Gentle algebras arising from surface triangulations

Ibrahim Assem   Thomas Brüstle
Gabrielle Charbonneau-Jodoin   Pierre-Guy Plamondon*

Abstract

In this paper, we associate an algebra $A(\Gamma)$ to a triangulation Γ of a surface S with a set of boundary marking points. This algebra $A(\Gamma)$ is gentle and Gorenstein of dimension one. We also prove that $A(\Gamma)$ is cluster-tilted if and only if it is cluster-tilted of type $\mathbb{A}$ or $\widetilde{\mathbb{A}}$, or if and only if the surface S is a disc or an annulus. Moreover all cluster-tilted algebras of type $\mathbb{A}$ or $\widetilde{\mathbb{A}}$ are obtained in this way.

1 Introduction

Among the main recent results in the fast-growing theory of cluster algebras is the paper of Fomin, Shapiro and Thurston [16], relating triangulations of oriented surfaces to cluster algebras. This approach, which existed since the beginning of the theory [12], was followed, among others, in [24, 28]. In the same spirit, we consider in the present paper an unpunctured oriented surface S, and a finite set of points M lying on the boundary of S, and intersecting every boundary component of S. We then associate to a triangulation Γ of the marked surface (S, M) a quiver $Q(\Gamma)$, and a potential on $Q(\Gamma)$ (in the sense of [13]), thus defining an algebra $A(\Gamma)$, namely the (non-completed) Jacobian algebra defined by $Q(\Gamma)$ and the associated potential.

Such an algebra $A(\Gamma)$ has some very nice properties: it is always Gorenstein of dimension one, and also it is a gentle algebra in the sense of [5]. In the unpunctured case studied here, our definition coincides with Labardini's definition of a quiver with potential associated to a (possibly punctured) surface [24]. But in the punctured case, one does not get gentle algebras, or even string algebras. For instance, a once-punctured disc gives rise to cluster-tilted algebras of type $\mathbb{D}$, see [28].

Gentle algebras form a particularly nice subclass of the class of string algebras of [11] and are much investigated in the representation theory of algebras. For instance, this subclass contains the tilted algebras of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ (see [1] and [5], respectively) and it is closed under tilting and even under derived equivalence (see [29] and [30], respectively).

Our objective in this paper is twofold. Firstly, we ask which gentle algebras arise in this way, that is, are induced from triangulations of an unpunctured surface with boundary marked points. We show in (2.8) below that this is the case for every gentle algebra where every relation lies on what we call a 3-cycle with radical-square zero (see (2.2) or [10] for the definition). Secondly, we ask which gentle algebras are cluster-tilted. The class of cluster-tilted algebras, introduced in [9], has been much investigated and is by now well-understood (see, for instance, [2, 6, 8, 10, 12, 22, 23, 28]). In particular, it was shown in [2] that every cluster-tilted algebra is the relation-extension of a tilted algebra, that is, it is the trivial extension of a tilted algebra C by the $C - C-$bimodule $\mathrm{Ext}^2_C(DC, C)$. We may now state the main result of this paper.

Theorem 1.1 *Let $A(\Gamma)$ be the algebra associated to the triangulation Γ of an unpunctured marked surface (S, M). Then the following statements are equivalent:*

(1) $A(\Gamma)$ is cluster-tilted

(2) $A(\Gamma)$ is cluster-tilted of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$

(3) $A(\Gamma)$ is the relation-extension of a tilted algebra of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$

(4) the surface S is a disc or an annulus.

Moreover, all cluster-tilted algebras of type $\mathbb{A}$ (or $\widetilde{\mathbb{A}}$) are of the form $A(\Gamma)$ for some triangulation of a disc S (or an annulus S, respectively).

2

Actually, we prove in (3.3) that a cluster-tilted algebra is gentle if and only if it is of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$, or if and only if it is the relation-extension of a gentle tilted algebra, and the latter coincide with the tilted algebras of type $\mathbb{A}$ or $\widetilde{\mathbb{A}}$, respectively.

The case where S is a disc has already been studied in [12], and it is known that the bound quivers of all cluster-tilted algebras of type $\mathbb{A}$ arise from triangulations of the (unpunctured) disc. These algebras have also been described explicitly in [10]. Also, the potential we use for defining the cluster-tilted algebras of type $\widetilde{\mathbb{A}}$ is a particular case of the potential recently defined by B. Keller [22]. However, we do not use this fact, but rather present another proof (anterior to Keller's result), which uses [2] and properties of the second extension group.

The paper is organised as follows: in section 2, we define our algebras $A(\Gamma)$ and prove their main properties in (2.7). Section 3 is devoted to the classification of the gentle cluster-titled algebras and section 4 to the proof of our main theorem and some of its consequences. We end the paper with an example of an algebra $A(\Gamma)$ which is not of polynomial growth.

2 Algebras arising from surface triangulations

Throughout this paper, the algebras we consider are basic connected algebras over a fixed algebraically closed field k. Unless otherwise stated, all algebras are finite-dimensional. Consequently, they are given in the form $A = kQ/I$ where Q is a quiver and I is an admissible ideal of the path algebra kQ, see [4]. The pair (Q, I) is called a bound quiver, and the algebra $A = kQ/I$ is referred to as a bound quiver algebra.
Given a bound quiver algebra $A = kQ/I$, for every vertex x of Q we denote by e_x the idempotent of A associated to x. Also, P_x, I_x and S_x will be the corresponding indecomposable projective module, indecomposable injective module and simple module, respectively.

We study in this section the algebra associated with a surface triangulation. For background material on oriented surfaces we refer to [25].

2.1 The medial quiver $Q(\Gamma)$

We first recall from [16] the construction of a quiver for every triangulation of a marked surface: Let S be an oriented surface with boundary ∂S, and let M be a finite set of points on ∂S intersecting each connected component of the boundary ∂S. In this paper, we only consider the case where there are no punctures, that is we request that the set of marked points M be contained in the boundary ∂S. The pair (S, M) is referred to as an unpunctured bordered surface with marked points.
An arc γ in (S, M) is a curve in S such that

- the endpoints of γ are marked points in M

- γ does not intersect itself, except that its endpoints may coincide

- γ intersects the boundary of ∂S only in its endpoints

- γ does not cut out a monogon (that is, γ is not contractible into a point of M).

We call an arc γ a boundary arc if it cuts out a digon (that is, γ is homotopic to a curve δ on the boundary ∂S such that δ intersects M only in its endpoints). Otherwise, γ is said to be an internal arc. Each arc γ is considered up to homotopy in the class of such curves. A triangulation of (S, M) is a maximal collection Γ of arcs that do not intersect in the interior of S (more precisely, there are curves in their respective homotopy classes that do not intersect in the interior of S).

Proposition 2.1 ([16, (2.10)]) *In each triangulation of (S, M), the number of internal arcs is*

$$n = 6g + 3b + c - 6$$

where g is the genus of S, b is the number of boundary components, and $c = |M|$ is the number of marked points.

The proposition also indicates that in some cases a triangulation does not exist (for instance a disc with one marked point would give $n = -2$). We consider from now on only marked surfaces (S, M) that admit a triangulation. Given a triangulation Γ, we also refer to M as the set of vertices of Γ. The triangles are the components of $S \backslash \Gamma$ with the arcs of Γ as edges.

We denote by $Q(\Gamma)$ the medial quiver of internal arcs of Γ. That is, $Q(\Gamma)$ is the quiver whose set of points is the set of internal arcs of Γ, and the arrows are defined as follows: whenever there is a triangle T in Γ containing two internal arcs a and b, then $Q(\Gamma)$ contains an arrow $a \to b$ if a is a predecessor of b with respect to clockwise orientation at the joint vertex of a and b in T (here we use that S is an oriented surface, thus inducing an orientation on the triangle T which allows to talk about clockwise orientation).

Example. We illustrate the construction of $Q(\Gamma)$ when Γ is a triangulation of an octagon:
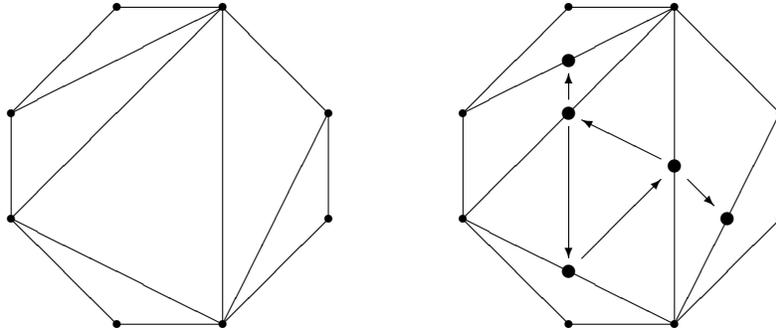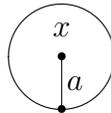

Lemma 2.2 *The quiver $Q(\Gamma)$ contains no oriented cycles of length ≤ 2.*

Proof. We first show that $Q(\Gamma)$ contains no loops. A loop α at the point a of $Q(\Gamma)$ would arise from a triangle T in Γ in the following way:
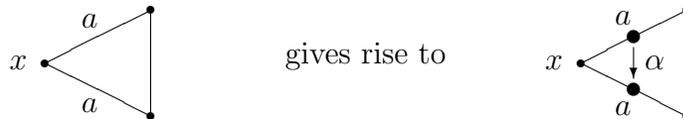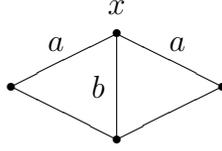

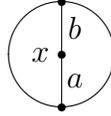But in this case the triangle T is homeomorphic to

which means that x is an internal vertex, contradicting our assumption that M is contained in the boundary of S.

We now show that $Q(\Gamma)$ contains no oriented cycles of length two. Indeed, such a cycle corresponds to the following situation in Γ:

Then a neighbourhood of x is homeomorphic to

which again contradicts the assumption that Γ contains no internal vertices.
$\square$

Remark. In [16] the authors associate a skew-symmetric matrix $B(\Gamma)$ to a triangulation Γ of (S, M). This construction is equivalent to the construction of the quiver $Q(\Gamma)$ we consider here. Since $Q(\Gamma)$ contains no oriented cycles of length ≤ 2, it is uniquely determined by a skew-symmetric matrix B (where the number of arrows between two vertices is given by the entries of B, and the direction of the arrows is determined by the sign of the matrix entries). It is easy to see that B coincides with $B(\Gamma)$. Thus all the results from [16] apply, in particular, mutations of the quiver $Q(\Gamma)$ correspond to flips of the triangulation Γ:

Let b be an internal arc of Γ. Thus b is one diagonal of the quadrilateral formed by the two triangles of Γ that contain b. A flip in b replaces the edge b by the other diagonal b^* of the same quadrilateral. Keeping all other edges unchanged, one obtains a new triangulation $\mu_b(\Gamma)$.
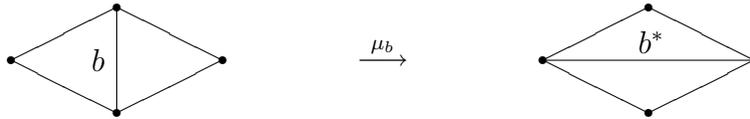
An essential ingredient in the definition of cluster algebras by Fomin and Zelevinsky [17] is the mutation of skew-symmetric matrices. Reformulated in the language of quivers, one obtains a mutation of quivers $Q \mapsto \mu_b(Q)$. The following proposition shows that flips of the triangulation commute with these quiver mutations:

Proposition 2.3 ([16, Prop 4.8]) *Suppose that the triangulation $\mu_b(\Gamma)$ is obtained from Γ by a flip replacing the diagonal labeled b. Then*

$$Q(\mu_b(\Gamma)) = \mu_b(Q(\Gamma))$$

6

2.2   The algebra $A(\Gamma)$

We define in this section an algebra $A(\Gamma)$ for each triangulation Γ of the unpunctured marked surface (S, M). Our construction generalizes the one given in [12] for polygons. An even more general case is considered by Labardini in [24], where such an algebra $A(\Gamma)$ is defined for a general marked surface (allowing punctures). If there are no punctures, the definitions coincide.
A triangle T in Γ is called an internal triangle if all edges of T are internal arcs. Every internal triangle T in Γ gives rise to an oriented cycle $\alpha_T \beta_T \gamma_T$ in $Q(\Gamma)$. We define

$$W = \sum_T \alpha_T \beta_T \gamma_T$$

where the sum runs over all internal triangles T of Γ. Then W is a potential on $Q(\Gamma)$ and we define $A(\Gamma)$ to be the (non-completed) Jacobian algebra of (Q, W) (see [13], [21]). Thus $A(\Gamma)$ can be described as a quotient $A(\Gamma) = kQ(\Gamma)/I(\Gamma)$ of the path algebra $kQ(\Gamma)$ by the ideal $I(\Gamma)$ generated by all paths $\alpha_T \beta_T$, $\beta_T \gamma_T$ and $\gamma_T \alpha_T$ whenever T is an internal triangle of Γ. In [24, Theorem 30] it is shown that flips in the triangulation correspond to mutations of the quiver with potential $(Q(\Gamma), W)$ as defined in [13].

The following result is shown in [24, Theorem 36] for the more general case of punctured marked surfaces.

Lemma 2.4 *Let Γ be a triangulation of an unpunctured marked surface (S, M). Then the algebra $A(\Gamma)$ is finite-dimensional.*

We show in Lemma 2.5 that the algebras $A(\Gamma)$ belong to a class of algebras called gentle algebras. Recall from [5] that a finite-dimensional algebra is gentle if it admits a presentation $A = kQ/I$ satisfying the following conditions on the quiver Q and the ideal I:

(G1) At each point of Q start at most two arrows and stop at most two arrows.

(G2) The ideal I is generated by paths of length 2.

(G3) For each arrow β there is at most one arrow α and at most one arrow γ such that $\alpha\beta \in I$ and $\beta\gamma \in I$.

(G4) For each arrow β there is at most one arrow α and at most one arrow γ such that $\alpha\beta \notin I$ and $\beta\gamma \notin I$.

If the pair (Q, I) satisfies conditions (G1) to (G4), we call it a gentle bound quiver, or a gentle presentation of $A = kQ/I$. Note that in contrast to [5], we do not assume that $A = kQ/I$ is triangular. An algebra $A = kQ/I$ where I is generated by paths and (Q, I) satisfies the two conditions (G1) and (G4) is called a string algebra (see [11]), thus every gentle algebra is a string algebra. The gentle algebras can be characterized by the fact that their repetitive category is special biserial [5, 26].

We recall here the classification of indecomposable modules over a string algebra $A = kQ/I$ which is given in [11] in terms of reduced walks in the quiver Q: A string in A is by definition a reduced walk w in Q avoiding the zero-relations, thus w is a sequence

$$w = x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_1} \cdots \xleftrightarrow{\alpha_{n-1}} x_n$$

where the x_i are vertices of Q and each α_i is an arrow between the vertices x_i and x_{i+1} in either direction such that w does not contain a sequence of the form $\xleftrightarrow{\beta}\xrightarrow{\beta}$ or $\xrightarrow{\beta_1} \cdots \xrightarrow{\beta_s}$ with $\beta_1 \cdots \beta_s \in I$ or their duals. A string is cyclic if the first and the last vertex coincide. A band is defined to be a cyclic string b such that each power b^n is a string, but b itself is not a proper power of some string c.

We refer to [11] for the definition of a string module $M(w)$ for every string w of A and a family of band modules $M(b, \lambda, n)$ for every band b, and each $\lambda \in k$ and $n \in \mathbb{N}$. The dimension vector $\underline{\dim} M(w)$ of the string module $M(w)$ is obtained by counting how often the string w passes through each vertex x of the quiver Q, thus

$$\underline{\dim} M(w) = (\sum_{1 \leq i \leq n} \delta_{x, x_i})_{x \in Q_0}$$

when w is the walk passing through the vertices $x_1, \ldots, x_n$ as above and $\delta_{x, x_i} = 1$ for $x = x_i$ and $\delta_{x, x_i} = 0$ otherwise.

All string and band modules are indecomposable, and in fact every indecomposable $A-$module is either a string module $M(w)$ or a band module $M(b, \lambda, n)$, see [11].

We now return to the study of algebras stemming from surface triangulations:

Lemma 2.5 *Let Γ be a triangulation of an unpunctured marked surface (S, M). Then $A(\Gamma)$ is a gentle algebra.*

Proof. By Lemma 2.4, the algebra $A(\Gamma)$ is finite-dimensional, so we only need to verify conditions (G1) to (G4) for the bound quiver $(Q(\Gamma), I(\Gamma))$ of $A(\Gamma)$.
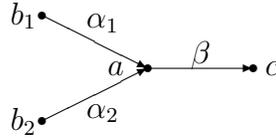
(G2) By definition, the ideal $I(\Gamma)$ is generated by paths of length two.

(G1) Let a be a point of $Q(\Gamma)$ corresponding to an internal arc a of Γ. Since Γ is a triangulation of a surface, the arc a is contained in at most two triangles:
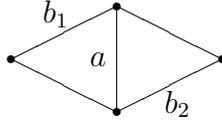
Hence there are at most two arrows $\alpha_1 : b_1 \to a$ and $\alpha_2 : b_2 \to a$ of $Q(\Gamma)$ ending in a. The same holds for arrows starting in a point a.

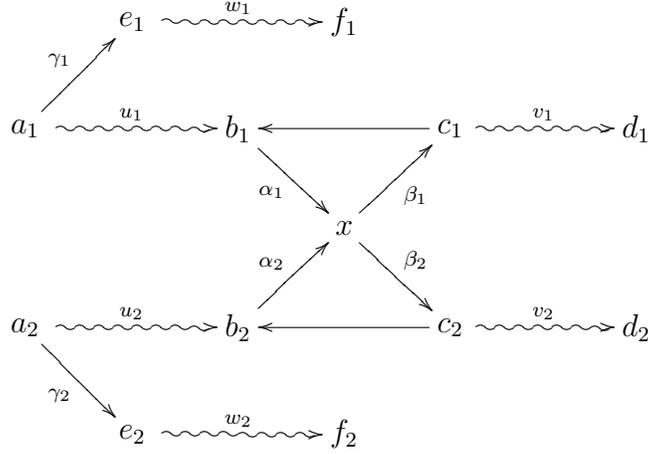(G3),(G4) Suppose now that $Q(\Gamma)$ contains $\alpha_1, \alpha_2, \beta$ as follows:

We have to show that precisely one of $\alpha_1\beta, \alpha_2\beta$ belongs to $I(\Gamma)$. In Γ, the internal arcs a, b_1, b_2 belong to two triangles as considered in the proof of (G1). The arrow β encodes that the arc c is a successor of a in one of these triangles, say the one formed by a, b_1, c. This gives rise to the relation $\alpha_1\beta$, and $\alpha_2\beta$ does not belong to $I(\Gamma)$ since α_2 and β arise from different triangles. □

From the construction of $A(\Gamma)$ it is clear that for each $\alpha\beta \in I(\Gamma)$ there is an arrow γ in $Q(\Gamma)$ such that $\beta\gamma \in I(\Gamma)$ and $\gamma\alpha \in I(\Gamma)$. In the following lemma we study a homological property of all gentle algebras satisfying this condition.

Lemma 2.6 *Let $A = kQ/I$ be a gentle algebra such that for each $\alpha\beta \in I$ there is an arrow γ in Q such that $\beta\gamma \in I$ and $\gamma\alpha \in I$. Then A is Gorenstein of dimension one.*

Proof. We have to show that for every vertex x of Q, the corresponding indecomposable injective A-module I_x has projective dimension at most one. To do so, we construct explicitly a projective resolution of I_x. We write

9

the string module I_x as $I_x = M(u_1\alpha_1\alpha_2^{-1}u_2^{-1})$ where u_1 and u_2 are oriented paths. Both paths might have length zero, and in this case, also the arrows α_1 and α_2 might not be present. The following figure is used throughout the proof:

$$
\begin{array}{ccccccc}
 & & e_1 & \xrightsquigarrow{w_1} & f_1 & & \\
 & \nearrow{\gamma_1} & & & & & \\
a_1 & \xrightsquigarrow{u_1} & b_1 & \longleftarrow & c_1 & \xrightsquigarrow{v_1} & d_1 \\
 & & & {\alpha_1}\searrow \quad \nearrow{\beta_1} & & & \\
 & & & x & & & \\
 & & & {\alpha_2}\nearrow \quad \searrow{\beta_2} & & & \\
a_2 & \xrightsquigarrow{u_2} & b_2 & \longleftarrow & c_2 & \xrightsquigarrow{v_2} & d_2 \\
 & \searrow{\gamma_2} & & & & & \\
 & & e_2 & \xrightsquigarrow{w_2} & f_2 & &
\end{array}
$$

Note that $\{x, c_1, b_1\}$ and $\{x, c_2, b_2\}$ form oriented cycles in Q such that the composition of any two consecutive arrows is zero. Let $p_0 : P(0) \to I_x$ be a projective cover, then

$$P(0) = M(w_1^{-1}\gamma_1^{-1}u_1\alpha_1\beta_2 v_2) \oplus M(w_2^{-1}\gamma_2^{-1}u_2\alpha_2\beta_1 v_1)$$

and

$$\operatorname{Ker} p_0 = M(w_1) \oplus M(w_2) \oplus M(v_1^{-1}\beta_1^{-1}\beta_2 v_2)$$

(note that some summands of the terms of this sequence can be zero). We show that $\operatorname{Ker} p_0$ is projective, thus obtaining the desired projective resolution

$$0 \longrightarrow \operatorname{Ker} p_0 \longrightarrow P(0) \xrightarrow{p_0} I_x \longrightarrow 0$$

In order to see that the first two summands of $P(1)$ are projective (namely the indecomposable projectives P_{e_1} and P_{e_2}) one has to show that there are no other arrows starting at the vertices e_1, e_2. Suppose there is an arrow $\delta_1 : e_1 \to y$ in Q. Since the algebra A is gentle, the composition $\gamma_1\delta_1$ lies in the ideal I. The assumption of the lemma guarantees the existence of a cycle $\gamma_1\delta_1\epsilon_1$ such that $\gamma_1\delta_1, \delta_1\epsilon_1, \epsilon_1\gamma_1 \in I$. But then the simple $A-$module S_y would be a composition factor of I_x, contradicting the assumption $I_x = M(u_1\alpha_1\alpha_2^{-1}u_2^{-1})$. $\qquad\square$

10

In the following Theorem we describe coverings of the bound quiver of $A(\Gamma)$. As in [10] we say that an oriented cycle $\alpha\beta\gamma$ is a 3-cycle, and by a 3-cycle with radical square zero we mean a cycle $\alpha\beta\gamma$ in an algebra kQ/I such that $\alpha\beta, \beta\gamma, \gamma\alpha \in I$. For background on Galois coverings we refer to [14].

Theorem 2.7 *Let Γ be a triangulation of an unpunctured marked surface (S, M). Then*

(1) the algebra $A(\Gamma)$ is a gentle algebra

(2) the algebra $A(\Gamma)$ is Gorenstein of dimension one

(3) there is a relation in $A(\Gamma)$ from x to y only if there is an arrow $y \to x$

(4) $A(\Gamma)$ admits a Galois covering by a quiver $\tilde{Q}$ satisfying

  (T1) Every simple cycle in $\tilde{Q}$ is a 3-cycle with radical square zero

  (T2) The only relations are those in the 3-cycles

Proof. Part (1) is shown in Lemma 2.5. Part (3) and (4) follow directly from the definition of $A(\Gamma)$. Part (2) is shown in Lemma 2.6 since the condition imposed on the gentle algebra A there clearly holds for the algebra $A(\Gamma)$. $\square$

Note that the finite quivers satisfying conditions (T1) and (T2) from the previous theorem form precisely the class of quivers $\mathcal{Q}_n$ considered in [10], where also the same relations are imposed.

2.3 Recovering topological data from $A(\Gamma)$

The condition (4) in Theorem 2.7 is very strong. Combined with the fact that the algebra is gentle, it implies the remaining conditions (2) and (3). We show in this section that a gentle algebra satisfying condition (4) actually defines an unpunctured marked surface.

First we give a different combinatorial description of the algebras studied here. Consider the following two bound quivers, where type I is a quiver of type $\mathbb{A}_2$, and type II is a 3-cycle with radical square zero:

Type I   Type II

Using these bound quivers one can construct algebras in the following way: Suppose we start with a collection C of disjoint blocks of type I and II. Choose a partial matching (that is to say a partial bijection) π of the vertices in C, where matching a vertex to itself or to another vertex of the same block is not allowed. Identifying (or "gluing") the vertices within each pair of the matching we obtain an algebra $A(C, \pi)$. We consider only matchings where the algebra $A(C, \pi)$ is connected. Note that the algebra $A(C, \pi)$ might be infinite-dimensional.

The procedure of gluing blocks is considered in a more general situation (using plenty of building blocks) in [7], where the resulting algebras are called kit algebras. A similar construction to glue blocks of type I, II and four more types is described in [16].

We show below that the gentle algebras that admit a Galois covering satisfying conditions (T1) and (T2) from Theorem 2.7 are algebras of the form $A(C, \pi)$, thus results from [16] concerning these algebras can be applied.

Proposition 2.8 *Let $A = Q/I$ be a gentle algebra where every relation lies on a 3-cycle with radical square zero. Then there exists an unpunctured marked surface (S, M) with a triangulation Γ such that $A(\Gamma) = A$.*
Moreover, the topology of (S, M) is uniquely determined by A. More precisely, for any $(S', M') \neq (S, M)$ and any triangulation Γ' of (S', M'), the quiver $Q(\Gamma')$ is not mutation-equivalent to Q.

Proof. The statement follows from [16, (14.1)] once we show that the algebra A admits a unique block decomposition $A = A(C, \pi)$ using blocks of type I and II. We define thus C to be the disjoint union of all 3-cycles with radical square zero of A together with the disjoint union of all remaining arrows from A. Denote by f the quiver morphism $f : C \to Q$ that identifies the blocks of C with their images in Q.

We first show that $|f^{-1}(x)| \leq 2$ for each vertex $x \in Q$. Indeed, if $f^{-1}(x)$ contains three different vertices, then there are three different arrows in Q adjacent to the vertex x. But since the algebra A is gentle, there has to be one relation between these three arrows. However, the set C is constructed in such a way that all relations of A belong to one of the components in C, so there are no relations between arrows corresponding to different components of C, and so the fiber $f^{-1}(x)$ contains at most two vertices.

We now define a matching π on C relating x_1 to x_2 whenever $f^{-1}(x) = \{x_1, x_2\}$. As required in the definition of $A(C, \pi)$, we do not match a vertex to itself or to some vertex in the some block. It is clear from the construction that $A = A(C, \pi)$. Moreover, the choice of blocks of type I or II is unique since all relations have to correspond to a block of type II. $\qquad\square$

3 Gentle cluster-tilted algebras

3.1 Cluster-tilted algebras

Let Δ be an acyclic quiver. In [8] the cluster category $\mathcal{C}_\Delta$ is studied in order to obtain a categorical interpretation of the cluster variables of the cluster algebra associated with Δ. It is shown in [8] that clusters correspond bijectively to tilting objects T in $\mathcal{C}_\Delta$. Their endomorphism rings $\mathrm{End}_{\mathcal{C}_\Delta}(T)$ are called cluster-tilted algebras of type Δ. They were introduced and studied in [9].

We use here a different description that has been given in [2]. Denote by A the hereditary algebra $A = k\Delta$. An A-module T is called a tilting module provided $\mathrm{Ext}_A^1(T, T) = 0$ and the number of isomorphism classes of indecomposable summands of T equals the number of isomorphism classes of simple A-modules. In this case, the endomorphism ring $\mathrm{End}_A(T)$ is called a tilted algebra of type Δ.

Let C be an algebra of global dimension two. The trivial extension

$$\tilde{C} = C \ltimes \mathrm{Ext}_C^2(DC, C)$$

of C by the $C - C-$bimodule $\mathrm{Ext}_C^2(DC, C)$ is called the *relation-extension* of C.

The following theorem allows to view cluster-tilted algebras as relation-extensions of tilted algebras.

Theorem 3.1 ([2]) *An algebra Λ is cluster-tilted of type Δ if and only if there exists a tilted algebra C of type Δ such that Λ is isomorphic to the relation-extension $\tilde{C}$ of C.*

Every cluster-tilted algebra satisfies conditions (2) and (3) from Theorem 2.7, see [23]. The bound quivers of cluster-tilted algebras of type $\mathbb{A}$ are explicitly described in [10], Prop. 3.1. In fact they were already described in [12] as the

algebras $A(\Gamma)$ arising from a triangulation of an unpunctured polygon. The following proposition is contained in [10], but it also follows from theorem 3.3 below.

Proposition 3.2 ([10, (10)]) *An algebra A is cluster-tilted of type $\mathbb{A}$ precisely when A is gentle and there is a presentation $A = kQ/I$ which satisfies conditions (T1) and (T2) from theorem 2.7.*

In particular, the cluster-tilted algebras of type $\mathbb{A}$ are gentle. We describe in the following theorem which of the gentle algebras are cluster-tilted:

Theorem 3.3 *Let $C = kQ_C/I_C$ be a tilted algebra, and $\tilde{C}$ be its relation-extension. The following are equivalent.*

1. *C is gentle;*

2. *C is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$;*

3. *$\tilde{C}$ is gentle; and*

4. *$\tilde{C}$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$.*

The rest of this section is devoted to prove Theorem 3.3.

3.2 Preliminaries

One part of the proof of the main theorem follows from a result of [29] which says that the class of gentle algebras is stable under tilting.

Lemma 3.4 *If a tilted algebra is gentle, then it is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$.*

Proof. Let Δ be a quiver such that C is tilted of type Δ. Then there exists a tilting C-module T such that $\text{End } T = k\Delta$. According to [29], $k\Delta$ is a gentle algebra. This implies that the quiver Δ is of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. $\square$

Lemma 3.5 *If $\tilde{C}$ is gentle, then so is C.*

Proof. This follows from [3, (2.7)]. $\square$

Finally, another part of the proof follows directly from 3.1.

Lemma 3.6 *C is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$ if and only if $\tilde{C}$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$.*

All that is left to do is to suppose that C is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$ and show that $\tilde{C}$ is gentle. We do this in the next subsection.

3.3 Relation-extensions of tilted algebras of types $\mathbb{A}$ and $\tilde{\mathbb{A}}$

Suppose that $C = kQ_C/I_C$ is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. Applying Lemmas 3.6 and 3.5, we get that C is gentle. Moreover, the quiver of $\tilde{C}$ is known, as are some of its relations, namely those already in C (see [2] and [3]). The aim here is to study the remaining relations of $\tilde{C}$.

First, the bound quiver of a tilted algebra of type $\mathbb{A}$ has been described in [1] and that of a tilted algebra of type $\tilde{\mathbb{A}}$ in [27]. The criterion given here is derived from [20].

We recall that a double-zero in a gentle algebra is a walk of the form $\alpha\beta\omega\gamma\delta$, where α, β, γ and δ are arrows such that $\alpha\beta$ and $\gamma\delta$ are relations, while ω is a non-zero walk (that is, a walk which does not contain any relation). Note that ω may be trivial.

Example 3.7 The algebra
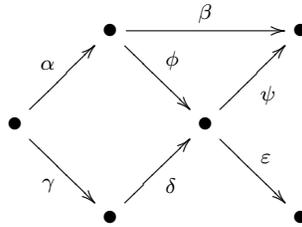
where $\alpha\beta = \phi\psi = \delta\varepsilon = 0$, is gentle with a double-zero (namely $\phi\psi\beta^{-1}\phi\psi$).
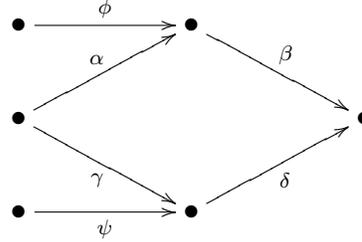
Proposition 3.8 ([1, 5])

1. *An algebra is tilted of type $\mathbb{A}$ if and only if it admits a bound quiver presentation kQ/I with (Q, I) a gentle tree with no double-zero.*

2. *An algebra is tilted of type $\tilde{\mathbb{A}}$ if and only if it admits a bound quiver presentation kQ/I with (Q, I) a gentle presentation with no double-zero and a unique (non-oriented) cycle such that, if the cycle is a band, then all arrows attached to the cycle either enter it or leave it.*

Example 3.9 Consider the algebras given by the bound quivers
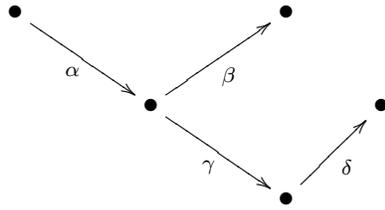
$$\alpha\beta = 0 \qquad\qquad\qquad \alpha\beta = \gamma\delta = 0$$

Using proposition 3.8, we see that the first one is tilted of type $\mathbb{A}$, while the second one is tilted of type $\tilde{\mathbb{A}}$.

3.3.1 A vanishing criterion

We need a criterion allowing to verify whether a given exact sequence represents the zero element in the second extension group. We recall the following result from [18, (II.1.3)].

Lemma 3.10 *Given a morphism $f : M \longrightarrow N$, the exact sequence*

$$0 \longrightarrow \mathrm{Ker}\, f \longrightarrow M \xrightarrow{\;f\;} N \longrightarrow \mathrm{Coker}\, f \longrightarrow 0$$

represents the zero element of $\mathrm{Ext}^2(\mathrm{Coker}\, f, \mathrm{Ker}\, f)$ if and only if there exists a module X such that the sequence

$$0 \longrightarrow M \xrightarrow{(p,g)^t} \mathrm{Im}\, f \oplus X \xrightarrow{(j,h)} N \longrightarrow 0$$

is exact, where p and j are the natural morphisms arising from f.

The following lemma will be used frequently.

Lemma 3.11 *Let (Q, I) be a gentle presentation of an algebra C, and let $\alpha : c \longrightarrow b$ and $\beta : b \longrightarrow a$ be arrows in Q. Let σ and η be strings, not passing through b, such that $\beta\sigma$ and $\eta\alpha$ are strings. Let $f : M(\beta\sigma) \longrightarrow M(\eta\alpha)$ be a morphism such that $\mathrm{Im}\, f = S_b$.*

16

Then the exact sequence

$$\mathbf{e}: \qquad 0 \longrightarrow \operatorname{Ker} f \longrightarrow M(\beta\sigma) \xrightarrow{\ f\ } M(\eta\alpha) \longrightarrow \operatorname{Coker} f \longrightarrow 0$$

represents a non-zero element of $\operatorname{Ext}_C^2(\operatorname{Coker} f, \operatorname{Ker} f)$ *if and only if* $\alpha\beta$ *lies in* I.

Proof. In view of 3.10, the sequence $\mathbf{e}$ represents a non-zero element of $\operatorname{Ext}_C^2(\operatorname{Coker} f, \operatorname{Ker} f)$ if and only if there exists no short exact sequence of the form

$$0 \longrightarrow M(\beta\sigma) \xrightarrow{\ (p,g)^t\ } \operatorname{Im} f \oplus X \xrightarrow{\ (j,h)\ } M(\eta\alpha) \longrightarrow 0,$$

where $f = jp$ is the canonical factorisation.

Assume such a sequence exists. Since S_b appears exactly once as a composition factor of $M(\beta\sigma)$ and $M(\eta\alpha)$, then it also appears exactly once as a composition factor of X. Therefore, there exists a unique indecomposable summand Y of X admitting S_b as a composition factor. Note that, since p is an epimorphism, while $(p,g)^t$ is a monomorphism, then $g : M(\beta\sigma) \longrightarrow X$ must be a monomorphism. Since the evaluation $M(\beta\sigma)_\beta$ of the module $M(\beta\sigma)$ on the arrow β is non-zero, we must have $X_\beta \neq 0$. Now, S_b is a composition factor of Y, hence $Y_\beta \neq 0$ as well. In the same way, we find that $Y_\alpha \neq 0$.

On the other hand, Y must be a string or a band module. The above reasoning implies that $\alpha\beta$ must then be a subpath of a string or a band, which implies that $\alpha\beta \notin I$, as required.

Conversely, if $\alpha\beta \notin I$, then we have a short exact sequence

$$0 \longrightarrow M(\beta\sigma) \longrightarrow S_b \oplus M(\eta\alpha\beta\sigma) \longrightarrow M(\eta\alpha) \longrightarrow 0,$$

and hence $\mathbf{e}$ represents the zero element in $\operatorname{Ext}_C^2(\operatorname{Coker} f, \operatorname{Ker} f)$. $\qquad\square$

3.3.2   Arrows

From now on, let C be a tilted algebra. We give a description of the elements of $\tilde{C} = C \ltimes \operatorname{Ext}_C^2(DC, C)$ corresponding to the arrows of its ordinary quiver. In [2, (2.4)], it is proved that the quiver of $\tilde{C}$ is obtained from that of C by adding an arrow from x to y for each relation from y to x. The elements of $\tilde{C}$

corresponding to the arrows of C are of the form $(\alpha, 0)$, where α is an arrow of C.

The other arrows correspond to relations in C. Let $\alpha\beta$ be a relation from x to y in C, and let $\xi_{\alpha\beta}$ be the corresponding new arrow in $\tilde{C}$.

Lemma 3.12 *The new arrow $\xi_{\alpha\beta}$ lies in $0 \oplus \mathrm{Ext}^2_C(I_x, P_y)$.*

Proof. This new arrow lies in $e_y\tilde{C}e_x$, which can be written as the direct sum $e_y C e_x \oplus e_y\mathrm{Ext}^2_C(DC, C)e_x$. We know from (3.8) that the quiver of C contains no double-zero. Consequently, there are no paths from y to x, and hence $e_y C e_x = 0$. Moreover, $e_y\mathrm{Ext}^2_C(DC, C)e_x = \mathrm{Ext}^2_C(I_x, P_y)$. The element $\xi_{\alpha\beta}$ thus lies in $0 \oplus \mathrm{Ext}^2_C(I_x, P_y)$. $\square$

The following lemma gives the dimension and a basis of the extension space involved in the last expression.

Lemma 3.13 *Let $\alpha : c \longrightarrow b$ and $\beta : b \longrightarrow a$ be two arrows of C such that $\alpha\beta \in I_C$.*

(a) *The dimension of the vector space $\mathrm{Ext}^2_C(I_c, P_a)$ is 1 or 2. Moreover, its dimension is 2 if, and only if, the following situation occurs in the bound quiver of C :*
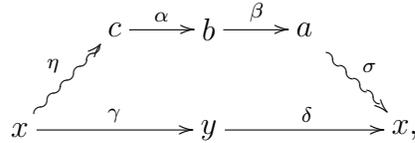
where γ and δ are arrows, η and σ are paths, possibly stationary, without relations, and $\alpha\beta, \gamma\delta$ are relations.

(b) *If the dimension of the space is 1, then a basis is given by the sequence*

$$\mathbf{e}_1 : \qquad 0 \longrightarrow P_a \longrightarrow M(\beta\sigma) \longrightarrow M(\eta\alpha) \longrightarrow I_c \longrightarrow 0,$$

where η and σ are paths such that $I_c = M(\eta)$ and $P_a = M(\sigma)$.

**(c)** *If the dimension of the space is 2, then a basis is given by the sequences*

$$\mathbf{e}_1 : \qquad 0 \longrightarrow P_a \longrightarrow M(\beta\sigma) \longrightarrow M(\eta\alpha) \longrightarrow I_c \longrightarrow 0$$

$$\mathbf{e}_2 : \qquad 0 \longrightarrow P_a \longrightarrow M(\sigma\delta^{-1}) \longrightarrow M(\gamma^{-1}\eta) \longrightarrow I_c \longrightarrow 0,$$

where γ, δ, η and σ are as in the figure in 1.

18

Proof.

(a) It is known from [2] that there is a new arrow from a to c; thus the dimension cannot be zero. On the other hand, since C is gentle and without double-zero, the situation of the relation $\alpha\beta$ can be described by the following figure, where dotted lines represent relations.
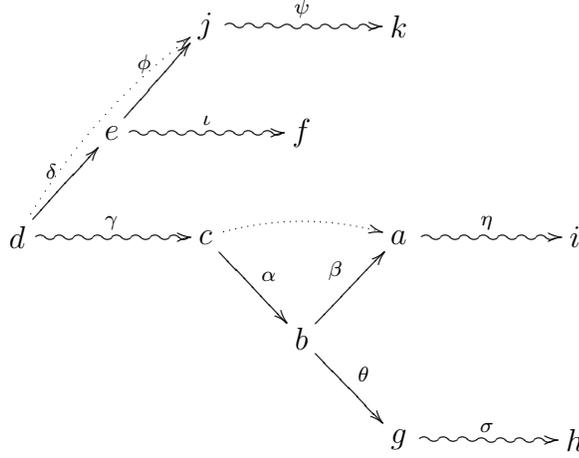
This diagram allows us to compute a projective resolution of I_c in mod C :

$$0 \xrightarrow{\ p_3\ } P(2) \xrightarrow{\ p_2\ } P(1) \xrightarrow{\ p_1\ } P(0) \xrightarrow{\ p_0\ } I_c \longrightarrow 0.$$

where $P(2) = M(\psi) \oplus M(\eta)$, $P(1) = M(\iota^{-1}\phi\psi) \oplus M(\sigma^{-1}\theta^{-1}\beta\eta)$ and $P(0) = M(\iota^{-1}\delta^{-1}\gamma\alpha\theta\sigma)$. Note that some direct summands of the terms of this sequence can be zero. Applying $\mathrm{Hom}_C(-, P_a)$, we get a complex

$$0 \longrightarrow \mathrm{Hom}_C(I_c, P_a) \xrightarrow{(p_0, P_a)} \mathrm{Hom}_C(P(0), P_a) \xrightarrow{(p_1, P_a)} \quad .$$

$$\xrightarrow{(p_1, P_a)} \mathrm{Hom}_C(P(1), P_a) \xrightarrow{(p_2, P_a)} \mathrm{Hom}_C(P(2), P_a) \xrightarrow{(p_3, P_a)} 0.$$

This yields

$$\mathrm{Ext}_C^2(I_c, P_a) = \frac{\mathrm{Ker}\,\mathrm{Hom}(p_3, P_a)}{\mathrm{Im}\,\mathrm{Hom}(p_2, P_a)} = \frac{\mathrm{Hom}(M(\psi), P_a) \oplus \mathrm{Hom}(M(\eta), P_a)}{\mathrm{Im}\,\mathrm{Hom}(p_2, P_a)}.$$

19

Since $P_a = M(\eta)$, then $\dim \operatorname{Hom}(M(\eta), P_a) = 1$, and since $\operatorname{Hom}(M(\sigma^{-1}\theta^{-1}\beta\eta), P_a) = 0$, no morphism in $\operatorname{Hom}(M(\eta), P_a)$ factors through p_2.

Moreover, $\operatorname{Hom}(M(\psi), P_a)$ is non-zero if, and only if, $j = i$. In this case, ψ has no choice but to be the trivial path in i, and $\dim \operatorname{Hom}(M(\psi), P_a) = 1$. Since $\operatorname{Hom}(M(\iota^{-1}\phi\psi), P_a) = 0$, no morphism in $\operatorname{Hom}(M(\psi), P_a)$ factors through p_2.

Hence no morphism in $\operatorname{Hom}(M(\psi), P_a) \oplus \operatorname{Hom}(M(\eta), P_a)$ factors through p_2. Thus the dimension of this space is either 1 or 2, and it is 2 exactly when $i = j$. In this case, and in this case only, we have

$$
\begin{array}{ccc}
c \xrightarrow{\alpha} b \xrightarrow{\beta} a \\
\end{array}
$$

as desired.

(b) It follows from (3.11) that $\mathbf{e}_1$ is non-zero. The result follows.

(c) It follows from (3.11) that $\mathbf{e}_1$ and $\mathbf{e}_2$ are non-zero.

It remains to be shown that $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independant. Suppose there exists a non-zero scalar λ such that $\mathbf{e}_2 + \lambda\mathbf{e}_1 = 0$. Computing this sum, we get the sequence

$$
0 \longrightarrow P_a \longrightarrow M(\beta\sigma\delta^{-1}) \xrightarrow{f} M(\gamma^{-1}\eta\alpha) \longrightarrow I_c \longrightarrow 0
$$

where all morphisms are multiples of the natural morphisms between string modules.

Here, applying (3.11) is not possible, since $\operatorname{Im} f = S_b \oplus S_y$, but a similar technique of proof can be used.

Suppose there exists a module X and morphisms g and h such that the sequence

$$
0 \longrightarrow M(\beta\sigma\delta^{-1}) \xrightarrow{(p,g)^t} (S_b \oplus S_y) \oplus X \xrightarrow{(j,h)} M(\gamma^{-1}\eta\alpha) \longrightarrow 0
$$

is exact, where $f = jp$ is the canonical factorisation. Since S_b appears exactly once as a composition factor of $M(\beta\sigma\delta^{-1})$ and $M(\gamma^{-1}\eta\alpha)$, then

it also appears exactly once as a composition factor of X. Therefore, there exists a unique indecomposable summand Y of X admitting S_b as a composition factor. As in the proof of 3.11, we show that $Y_\beta \neq 0$ and $Y_\alpha \neq 0$. Therefore, $\alpha\beta$ must be a subpath of a string or a band, which is a contradiction, since it is a relation.

The sequences $\mathbf{e}_1$ and $\mathbf{e}_2$ thus form a basis of the extension space. $\square$

It remains to determine which of the basis elements are represented by arrows of $\tilde{C}$.

Lemma 3.14 *Let $\alpha : c \longrightarrow b$ and $\beta : b \longrightarrow a$ be two arrows of the quiver of C such that $\alpha\beta$ is a relation. Let $\xi_{\alpha\beta}$ be the corresponding new arrow in $\tilde{C}$. With the notation of (3.13), the element $(0, \mathbf{e}_1)$ can be chosen to represent $\xi_{\alpha\beta}$.*

Proof. The space $0 \oplus \operatorname{Ext}^2_C(I_c, P_a)$ contains at least one arrow.
If its dimension is 1, then the result is obvious.
If its dimension is 2, then lemma (3.13) describes the situation of $\alpha\beta$ in the quiver of C. Two cases arise.
First, suppose that η and σ are both trivial paths.
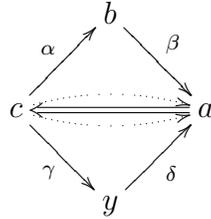
In this case, two arrows from a to c are added to the quiver. Both $(0, \mathbf{e}_1)$ and $(0, \mathbf{e}_2)$ must thus represent arrows of $\tilde{C}$.
Second, suppose η and σ are not both trivial. In this case, lemma (3.13) implies that $\operatorname{Ext}^2_C(I_x, P_z)$ is of dimension 1, and that a basis is given by

$$\underline{e'}: \qquad 0 \longrightarrow P_z \longrightarrow M(\delta) \longrightarrow M(\gamma) \longrightarrow I_x \longrightarrow 0.$$

Reasoning as above, we get that $(0, \underline{e'})$ represents the new arrow from z to x. Moreover, a straightforward calculation yields $(\sigma, 0)(0, \underline{e'})(\eta, 0) = (0, \mathbf{e}_2)$. Since one of η and σ is not trivial, one of $(0, \eta)$ and $(0, \sigma)$ must lie in rad $\tilde{C}$. Therefore $(0, \mathbf{e}_2) \in \operatorname{rad}^2\tilde{C}$, and $(0, \mathbf{e}_1) \in \operatorname{rad} \tilde{C} \backslash \operatorname{rad}^2\tilde{C}$; in other words, $(0, \mathbf{e}_1)$ represents an arrow from a to c. $\square$

3.3.3 Relations

Knowing how to write arrows in $\tilde{C}$ allows us to compute the relations.

Lemma 3.15 *Let $C = kQ_C/I_C$ and $\tilde{C} = kQ_{\tilde{C}}/I_{\tilde{C}}$.*

1. *Let $\omega_1, \omega_2, \ldots, \omega_n$ be paths from x to y in the quiver of C, and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in k$. Then $\sum_{i=1}^{n} \lambda_i(\omega_i, 0) = 0$ in $\tilde{C}$ if, and only if, $\sum_{i=1}^{n} \lambda_i \omega_i = 0$ in C.*

2. *Let $\alpha : c \longrightarrow b$ and $\beta : b \longrightarrow a$ be two arrows in the quiver of C such that $\alpha\beta$ is a relation. Let $(0, \mathbf{e}_1)$ be the element representing the corresponding new arrow, where $\mathbf{e}_1$ is as in lemma (3.13). Then $(0, \mathbf{e}_1)(\alpha, 0) = 0$ and $(\beta, 0)(0, \mathbf{e}_1) = 0$.*

3. *The ideal $I_{\tilde{C}}$ is generated by the relations of C and those described in 2.*

Proof.

1. This is shown in [3].

2. Viewing α as an element of End DC, more precisely as a morphism from I_b to I_c, we can compute $\mathbf{e}_1\beta$:

$$\mathbf{e}_1\beta : \qquad 0 \longrightarrow P_a \longrightarrow M(\beta\sigma) \longrightarrow M(\eta\alpha) \oplus M(\varphi\gamma) \longrightarrow I_b \longrightarrow 0,$$

   where $I_b = M(\eta\alpha\gamma^{-1}\varphi^{-1})$. This sequence represents the zero element, because of (3.10) and exactness of the sequence

$$0 \longrightarrow M(\beta\sigma) \longrightarrow S_b \oplus M(\varphi\gamma\beta\sigma) \oplus M(\eta\alpha) \longrightarrow M(\varphi\gamma) \oplus M(\eta\alpha) \longrightarrow 0$$

   Therfore $(0, \mathbf{e}_1)(\alpha, 0) = 0$.

   In a dual way, we prove that $(\beta, 0)(0, \mathbf{e}_1) = 0$.

3. It is sufficient to show that new arrows in the quiver of $\tilde{C}$ are not involved in other relations than those described in 2.

   First suppose that w is a monomial relation involving new arrows and other than those relations described in 2. Then it must contain exactly

one new arrow ξ, corresponding to a relation $\alpha\beta$; otherwise the quiver of C would contain a double-zero. Write $w = u\xi v$, where u and v are non-zero paths consisting of arrows of C. Let $\mathbf{e}_1$ be the sequence as in (3.13) corresponding to ξ. Then $(u,0)(0,\mathbf{e}_1)(v,0) = (0, u\mathbf{e}_1 v)$, where $u\mathbf{e}_1 v$ is the sequence

$$0 \longrightarrow M(u^{-1}u') \longrightarrow M(\beta u^{-1}u') \longrightarrow M(v'v^{-1}\alpha) \longrightarrow M(v'v^{-1}) \longrightarrow 0,$$

where u' and v' are paths in the quiver of C. The figure below illustrates the local situation, where $\alpha\beta = \gamma'\delta' = 0$, the last arrow of u and the first of t form a relation, as for the last of v and v' and the first of w' and w, respectively.
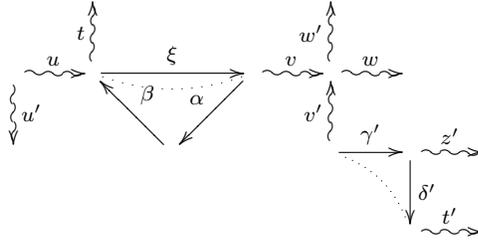
This yields the following commutative diagram, where the first line is a projective resolution of $M(v'v^{-1})$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P(2) & \xrightarrow{p_2} & P(1) & \xrightarrow{p_1} & P(0) & \xrightarrow{p_0} & M(v'v^{-1}) & \longrightarrow 0 \\
& & \downarrow{\scriptstyle (f,0)} & & \downarrow{\scriptstyle (0,g,0)} & & \downarrow{\scriptstyle (h,\ell)} & & \| & \\
0 & \longrightarrow & M(u^{-1}u') & \longrightarrow & M(\beta u^{-1}u') & \longrightarrow & M(v'v^{-1}\alpha) & \longrightarrow & M(v'v^{-1}) & \longrightarrow 0,
\end{array}
$$

where $P(2) = M(t)\oplus M(t')$, $P(1) = M(w'w)\oplus M(z^{-1}\gamma^{-1}\beta t)\oplus M(z'^{-1}\delta't')$ and $P(0) = M(z^{-1}\gamma^{-1}\alpha^{-1}vw)\oplus M(z'^{-1}\gamma'^{-1}v'w')$, and all non-zero morphisms are the natural morphisms between string modules. It is then seen that $(f,0)$ cannot factor through p_2, and thus the lower exact sequence is non-zero. Hence there are no other monomial relations than those in 2.

Now suppose we have a minimal relation of the form $\sum_{i=1}^{m} \lambda_i w_i$, where each λ_i is a non-zero scalar, each w_i is a path in the quiver of $\tilde{C}$ and

$m \geq 2$. At least one of the w_i must pass through a new arrow, and since C contains no double zero, this implies that each w_i must pass through exactly one new arrow, say ξ_i, corresponding to a relation $\alpha_i\beta_i$. Write $w_i = u_i\xi_iv_i$, where u_i and v_i are paths of the quiver of C.

Since the quiver of C contains at most one cycle, we must have $m = 2$. Since k is a field, we may suppose that $\lambda_1 = 1$. Letting $\mathbf{e}_1$ and $\mathbf{e}_2$ be the sequences associated to ξ_1 and ξ_2, respectively, we get that $u_1\mathbf{e}_1v_1$ and $\lambda_2u_2\mathbf{e}_2v_2$ both are sequences of the form above. Their sum is the sequence

$$0 \longrightarrow M(u_2^{-1}u_1) \longrightarrow M(\beta_2u_2^{-1}u_1\beta_1^{-1}) \longrightarrow$$

$$\longrightarrow M(\alpha_2^{-1}v_2v_1^{-1}\alpha_1) \longrightarrow M(v_1v_2^{-1}) \longrightarrow 0.$$

By an argument similar to the one given in the proof of (3.13)(c), this element is not zero, a contradiction. Hence no binomial relations exist in $\tilde{C}$.

$\square$

The relations described in the preceding lemma make $\tilde{C}$ a gentle algebra.

Lemma 3.16 *If $\tilde{C}$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$, then $\tilde{C}$ is gentle.*
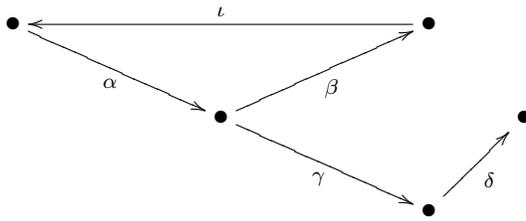
Proof. The relations of $\tilde{C}$ are known (see 3.15). Moreover, C is gentle. Suppose that there are r new arrows. Let us add the new arrows and the corresponding new relations one by one, thus obtaining a sequence $C = C_0, C_1, \ldots, C_r = \tilde{C}$ of algebras. We show that C_i is gentle for all i in $\{0, 1, 2, \ldots, r\}$.
Since C is gentle, then so is C_0. Suppose that C_i is gentle, where i is in $\{0, 1, 2, \ldots, r-1\}$. To get C_{i+1}, we add one new arrow, say γ from x to y. This arrow comes from a relation $\alpha\beta$ from y to x in C. We must add the relations $\beta\gamma$ and $\gamma\alpha$ to obtain C_{i+1}.
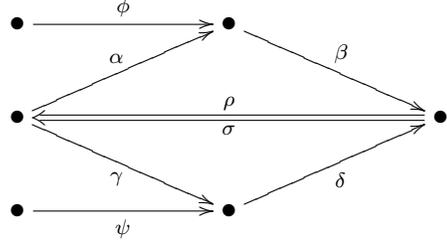Since C_i is gentle, there were already at most two arrows starting from x in C_i. Suppose that there were two, say η_1 and η_2. Since C_i is gentle, then β is involved in a relation with one of the two, say η_1. The arrow η_1 cannot be in C, otherwise there would be a double zero involving $\alpha\beta$ and $\beta\eta_1$. So

the arrow η_1 comes from a relation $\sigma\beta$ in C. Since C is gentle, we must have that $\sigma = \alpha$, so that $\eta_1 = \gamma$, which is absurd because γ is not in C_i.

Therefore, in C_i, there is at most one outgoing arrow from x, and this arrow is not involved in a relation with β. This shows that in C_{i+1}, there are at most two arrows starting from x, say η and γ, and that $\beta\eta$ is not a relation while $\beta\gamma$ is. Moreover, there is at most one more arrow ending in x, say δ, and since C_i is gentle, we have that $\delta\eta$ is a relation, while $\delta\gamma$ is not. So the relations at x are those found in a gentle algebra.

Using a similar argument for the vertex y, we get that C_{i+1} is a gentle algebra. By induction, $\tilde{C}$ is a gentle algebra. $\qquad\square$

Example 3.17 Lemma 3.15 allows us to compute the relation-extension of any gentle tilted algebra. As an illustration, consider the two algebras given in example 3.9. The relation-extension of each is given in the following diagram:

$$\alpha\beta = \iota\alpha = \beta\iota = 0 \qquad\qquad \alpha\beta = \gamma\delta = \rho\alpha = 0$$

$$\beta\rho = \sigma\gamma = \delta\sigma = 0 \qquad\qquad \cdot$$

3.3.4 Proof of the main theorem

Now the proof of theorem 3.3, which has been written in separate parts during the last sections, can be stated properly.

Proof (of theorem 3.3): 1 implies 2 is shown in (3.4), 2 implies 4 in (3.6), 4 implies 3 in (3.16) and 3 implies 1 in (3.5). $\qquad\square$

4 Geometry of surfaces and $A(\Gamma)$

We study in this section more connections between geometric properties of the marked surface (S, M) and properties of the algebra $A(\Gamma)$ given by a triangulation of (S, M).

4.1 Cluster-tilted algebras arising from surfaces

We first address the question which of the algebras $A(\Gamma)$ are cluster-tilted. Recall that all algebras $A(\Gamma)$ share the properties (2) and (3) from Theorem 2.7 with every cluster-tilted algebra. Moreover, it is shown in [12] and [10] that the cluster-tilted algebras of type $\mathbb{A}$ are algebras $A(\Gamma)$ arising from a triangulation of an unpunctured polygon. In this section, we show the following generalization:

Theorem 4.1 *Let $A(\Gamma)$ be the algebra associated to the triangulation Γ of an unpunctured marked surface (S, M). Then the following statements are equivalent:*

(1) the algebra $A(\Gamma)$ is cluster-tilted

(2) the algebra $A(\Gamma)$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$

(3) S is a disc or an annulus

Moreover, all cluster-tilted algebras of type $\mathbb{A}$ (or $\tilde{\mathbb{A}}$) are of the form $A(\Gamma)$ for some triangulation Γ of a disc S (or an annulus S, respectively).

Proof. It is clear that (2) implies (1). Let us show the converse: Suppose that the algebra $A(\Gamma)$ is cluster-tilted. Thus there is a sequence of mutations transforming the quiver with potential defining $A(\Gamma)$ into some quiver Q with zero potential. This sequence of mutations corresponds to a sequence of flips, transforming the triangulation Γ of (S, M) into a triangulation T with $Q(T) = Q$ and zero potential. Hence $A(T) = kQ$ is hereditary. Since we know from (2.7) that $A(T)$ is gentle, this leaves only the possibilities that Q is of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. Therefore the algebra $A(\Gamma)$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$ We prove now the equivalence of (2) and (3). Since all triangulations on (S, M) are flip-equivalent (see [19]) and flips of the triangulation correspond to mutations of the corresponding quiver with potential, it is sufficient to

consider one particular triangulation. In case S is a disc, we choose the triangulation to be in the form of a fan, giving rise to a linear oriented quiver of type $\mathbb{A}$. In case where S is an annulus, we choose the triangulation given by two fans in opposite direction as shown in the following figure (identify the left and right vertical edge):
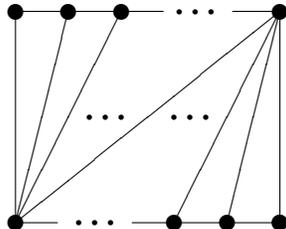
The corresponding quiver is of type $\widetilde{\mathbb{A}}$ with zero potential, thus (3) implies (2). Conversely, we know from proposition 2.8 that the quivers $Q(\Gamma)$ uniquely determine the topology of the unpunctured marked surface (S, M). Therefore S is a disc or an annulus, respectively, and since all triangulations are flip-equivalent, it is clear that all cluster-tilted algebras of the corresponding type occur. $\qquad\square$

4.2   Curves in (S, M) and string modules

In this section we are comparing strings in $A(\Gamma)$ to curves in (S, M). By a curve in (S, M) we mean a curve γ in S whose endpoints lie in M and where all points except the endpoints lie in the interior of S. We usually consider curves up to homotopy. For instance, for two distinct curves γ and δ in (S, M), the intersection number $I_\Gamma(\gamma, \delta)$ is defined as the minimal number of transversal intersections of two representatives of the homotopy classes of γ and δ. Denote the internal arcs of the triangulation Γ by $\{a_1, \ldots, a_n\}$. Then we define the intersection vector $I_\Gamma(\gamma)$ of a curve γ as

$$I_\Gamma(\gamma) = (I_\Gamma(\gamma, a_1) \ldots, I_\Gamma(\gamma, a_n))$$

Proposition 4.2 *Let Γ be a triangulation of an unpunctured marked surface (S, M). Then there exists a bijection $\{\gamma\} \mapsto w(\gamma)$ between the homotopy classes of curves in (S, M) not homotopic to an arc in Γ and the strings of $A(\Gamma)$. Under this bijection, the intersection vector corresponds to the dimension vector of the corresponding string module, that is*

$$I_\Gamma(\gamma) = \underline{\dim}\, M(w(\gamma))$$

Proof. Let $w = x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_{s-1}} x_s$ be a string in $A(\Gamma)$. We define a curve $\gamma(w)$ in (S, M) as follows: The arcs x_1 and x_2 belong to the same triangle T_1 since they are joined by an arrow in $A(\Gamma)$. We connect the midpoints of x_1 and x_2 by a curve γ_1 in the interior of T_1. Proceeding in the same way with the remaining arcs $x_2, \ldots, x_s$ we obtain curves $\gamma_2, \ldots, \gamma_{s-1}$ connecting the midpoints of the respective arcs. The internal arc x_1 belongs to two triangles, the triangle T_1 which we considered above and another triangle T_0. Let $P \in M$ be the marked point in T_0 opposite to the arc x_1. We now connect P with the midpoint of x_1 by a curve γ_0 in the interior of T_0, and proceed in the same way on the other end of the string w, connecting the midpoint of x_s with a marked point Q by some curve γ_s. The curve $\gamma(w)$ is then defined as the concatenation of the curves $\gamma_0, \ldots, \gamma_s$.
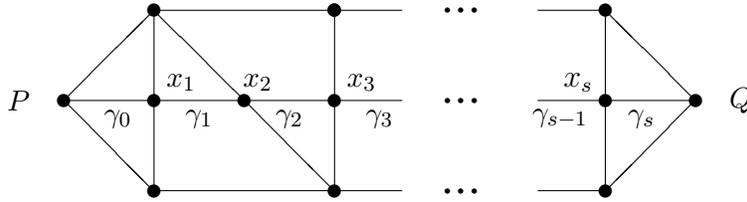
By construction the points of intersection of the curve $\gamma(w)$ with arcs in Γ are indexed by the vertices of the string w. The curve intersects the arcs of Γ transversally, and since the string w is reduced, none of the γ_i is homotopic to a piece of an arc in Γ. Thus the intersection numbers are minimal, and $I_\Gamma(\gamma(w)) = \underline{\dim}M(w)$. Since $\gamma(w)$ has non-trivial intersection with arcs of Γ, it is clear that it is not homotopic to an arc in the triangulation Γ.

Conversely, let $\gamma : [0, 1] \to S$ be a curve in (S, M) which is not homotopic to an arc in Γ. We assume that the curve γ is chosen (in its homotopy class) such that it intersects the arcs a of Γ transversally (if at all) and such that the intersection numbers $I_\Gamma(\gamma, a)$ are minimal. Orienting γ from $P = \gamma(0) \in M$ to $Q = \gamma(1) \in M$, we denote by x_1 the first internal arc of Γ that intersects γ, by x_2 the second arc, and so on. We thus obtain a sequence $x_1, \ldots, x_s$ of (not necessarily different) internal arcs in Γ. Since the intersection numbers are minimal, we know that $x_i \neq x_{i+1}$. Thus there are arrows, either $\alpha_i : x_i \to x_{i+1}$ or $\alpha_i : x_{i+1} \to x_i$ in $Q(\Gamma)$, and we obtain a walk $w(\gamma) = x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_1} \cdots \xleftrightarrow{\alpha_{s-1}} x_s$ in $Q(\Gamma)$. The fact that γ intersects the

arcs of Γ transversally implies that the walk $w(\gamma)$ is reduced and avoids the zero-relations, thus $w(\gamma)$ is a string in $A(\Gamma)$.

It follows from their construction that the two maps between strings and homotopy classes of curves defined above are mutually inverse. $\qquad\square$

Remark: Recall that two string modules $M(w)$ and $M(v)$ are isomorphic precisely when $v = w$ or $v = w^{-1}$. The inverse string w^{-1} corresponds to orienting the curve in opposite direction.

Proposition 4.3 *Let Γ be a triangulation of an unpunctured marked surface (S, M). Then there exists a bijection between the homotopy classes of closed curves in (S, M) and powers b^n of bands b of $A(\Gamma)$.*

The proof is analogous to the proof the previous proposition.

4.3 An example of non-polynomial growth

We finally present in this section an example of a gentle algebra $A(\Gamma)$ which is of non-polynomial growth. Let S be a sphere with three holes and choose one marked point in each boundary component. We fix the following triangulation Γ of (S, M) :
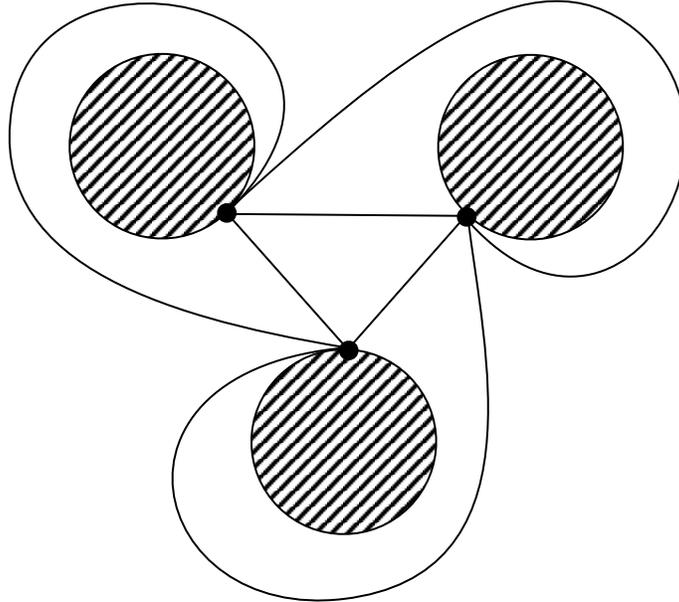
Then the algebra $A(\Gamma)$ is given by the following quiver with relations $\epsilon_i \rho_i = 0, \rho_i \sigma_i = 0$ and $\sigma_i \epsilon_i = 0$ for $i = 1$ and $i = 2$.
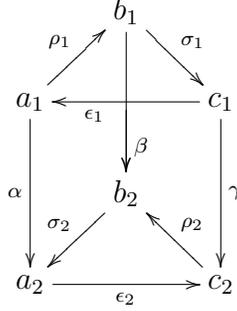
The string algebra $A(\Gamma)$ admits the following two bands

$$\xi = \; b_2 \xrightarrow{\sigma_2} a_2 \xleftarrow{\alpha} a_1 \xrightarrow{\rho_1} b_1 \xrightarrow{\beta} b_2$$

and

$$\eta = \; b_2 \xleftarrow{\rho_2} c_2 \xleftarrow{\gamma} c_1 \xleftarrow{\sigma_1} b_1 \xrightarrow{\beta} b_2$$

Since ξ and η can be composed arbitrarily, the algebra $A(\Gamma)$ is of non-polynomial growth, see [31]. Note that the algebras of non-polynomial growth are never cluster-tilted.

References

[1] I.Assem, *Tilted algebras of type A_n*, Comm. Algebra 10 (19) (1982) 2121–2139.

[2] I.Assem, Th.Brüstle and R.Schiffler *Cluster-tilted algebras as trivial extensions*, Bull. London Math. Soc. 40 (2008), 151–162.

[3] I.Assem, F.Coelho and S.Trepode, *The Bound Quiver of a Split Extension*, J. Algebra Appl. 7 (2008), no. 4, 405–423.

[4] I.Assem, D.Simson, and A.Skowroński, Elements of representation theory of associative algebras, London Math. Soc. Student Texts 65, Cambridge University Press (Cambridge) 2006.

[5] I. Assem, A.Skowroński, *Iterated tilted algebras of type* $\tilde{A}_n$, Math. Z. **195** (1987), no. 2, 269–290.

[6] M. Barot, E. Fernández, M. I. Platzeck, N. I. Pratti and S. Trepode, *From iterated tilted algebras to cluster-tilted algebras*, arXiv:0811.1328v1 [math.RT].

[7] Th. Brüstle, *Kit algebras,* Journal of Algebra 240 (2001), no. 1, 1–24.

[8] A.B.Buan, R.Marsh, M.Reineke, I.Reiten and G.Todorov, *Tilting theory and cluster combinatorics, Adv. Math.* **204** (2006), 572-612.

[9] A.B.Buan, R.Marsh and I.Reiten, *Cluster-tilted algebras*, Trans. Amer. Math. Soc. 359, No 1 (2007) 323-332.

[10] A.B. Buan and D.F.Vatne, *Derived equivalence classification for cluster-tilted algebras of type A_n*, J. Algebra 319 (2008) 2723-2738

[11] M.C.R.Butler and C.M.Ringel, *Auslander-Reiten sequences with few middle terms*, Comm. in Algebra, 15(1&2), 145-179 (1987).

[12] P. Caldero, F. Chapoton and R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. 358 , no. 3, (2006) 1347-1364.

[13] Derksen, Weyman, Zelevinsky, *Quivers with potentials and their representations I: Mutations*, Selecta Math., New Series. 14 (2008), 59–119.

[14] P.Gabriel, *The universal cover of a representation-finite algebra*, Proc. ICRA Puebla 1980, Springer Lecture Notes 903 (1981), 68–105.

[15] V.V.Fock, A.B.Goncharov, *Dual Teichmüller and lamination spaces*, Handbook of Teichmüller theory, Vol.I, 647–684.

[16] S.Fomin, M.Shapiro and D.Thurston, *Cluster algebras and triangulated surfaces. Part I: Cluster complexes*, Acta Mathematica 201 (2008), 83–146.

[17] S. Fomin, A.Zelevinsky, *Cluster algebras I. Foundations,* J. Amer. Math. Soc. 15(2) (2002) 497–529 (electronic).

[18] D.Happel, I.Reiten and S.O.Smalo, *Tilting in Abelian Categories and Quasitilted Algebras*, Memoirs of the AMS, 1996, vol. 120, 575 (3/4).

[19] A.Hatcher, *On triangulations of surfaces*, Topology and its Applications 40 (1991), 189–194.

[20] F.Huard and S.Liu, *Tilted string algebras*, J. of Pure and Applied Algebra, Volume 153, Number 2, 25 October 2000, p. 151-164(14).

[21] B.Keller, *Triangulated Calabi-Yau categories*, Proceedings of the workshop of the ICRA 12, Toruń, August 2007

[22] B.Keller, *Deformed Calabi-Yau completions and their duals,* in preparation.

[23] B.Keller, I.Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. 211 (2007), 123–131.

[24] D. Labardini-Fragoso, *Quivers with potentials associated to triangulated surfaces*, arXiv:math.RT/0608264v3.

[25] Massey, William S. *A Basic Course in Algebraic Topology*, Springer Verlag, New York, 1991.

[26] Z. Pogorzały and A. Skowroński, *Selfinjective biserial standard algebras,* J. Algebra **138** (1991), no. 2, 491–504.

[27] O.Roldán, *Tilted algebras of types* $\widetilde{A_n}, \widetilde{B_n}, \widetilde{C_n}, \widetilde{BC_n}$, Ph.D. Thesis, Carleton U. (1983).

[28] R.Schiffler, *A geometric model for cluster categories of type D_n,* J. Algebr. Comb. 27 (2008), 1–21.

[29] J.Schröer, *Modules without self-extensions over gentle algebras*, J. Algebra 216 (1999) 178–189.

[30] J.Schröer, A.Zimmermann, *Stable endomorphism algebras of modules over special biserial algebras*, Mathematische Zeitschrift 244 (2003), 515–530.

[31] A.Skowroński, *Algebras of polynomial growth,* Topics in Algebra, Banach Center Publications Vol. 26, Part 1, PWN, Warsaw (1990), 535–568.

Département de Mathématiques, Université de Sherbrooke
Sherbrooke (Québec), J1K 2R1, Canada

Department of Mathematics, Bishop's University
2600 College St., Sherbrooke, Quebec, Canada J1M 0C8

Université Denis Diderot (Paris VII) Institut de Mathmatiques
Case 7012-2, place Jussieu 75251 Paris Cedex 05

E-mail addresses:
ibrahim.assem@usherbrooke.ca
thomas.brustle@usherbrooke.ca   and   tbruestl@ubishops.ca
gabrielle.charbonneau-jodoin@usherbrooke.ca
plamondon@math.jussieu.fr