Dually, every indecomposable module not in $\mathcal{R}_\Lambda$ is cogenerated by the cotilting module $W = \tau_\Lambda U_2 \oplus U_1 \oplus Q$, where U_1 (or U_2) is the direct sum of the indecomposable Ext-projectives of the first kind (or of the second kind, respectively).

Let $X \notin \mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda$. Since $X \notin \mathcal{L}_\Lambda$, there exists an epimorphism $V^m \longrightarrow X$ (for some $m > 0$), and so $X \in$ Succ$(E \oplus F)$. Similarly, $X \notin \mathcal{R}_\Lambda$ gives a monomorphism $X \longrightarrow W^n$ (for some $n > 0$), implying $X \in$ Pred$(U \oplus Q)$. This shows that X lies in the convex hull Δ of ind$(E \oplus F \oplus U \oplus Q)$.

We now want to show that Δ is a core. Since $\Delta \subset \Gamma$, and Γ is quasi-directed, then Δ is almost directed. By definition, it is convex in indΛ. Thus, there remains to show that Δ crosses each τ_Λ-orbit of Γ and only finitely many times. By [2], $E \neq 0$. Clearly, $\Gamma \cap$ ind$E \neq \emptyset$ and so, by [1](6.6), each τ_Λ-orbit of $\Gamma \cap \mathcal{L}_\Lambda$ crosses ind$E$ exactly once. Dually, $U \neq 0$ and each τ_Λ-orbit of $\Gamma \cap \mathcal{R}_\Lambda$ crosses ind$U$ exactly once. Suppose $X \notin \mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda$. By the previous argument, $X \in \Delta$ and so the τ_Λ-orbit $\mathcal{O}(X)$ of X intersects Δ. Therefore, $X \in \Gamma$ implies that $|\mathcal{O}(X) \cap \Delta| \geq 1$. We have to show that $|\mathcal{O}(X) \cap \Delta| < \infty$. Since Λ is strict laura, then $|\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda| < \infty$ and $|(\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)^c| < \infty$. If there exists $X \in \Gamma$ such that $|\mathcal{O}(X) \cap \Delta| = \infty$, then there exists in $\mathcal{O}(X)$ an infinity of non-isomorphic

indecomposable modules M_i in $(\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda) \cap \Delta$ or in $(\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda) \cap \Delta$ (here, we use the fact that

$$[(\mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda) \cup (\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)]^c = (\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)^c \cup (\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda)$$

which is finite). Suppose the former case holds. Then an infinity of such modules belongs neither to $\mathrm{ind}U$, nor to $\mathrm{ind}Q$, nor to $\mathrm{ind}F$ nor to $\mathrm{ind}E$ (since the latter are finite). For each such $M_i \in \Delta$, we have a path $Y \rightsquigarrow M_i \rightsquigarrow Z$, with $Y, Z \in \mathrm{ind}(E \oplus F \oplus U \oplus Q)$. Since $M_i \in \mathcal{L}_\Lambda \setminus \mathcal{R}_\Lambda$, we have $Y \in \mathrm{ind}(F \oplus U)$. Moreover, if $Y \in \mathrm{ind}E$, then infinitely many of the M_i lie in $\mathrm{ind}E$, an absurdity. Hence $Y \in \mathrm{ind}Q$, that is, Y is injective. Since $Y \in \mathcal{L}_\Lambda$ (because $M_i \in \mathcal{L}_\Lambda$), then it is Ext-injective in $\mathcal{L}_\Lambda$ so $Y \in \mathrm{ind}E$ which is impossible as we just saw. This contradiction proves that $|\mathcal{O}(X) \cap \Delta| < \infty$ for $X \in \Gamma$. Therefore, Δ is a core. Finally, observe that $\mathrm{ind}\Lambda = (\mathrm{Pred}E \setminus \mathrm{ind}E) \cup \Delta \cup (\mathrm{Succ}U \setminus \mathrm{ind}U)$.

Sufficiency. In order to show that Λ is laura, we show that the core Δ is finite. This suffices since $\mathrm{Pred}E \subset \mathcal{L}_\Lambda$ and $\mathrm{Succ}U \subset \mathcal{R}_\Lambda$ (so that $(\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)^c \subset \Delta$). Since Λ is not quasi-tilted, there exists an indecomposable injective module I which is not in $\mathcal{R}_\Lambda$. Since $\mathrm{Succ}U \subset \mathcal{R}_\Lambda$ then $I \in \mathcal{L}_\Lambda$ or $I \in \Delta$. In both cases, $I \in \Delta$. Hence, Γ contains an injective. By (6.1), $\mathcal{L}_\Lambda \cap \Gamma$ is directed. Similarly, $\mathcal{R}_\Lambda \cap \Gamma$ is directed. Since Δ is almost directed, we infer that Γ is almost directed. We claim that Γ is convex in $\mathrm{ind}\Lambda$. Let

$$X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t$$

be a path in $\mathrm{ind}\Lambda$ with $X_0, X_t \in \Gamma$, and $X_1, \cdots, X_{t-1} \notin \Gamma$ (hence $t \geq 2$). Since $X_0 \in \Gamma$, $X_1 \notin \Gamma$, then $f_1 \in \mathrm{rad}_\Lambda^\infty(X_0, X_1)$. Since, by [1](6.7), $\Gamma \cap \mathrm{ind}E$ is a left section of Γ, then there exist $E_1 \in \mathrm{ind}E$ and a path

$$X_0 = X_0' \xrightarrow{h_1} X_1' \longrightarrow \cdots \longrightarrow X_{s-1}' \xrightarrow{h_s} X_s' = E_1 \xrightarrow{g_s} X_1$$

with the h_i irreducible, $g_s \in \mathrm{rad}_\Lambda^\infty(E_1, X_1)$ and $g_s h_s \cdots h_1 \neq 0$. Dually, there exist $U_1 \in \mathrm{ind}U$ and a path

$$X_{t-1} \xrightarrow{l_r} U_1 = X_0'' \xrightarrow{k_1} X_1'' \longrightarrow \cdots \longrightarrow X_{r-1}' \xrightarrow{k_r} X_r'' = X_t$$

with the k_i irreducible, $l_r \in \mathrm{rad}_\Lambda^\infty(X_{t-1}, U_1)$ and $k_r \cdots k_1 l_r \neq 0$. Since $E_1, U_1 \in \Delta$, the convexity of Δ and the path

$$E_1 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{t-1} \longrightarrow U_1$$

yield $X_1, \cdots, X_{t-1} \in \Delta$, hence $X_1, \cdots, X_{t-1} \in \Gamma$, a contradiction which establishes our claim.

Applying [14](4.2.3), we get that Γ is quasi-directed. In particular, Γ has only finitely many τ_Λ-orbits. Hence, so does Δ. Since Δ is a core, then it intersects each τ_Λ-orbit in Γ finitely many times. Since Δ is almost directed, then it is finite. This finishes the proof of the theorem. $\qquad\square$

6.4. COROLLARY. *Let Λ be a representation-infinite artin algebra which is not quasi-tilted. Then Λ is strict weakly shod if and only if*

  (a) *$\mathcal{L}_\Lambda \neq \emptyset$ or $\mathcal{R}_\Lambda \neq \emptyset$.*
  (b) *There exists a component Γ of $\Gamma(mod\Lambda)$ which contains a directed core Δ such that $ind\Lambda = (PredE \setminus indE) \cup \Delta \cup (SuccU \setminus indU)$.*

6.5. REMARK. If Λ is representation-infinite and strict weakly hod, then the convex hull Δ of $ind(E \oplus F \oplus U \oplus Q)$ is equal to the convex hull Δ' of $ind(E \oplus U)$. Indeed, since Λ is strict weakly shod, then $\mathcal{L}_\Lambda = PredE$ and $\mathcal{R}_\Lambda = SuccU$ by [2]. Let us show that $(\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)^c \subset \Delta'$. Let $X \notin \mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda$ be indecomposable. By [21](p.74), there exist an indecomposable P_1, an indecomposable injective I_1 and two non-sectional paths $I_1 \rightsquigarrow X$ and $X \rightsquigarrow P_1$. Note first that, if $I_1 \in \mathcal{L}_\Lambda$, then $I_1 \in indE$. Under this assumption, suppose that $P_1 \notin \mathcal{R}_\Lambda$ (otherwise, X would belong to Δ'). Then there exist an indecomposable projective P_2 and a non-sectional path $P_1 \rightsquigarrow P_2$. If $P_2 \notin \mathcal{R}_\Lambda$, we get a longer path from I_1 to P_2. Since there are only finitely many projectives, and the lengths of paths from injectives to projectives are bounded (because Λ is weakly shod), we get some $P_s \in \mathcal{R}_\Lambda$ hence $X \in \widetilde{\Delta}$.
This shows that $ind\Lambda = (PredE \setminus indE) \cup \Delta' \cup (SuccU \setminus indU)$. Since $\Delta' \subset \Delta$ and $ind\Lambda = (PredE \setminus indE) \cup \Delta \cup (SuccU \setminus indU)$, we get $\Delta' = \Delta$.

7.   QUASI-LAURA ALGEBRAS AND FURTHER GENERALISATIONS

7.1. We define a class of algebras which contains the laura algebras.

DEFINITION.   An algebra Λ is *quasi-laura* if it admits a compact trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ such that the support algebras A and C of $\mathcal{A}, \mathcal{C}$, respectively, are quasi-tilted.

  Observe that, by [15, 11], the only generalised standard component with sections of A, or C, are the postprojective, the preinjective and (eventually) the connecting component. The following is an example of a quasi-laura algebra which is not laura.

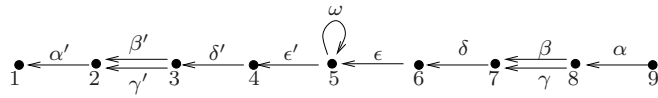EXAMPLE. Let Λ be given by the quiver

FIGURE 5.

bound by $\alpha\beta = 0, \gamma\delta = 0, \delta\epsilon = 0, \epsilon\omega = 0, \omega\epsilon' = 0, \epsilon'\delta' = 0, \delta'\gamma' = 0$ and $\beta'\alpha' = 0$.

7.2. The results of the previous sections allow us to describe the Auslander-Reiten components of a quasi-laura algebra Λ.

COROLLARY. *Let Λ be a quasi-laura algebra and Γ be a component of $\Gamma(mod\Lambda)$. Then Γ has one of the following shapes:*

(a) *It is the unique separating quasi-directed component, which is either non-semiregular or has a section.*

(b) *It is a directed component, namely postprojective, preinjective or connecting component.*

(c) *It is a stable tube or a component of type $\mathbb{Z}A_\infty$.*

(d) *It is a component obtained from a stable tube or a component of type $\mathbb{Z}A_\infty$ by finitely many ray insertions and co-insertions.*

Proof. This is a direct consequence of (4.2) and (4.4) using the well-known description of the Auslander-Reiten components of quasi-tilted algebras [16, 15, 11]. □

As can be easily seen in the above example, not only do we know the Auslander-Reiten components of a quasi-laura algebra Λ, but also the existence of a trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ as in the definition yields the global structure of $\Gamma(mod\Lambda)$. Namely, inside each of the subcategories $\mathcal{A}$ or $\mathcal{C}$, morphisms behave as in their support algebras, which are quasi-tilted. Inside $\mathcal{B}$, which is a finite subcategory of indΛ, morphisms behave as in a representation finite algebra. Also, morphims are from $\mathcal{A}$ to $\mathcal{B}$ and from $\mathcal{B}$ to $\mathcal{C}$ (and any morphism from $\mathcal{A}$ to $\mathcal{C}$ factors through add$\mathcal{B}$).

7.3. DEFINITION. Let $m \geq 0$. An algebra Λ is called *m-quasi-laura* provided

(a) If $m = 0$, then Λ is a quasi-tilted algebra.

(b) If $m \geq 1$, then Λ admits a compact trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ such that the support algebra A of $\mathcal{A}$ is a direct product of *n*-quasi-laura algebras, with $n < m$, at least one connected component of which is an $(m-1)$-quasi-laura algebra, and the support algebra C of $\mathcal{C}$ is a direct product of quasi-tilted algebras.

7.4. EXAMPLES. (a) It follows directly from the definition that the 1-quasi-laura algebras are simply the quasi-laura algebras.

(b) The following is easily seen to be an example of a 2-quasi-laura algebra. Let Λ be given by the quiver
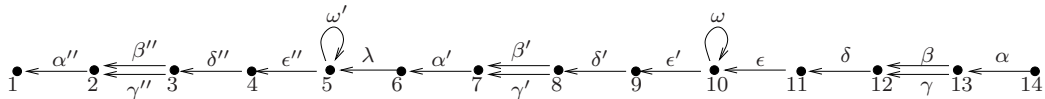

FIGURE 6.

bound by $\alpha\beta = 0, \gamma\delta = 0, \delta\epsilon = 0, \epsilon\omega = 0, \epsilon'\delta' = 0, \delta'\gamma' = 0, \beta'\alpha' = 0, \alpha'\lambda = 0, \lambda\omega' = 0, \omega'\epsilon'' = 0, \epsilon''\delta'' = 0, \delta''\beta'' = 0, \beta''\alpha'' = 0.$

7.5. COROLLARY. *Let Λ be an m-quasi-laura algebra and Γ be a component of $\Gamma(mod\Lambda)$. Then Γ has one of the following shapes:*

(a) *It is a separating quasi-directed convex component, which is either non-semiregular or has a section.*
(b) *It is a directed component, namely postprojective, preinjective or connecting component.*
(c) *It is a stable tube or a component of type $\mathbb{Z}A_\infty$.*
(d) *It is a component obtained from a stable tube or a component of type $\mathbb{Z}A_\infty$ by finitely many ray insertions and co-insertions.*

Proof. This follows from (7.2) and an obvious induction. $\square$

7.6. PROPOSITION. *Let Λ be an m-quasi-laura algebra. Then:*

(a) *there exists a separating quasi-directed convex component Γ of $\Gamma(mod\Lambda)$ such that, if $Hom_\Lambda(\Gamma', \Gamma) \neq 0$ and $\Gamma' \neq \Gamma$, then Γ' is semiregular or regular.*
(b) *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a compact trisection of Λ and $\Gamma = \Gamma^\mathcal{B}$. Then the support algebra A of $\mathcal{A}$ is quasi-tilted, and the support algebra C of $\mathcal{C}$ is n-quasi-laura, with $n \leq m$.*

Proof. We prove the statement by induction on m. If $m = 0$, the statement is clear, because we can take Γ to be a connecting component if Λ is tilted or the preinjective component otherwise. Let $m \geq 1$ and assume that the result holds for all $n < m$. Because of (4.2), Λ admits a compact trisection $(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1)$ such that $\Gamma(mod\Lambda)$ has a separating quasi-directed convex component $\Gamma^{\mathcal{B}_1}$, and all other components are components of $\Gamma(mod\ A_1)$ or $\Gamma(modC_1)$, where A_1 (or C_1) is the support algebra of $\mathcal{A}_1$ (or $\mathcal{C}_1$, respectively). Now A_1 has a connected component A_1' which is $(m-1)$-quasi-laura. By the induction hypothesis, A_1' admits a component Γ verifying the required conditions in (a). Now Γ considered as a component of $\Gamma(mod\Lambda)$ also satisfies these conditions, because of the induction hypothesis and because for $\Gamma^{\mathcal{B}_1}$, and for any component Γ'' of $\Gamma(modC_1)$, we have $Hom_\Lambda(\Gamma^{\mathcal{B}_1}, \Gamma) = 0$, $Hom_\Lambda(\Gamma'', \Gamma) = 0$. Moreover, let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a trisection such that $\Gamma = \Gamma^\mathcal{B}$, then the support algebra A of $\mathcal{A}$ is quasi-tilted. $\square$

It seems reasonable that, if Λ is m-quasi-laura, then so is Λ^{op}. However, we do not know a proof of this fact.

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Centro Politécnico, Departamento de Matemática, Universidade Federal do Paraná, CP 019081, Jardim das Américas, Curitiba-PR, 81531-990, Brazil
E-mail address: rolo@ufpr.br

Département de mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada, J1K 2R1.
E-mail address: ibrahim.assem@usherbrooke.ca

Departamento de Matemática-IME, Universidade de São Paulo, CP 66281, São Paulo, SP, 05315-970, Brazil
E-mail address: fucoelho@ime.usp.br

Depto de Matemática, FCEyN, Universidad Nacional de Mar del Plata, 7600, Mar del Plata, Argentina
E-mail address: mapena@mdp.edu.ar

Depto de Matemática, FCEyN, Universidad Nacional de Mar del Plata, 7600, Mar del Plata, Argentina
E-mail address: strepode@mdp.edu.ar