Bayes estimation based on k-record data from a general class of distributions under balanced type loss functions

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Abstract

A semi-parametric class of distributions that includes several well-known lifetime distributions such as: Exponential, Weibull (one parameter), Pareto, Burr type XII and so on is considered in this paper. Bayes estimation of parameters of interest based on k-record data under balanced type loss functions are developed; and in some cases the admissibility or inadmissibility of the linear estimators are considered. The results are presented under the balanced versions of two well-known loss functions, namely squared error loss (SEL) and Varian's linear-exponential (LINEX) loss. Some recently published results on Bayesian estimation using record data are shown to be special cases of our results.

Keywords: Admissibility; Bayes estimation; Balanced loss; Squared error loss; LINEX loss; k-records.

1 Introduction

1.1 Record data

Let $\{X_i,\ i \geq 1\}$ be a sequence of iid absolutely continuous random variables distributed according to the cumulative distribution function (cdf) $F(\cdot;\theta)$ and probability density function (pdf) $f(\cdot;\theta)$, where θ is an unknown parameter. An observation X_j is called an *upper record value* if its value exceeds all previous observations. Thus, X_j is an *upper record* if $X_j > X_i$ for every $i < j$. Analogously, an upper k-record value is defined in terms of the k-th largest X yet seen. It is of interest to note that there are situations in which only records are observed, such as in destructive stress testing, meteorology, hydrology, seismology, and mining. For a more specific example, consider the situation of testing the

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breaking strength of wooden beams as described in Glick (1978). Interest in records has increased steadily over the years since Chandler's (1952) formulation. Useful surveys are given by the books of Arnold, Balakrishnan, and Nagaraja (1998), Nevzorov (2001) and the references therein. For a formal definition of k-records, we consider the definition in Arnold, Balakrishnan and Nagaraja (1998, page 43) for the continuous case. Let $T_{1,k} = k$ and, for $n \geq 2$,

$$T_{n,k} = \min\{j : j > T_{n-1,k}, X_j > X_{T_{n-1,k}-k+1:T_{n-1,k}}\},$$

where $X_{i:m}$ denotes the i-th order statistic in a sample of size m. The sequence of *upper k-records* are then defined by $R_{n(k)} = X_{T_{n,k}-k+1:T_{n,k}}$ for $n \geq 1$. For $k = 1$, note that the usual records are recovered. These sequences of k-records were introduced by Dziubdziela and Kopocinski (1976) and they have found acceptance in the literature. Nagaraja (1988) pointed out that k-records with an underlying distribution function F can be viewed as ordinary record values (k=1) based on the distribution of the minimum with cdf $F_{1:k} \equiv 1 - \{1 - F\}^k$. Using the joint density of usual records, the marginal density of $R_{n(k)}$ is obtained as

$$f_{n,k}(r_{n(k)};\theta) = \frac{k^n}{(n-1)!}\left[-\log \bar{F}(r_{n(k)};\theta)\right]^{n-1}\{\bar{F}(r_{n(k)};\theta)\}^{k-1} f(r_{n(k)};\theta), \tag{1}$$

and the joint pdf of the first n k-records is given by

$$f_{1,\ldots,n}(\mathbf{r};\theta) = k^n[\bar{F}(r_{n(k)};\theta)]^k \prod_{i=1}^{n} \frac{f(r_{i(k)};\theta)}{\bar{F}(r_{i(k)};\theta)}, \tag{2}$$

where $\mathbf{r} = (r_{1(k)}, \ldots, r_{n(k)})$ and $\bar{F} = 1 - F$ (see Arnold et al., 1998). The problem of estimation based on record data has been previously studied in the literature, in particular under a Bayesian framework (e.g., Ali Mousa, Jaheen and Ahmad, 2002; Jaheen, 2003 and 2004; Ahmadi, Doostparast and Parsian, 2005; Ahmadi and Doostparast, 2006). The goal of this paper is to develop Bayes estimation of functions of θ based on k-record data from a general class of distributions under balanced type loss functions, which we now describe.

1.2 Balanced type loss functions

To reflect both goodness of fit and precision of estimation in estimating an unknown parameter θ under the model $\mathbf{X} = (X_1, ..., X_n) \sim F_\theta$, Zellner (1994) introduced a Balanced Loss Function (BLF) as follows:

$$\frac{\omega}{n}\sum_{i=1}^{n}(X_i - \delta)^2 + (1-\omega)(\delta - \theta)^2,$$

where $\omega \in [0, 1)$, and considered optimal estimates relative to BLF for estimation of a scalar mean, a vector mean and a vector regression coefficients. Dey, Ghosh and Strawderman (1999), as well as Jafari Jozani, Marchand and Parsian (2006a) studied the notion of a BLF from the perspective of unifying a variety of results both frequentist and Bayesian. They showed in broad generality that frequentist and Bayesian results for BLF follow from and also imply related results for SEL functions. Jafari Jozani, Marchand and Parsian (2006b) introduced an extended class of balanced type loss functions of the form

$$L^q_{\rho,\omega,\delta_0}(\gamma(\theta),\delta) = \omega q(\theta)\rho(\delta_0,\delta) + (1-\omega)q(\theta)\rho(\gamma(\theta),\delta), \tag{3}$$

with $q(\cdot)$ being a suitable positive weight function, $\rho(\gamma(\theta),\delta)$ being as arbitrary loss function in estimating $\gamma(\theta)$ by δ, and δ_0 a chosen a priori "target" estimator of $\gamma(\theta)$, obtained for instance from the criterion of maximum likelihood, least squares or unbiasedness among others. They give a general Bayesian connection between the cases $\omega > 0$ and $\omega = 0$. For the case of squared error ρ, a least squares δ_0, $\gamma(\theta) = \theta$, and $q(\theta) = 1$ in (3) is equivalent to Zellner's (1994) BLF, and the introduction of an arbitrary ρ extends the squared error version of (3) introduced by Jafari Jozani, Marchand and Parsian (2006a). In this paper, we shall use balanced squared error loss (balanced SEL) and balanced LINEX loss to illustrate Bayesian estimation of parameters of interest in a class of distributions (described in 1.3) based on a sample of k-record values. A companion paper (Ahmadi et al., 2008) considers a similar framework but for prediction of future k-records. Much of the analysis below is unified with respect to the choice of the target estimator δ_0, the weight ω, and to some extent, with respect to the parameter being estimated.

1.3 Proportional hazard rate models

Let $X_1, X_2, ...$ be a sequence of iid random variables from the family of continuous distribution functions with

$$F(x;\theta) = 1 - [\bar{G}(x)]^{\alpha(\theta)}, \quad -\infty \le c < x < d \le \infty, \tag{4}$$

where $\alpha(\theta) > 0$, $G \equiv 1 - \bar{G}$, and G is an arbitrary continuous distribution function with $G(c) = 0$ and $G(d) = 1$. The family in (4) is well-known in lifetime experiments as proportional hazard rate models (see for example Lawless, 2003), and includes several well-known lifetime distributions such as: Exponential, Pareto, Lomax, Burr type XII, and so on.
Let $g(x) = \frac{d}{dx}G(x)$ be the corresponding pdf, then

$$f(x;\theta) = \alpha(\theta)g(x)[\bar{G}(x)]^{\alpha(\theta)-1}, -\infty \le c < x < d \le \infty. \tag{5}$$

From (4), it can be easily shown that: **(i)** the Fisher information contained in a single observation X is $I_X(\theta) = \left(\frac{\alpha'(\theta)}{\alpha(\theta)}\right)^2$, **(ii)** $E[-\log \bar{G}(X)] = 1/\alpha(\theta)$, and **(iii)** $T(\mathbf{X}) = -\sum_{i=1}^n \log \bar{G}(X_i)$ is a complete sufficient statistic for θ. In this paper we consider the estimation of $\gamma_1(\theta) = \theta$, $\gamma_2(\theta) = \frac{1}{\theta}$, the hazard rate function at t: $\gamma_3(\theta) = h_\theta(t) = f(t;\theta)/\bar{F}(t;\theta)$, and the survival function at t: $\gamma_4(\theta) = R_\theta(t) = \bar{F}(t;\theta)$ of the parent distribution in (4) based on observed k-record data.

2 Frequentist estimation

Suppose we observe the first n upper k-record values $R_{1(k)} = r_{1(k)},\ R_{2(k)} = r_{2(k)}, \cdots, R_{n(k)} = r_{n(k)}$ from a distribution with cdf given by (4) with $\alpha(\theta) = \theta$. Using (4), (5) and (2), it is easy to verify that the joint density function of the n upper k-record values is given by

$$f_{1,\ldots,n}(\mathbf{r};\theta) = [k\alpha(\theta)]^n \{\bar{G}(r_{n(k)})\}^{k\alpha(\theta)} \prod_{i=1}^{n} \frac{g(r_{i(k)})}{\bar{G}(r_{i(k)})}, \tag{6}$$

where $\mathbf{r} = (r_{1(k)}, \ldots, r_{n(k)})$. Let $U_{n(k)} = -\log \bar{G}(R_{n(k)})$. Then from (6), $U_{n(k)}$ is seen to be a complete and sufficient statistic for θ with a $Gamma(n, k\theta)$ distribution. Also, the maximum likelihood estimator (MLE) of θ is given by $\frac{n}{k}U_{n(k)}^{-1}$. With the completeness and sufficiency of $U_{n(k)}$ based on the first n k-records for samples from (4), we obtain with Rao-Blackwell-Lehmann-Scheffe's Theorem that,

- $\frac{n-1}{k}U_{n(k)}^{-1}$ is the UMVUE of $\gamma_1(\theta) = \theta$,
- $\frac{k}{n}U_{n(k)}$ is the UMVUE of $\gamma_2(\theta) = \frac{1}{\theta}$,
- $\frac{n-1}{k}\frac{g(t)}{\bar{G}(t)}U_{n(k)}^{-1}$ is the UMVUE of $\gamma_3(\theta) = h_\theta(t)$,
- $\left(1 + \frac{\log \bar{G}(t)}{k}U_{n(k)}^{-1}\right)^{n-1}$ is the UMVUE of $\gamma_4(\theta) = R_\theta(t)$.

3 Bayesian estimation under balanced SEL

In this section we obtain Bayes estimators of $\gamma_i(\theta)$; $i = 1, 2, 3, 4$; based on observed k-record data generated from (4) with $\alpha(\theta) = \theta$, using balanced SEL given in (3) with $\rho(\gamma, \delta) = (\delta - \gamma)^2$ and equal to

$$L^q_{\omega,\delta_0}(\gamma(\theta), \delta) = \omega q(\theta)(\delta - \delta_0)^2 + (1 - \omega)q(\theta)(\delta - \gamma(\theta))^2. \tag{7}$$

Using a conjugate $Gamma(\alpha, \beta)$ prior for θ with density,

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\ \theta^{\alpha-1}e^{-\beta\theta}, \quad \theta,\ \alpha,\ \beta > 0, \tag{8}$$

the posterior pdf of θ is given by

$$\pi(\theta|r_{1(k)}, ..., r_{n(k)}) = \frac{(\beta + ku_{n(k)})^{n+\alpha}}{\Gamma(n+\alpha)}\ \theta^{n+\alpha-1}e^{-\theta(\beta+ku_{n(k)})}, \quad \theta > 0, \tag{9}$$

where $u_{n(k)} = -\log \bar{G}(r_{n(k)})$. The Bayes estimator of $\gamma(\theta)$ under $L^q_{\omega,\delta_0}(\gamma(\theta), \delta)$ is given by (see Jafari Jozani, Marchand and Parsian, 2006a)

$$\delta^q_{\omega,\gamma}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) + (1 - \omega)\frac{E[q(\theta)\gamma(\theta)|\mathbf{R}_{(k)}]}{E[q(\theta)|\mathbf{R}_{(k)}]}, \tag{10}$$

where $\mathbf{R}_{(k)} = (R_{1(k)}, ..., R_{n(k)})$. As apparent from (10), $\delta^q_{\omega,\gamma}(\mathbf{R}_{(k)})$ is a convex linear combination of the target estimator $\delta_0(\mathbf{R}_{(k)})$ (weight ω), and the Bayes estimator of $\gamma(\theta)$ under weighted SEL function. In the sequel, we pursue with the study of Bayes estimators of $\gamma_i(\theta)$, $i = 1, 2, 3, 4$ for $q \equiv 1$ in (7).

3.1 Bayes estimator of $\gamma_1(\theta)=\theta$ under balanced SEL

With a Gamma prior as in (8), the Bayes estimator of $\gamma_1(\theta)=\theta$ under SEL function is given by

$$\delta_{0,\gamma_1}(\mathbf{R}_{(k)}) = E[\theta|\mathbf{R}_{(k)}] = \frac{n+\alpha}{\beta - k\log\bar{G}(R_{n(k)})}.$$

Using (10), the Bayes estimator of $\gamma_1(\theta)=\theta$ under $L_{\omega,\delta_0}(\gamma_1(\theta),\delta)$ with respect to a $Gamma(\alpha,\beta)$ prior is given by

$$\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) + (1-\omega)\frac{n+\alpha}{\beta - k\log\bar{G}(R_{n(k)})}.$$

Among others, a plausible target estimator $\delta_0(\mathbf{R}_{(k)})$ is the MLE of $\gamma_1(\theta)$, given by $-n/\left(k\log\bar{G}(R_{n(k)})\right)$. It may be noted that $\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ is the unique Bayes estimator of $\gamma_1(\theta)=\theta$ under $L_{\omega,\delta_0}(\gamma_1(\theta),\delta)$ with finite Bayes risk and hence is admissible. Notice that as $(\alpha,\beta)\to(0,0)$,

$$\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)}) \to \delta^N_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) - (1-\omega)\frac{n}{k\log\bar{G}(R_{n(k)})}.$$

where $\delta^N_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ is the Bayes estimator of $\gamma_1(\theta)=\theta$ under $L_{\omega,\delta_0}(\gamma_1(\theta),\delta)$ with respect to the non informative prior $\pi(\theta)=\frac{1}{\theta}$. It is easy to see that under $L_{0,\delta}(\gamma_1(\theta),\delta)$, the estimator $\delta^N_{0,\gamma_1}(\mathbf{R}_{(k)})$ is dominated by $-\frac{n-2}{k\log\bar{G}(R_{n(k)})}$, provided that $n>2$, and is an inadmissible estimator of $\gamma_1(\theta)$. In fact, $\delta^a_{0,\gamma_1}(\mathbf{R}_{(k)}) = -\frac{n-2}{k\log\bar{G}(R_{n(k)})}$ is the unique admissible estimator of $\gamma_1(\theta)$ under $L_{0,\delta}(\gamma_1(\theta),\delta)$ in the class of all estimators of the form of $-\frac{A}{k\log\bar{G}(R_{n(k)})}$ (e.g., Ghosh and Singh, 1970). Let $\delta^a_{0,\gamma_1}(\mathbf{R}_{(k)}) = \delta_0(\mathbf{R}_{(k)}) + g(\mathbf{R}_{(k)})$, where $g(\mathbf{R}_{(k)}) = \delta^a_{0,\gamma_1}(\mathbf{R}_{(k)}) - \delta_0(\mathbf{R}_{(k)})$. Now, following Lemma 1 of Jafari Jozani, Marchand and Parsian (2006a), we conclude that

$$\delta^a_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) + (1-\omega)\delta^a_{0,\gamma_1}(\mathbf{R}_{(k)}),$$

is an admissible estimator of $\gamma_1(\theta)$ under $L_{\omega,\delta_0}(\gamma_1(\theta),\delta)$, and that $\delta^N_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ is an inadmissible estimator of $\gamma_1(\theta)$ under BLF (regardless to the choice of target estimator δ_0, $\delta_0\neq\delta^N_{\omega,\gamma_1}$). In general (see Example 1 of Jafari Jozani, Marchand and Parsian, 2006a)

$$\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) + (1-\omega)(A_0\delta^a_{0,\gamma_1}(\mathbf{R}_{(k)}) + B_0),$$

is inadmissible under loss $L_{\omega,\delta_0}(\gamma_1(\theta),\delta)$ if

(i) $A_0\in[0,\frac{n}{n-2}]$, $B_0\le 0$, or,

(ii) $A_0>\frac{n}{n-2}$, or $A_0<0$, or $A_0=\frac{n}{n-2}$, $B_0\neq 0$.

Alternatively, if the target estimator is chosen as $\delta^a_{0,\gamma_1}(\mathbf{R}_{(k)})$, the linear estimators $A_\omega\delta^a_{0,\gamma_1}(\mathbf{R}_{(k)})+B_\omega$ are inadmissible under $L_{\omega,\delta_0}(\gamma_1(\theta),\delta)$ whenever one of the following conditions holds

(i) $A_\omega\in[\omega,\frac{n-2\omega}{n-2}]$, $B_\omega\le 0$,

(ii) $A_\omega>\frac{n-2\omega}{n-2}$, or $A_\omega<0$, or $A_\omega=\frac{n-2\omega}{n-2}$, $B_\omega\neq 0$.

Remark 1 *The results of this section tell us that for estimating $\gamma_1(\theta)$ for a model in (4), the estimator $\delta^a_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ is admissible under information-weighted balanced SEL function, i.e.,*

$$L^I_{\omega,\delta_0}(\theta,\delta) = \omega\frac{(\delta-\delta_0)^2}{\theta^2} + (1-\omega)(\frac{\delta}{\theta}-1)^2.$$

Under $L^I_{0,\delta_0}(\theta,\delta) = (\frac{\delta}{\theta}-1)^2$ it is well known that $\delta^a_{0,\gamma_1}(\mathbf{R}_{(k)}) = -\frac{n-1}{k\log\bar{G}(R_{n(k)})}$ is, among the class of linear estimators in $-\frac{1}{\log\bar{G}(R_{n(k)})}$, the unique admissible and minimax estimator of $\gamma_1(\theta)$ (e.g., Ghosh and Singh, 1970). Suppose the target estimator δ_0 has constant risk under $L^I_{0,\delta_0}(\theta,\delta)$ as those proportional to $-\frac{1}{k\log\bar{G}(R_{n(k)})}$ such as δ^a_{0,γ_1}, or δ^N_{0,γ_1}. Now, Theorem 1 of Jafari Jozani, Marchand and Parsian (2006a) tells us that $\delta^a_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ is, among the class of linear estimators in $-\frac{1}{\log\bar{G}(R_{n(k)})}$, the unique admissible and minimax estimator of $\gamma_1(\theta)$ under $L^I_{\omega,\delta_0}(\gamma_1(\theta),\delta)$.

Example 1 ***(i)*** *Taking $\bar{G}(x) = e^{-\frac{x}{1-x}}$, $0<x<1$, in (4) (i.e., $X \sim \frac{Y}{Y+1}$ with $Y \sim Exp(1)$) and choosing $\delta_0(\mathbf{R}_{(k)}) = \frac{n-2}{k}\frac{1-R_{n(k)}}{R_{n(k)}}$ as an admissible target estimator of θ under $L_{0,\delta_0}(\gamma_1(\theta),\delta)$, the unique Bayes (and admissible estimator) of θ under $L_{\omega,\delta_0}(\gamma_1(\theta),\delta)$ is given by*

$$\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = \frac{(n-2)\omega}{k}\frac{1-R_{n(k)}}{R_{n(k)}} + \frac{(1-\omega)(n+\alpha)R_{n(k)}}{\beta+(\beta-k)R_{n(k)}}.$$

(ii) *(Exponential distribution). For an exponential survival function in (4); i.e., $\bar{G}(x) = e^{-x}$; and the admissible target estimator $\delta_0(\mathbf{R}_{(k)}) = \frac{n-2}{kR_{n(k)}}$ of θ, the unique Bayes estimator of θ under $L_{\omega,\delta_0}(\gamma_1(\theta),\delta)$ is given by*

$$\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = \frac{(n-2)\omega}{kR_{n(k)}} + \frac{(1-\omega)(n+\alpha)}{\beta+kR_{n(k)}}.$$

For $\omega=0$ the results of Jaheen (2004) are obtained as a special case, while, for $\omega=0$ and $k=1$, the results of Ahmadi and Doostparast (2006) are obtained as a special case.

3.2 Bayes estimator of $\gamma_2(\theta) = \frac{1}{\theta}$ under balanced SEL

In this case $E[\gamma_2(\theta)|\mathbf{r}_{(k)}] = \frac{1}{n+\alpha-1}(\beta - k\log\bar{G}(r_{n(k)}))$, so that the (unique) Bayes estimator of $\gamma_2(\theta) = \frac{1}{\theta}$ under $L_{\omega,\delta_0}(\gamma_2(\theta),\delta)$ and prior density in (8) is given by

$$\delta_{\omega,\gamma_2}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) + (1-\omega)\frac{\beta - kU_{n(k)}}{n+\alpha-1}.$$

It is worth mentioning that, for estimating $\gamma_2(\theta)$ under $L_{0,\delta_0}(\gamma_2(\theta),\delta)$, linear estimators of the form of $A_0U_{n(k)} + B_0$ are admissible whenever (Lehmann and Casella, 1998)

(a) $A_0 \in [0, \frac{1}{n+1}]$, $B_0 > 0$, or,

(b) $A_0 = \frac{1}{n+1}$, $B_0 = 0$.

So, by Lemma 1 of Jafari Jozani, Marchand and Parsian (2006a), for estimating $\gamma_2(\theta)$ under loss $L_{\omega,\delta_0}(\gamma_2(\theta),\delta)$, estimators of the form of $\omega\delta_0(\mathbf{R}_{(k)})+(1-\omega)(A_0U_{n(k)}+B_0)$ are admissible whenever one of the conditions (**a**) or (**b**) holds. In particular, when the target estimator is chosen to be $\delta_0(\mathbf{R}_{(k)})=\frac{U_{n(k)}}{n+1}$, we infer that linear estimators $A_\omega U_{n(k)}+B_\omega$ are admissible under loss $L_{\omega,\delta_0}(\gamma_2(\theta),\delta)$ if: **(a)** $A_\omega\in[\frac{\omega}{n+1},\frac{1}{n+1}]$, $B_\omega>0$; or **(b)** $A_\omega=\frac{1}{n+1}$, $B_\omega=0$. Alternatively, we can also show that $A_\omega U_{n(k)}+B_\omega$ is inadmissible under loss $L_{\omega,\delta_0}(\gamma_2(\theta),\delta)$, with $\delta_0(\mathbf{R}_{(k)})=\frac{U_{n(k)}}{n+1}$, if one of the following conditions holds:

1. $A_\omega\in[\frac{\omega}{n+1},\frac{n+1-\omega}{n(n+1)}]$, $B_\omega\leq 0$,

2. $A_\omega<\frac{\omega}{n+1}$, or $A_\omega>\frac{n+1-\omega}{n(n+1)}$, or $A_\omega=\frac{n+1-\omega}{n(n+1)}$, $B_\omega\neq 0$;

(Lehmann and Casella, 1998; Jafari Jozani, Marchand and Parsian, 2006a).

Remark 2 *In estimating $\gamma_2(\theta)$ under the balanced squared error loss, an alternative choice of the target estimator is the MLE, $\delta_0^M=\frac{k}{n}U_{n(k)}$. In such a case we show as above that the estimators $A_\omega^M U_{n(k)}+B_\omega^M$ are admissible under $L_{\omega,\delta_0^M}(\gamma_2(\theta),\delta)$ whenever* **(a)** $A_\omega^M\in[\frac{\omega}{n},\frac{\omega+n}{n+1}]$, $B_\omega^M>0$; *or* **(b)** $A_\omega^M=\frac{\omega+n}{n+1}$, $B_\omega=0$, *and are inadmissible whenever* **(i)** $A_\omega^M\in[\frac{\omega}{n},\frac{1}{n}], B_\omega^M\leq 0$, *or* **(ii)** $A_\omega^M>\frac{1}{n}$, *or* **(iii)** $A_\omega^M<\frac{\omega}{n}$, *or* **(iv)** $A_\omega^M=\frac{1}{n}, B_\omega^M\neq 0$.

Example 2 *(continued) (i) Again for the model with $\bar{G}(x)=e^{-\frac{x}{1-x}}$, $0<x<1$, in (4), the unique Bayes estimator of $\gamma_2(\theta)=\frac{1}{\theta}$ under $L_{\omega,\delta_0}(\gamma_2(\theta),\delta)$ when $\delta_0(\mathbf{R}_{(k)})=\frac{kR_{n(k)}}{(n+1)(1-R_{n(k)})}$ with respect to the prior density in (8) reduces to*

$$\delta_{\omega,\gamma_2}(\mathbf{R}_{(k)})=\left(\frac{k\omega}{n+1}+\frac{k-(1-\omega)\beta}{n+\alpha-1}\right)\frac{R_{n(k)}}{1-R_{n(k)}}+\frac{(1-\omega)\beta}{n+\alpha-1}.$$

(ii) Exponential distribution: The unique Bayes (hence admissible) estimator of $\gamma_2(\theta)=\frac{1}{\theta}$ under $L_{\omega,\delta_0}(\gamma_2(\theta),\delta)$ when $\delta_0(\mathbf{R}_{(k)})=\frac{kR_{n(k)}}{(n+1)}$ with respect to prior distribution (8) reduces to

$$\delta_{\omega,\gamma_2}(\mathbf{R}_{(k)})=\left(\frac{k\omega}{n+1}+\frac{k(1-\omega)}{n+\alpha-1}\right)R_{n(k)}+\frac{(1-\omega)\beta}{n+\alpha-1}.$$

For $\omega=0$ and $k=1$, the results of Ahmadi and Doostparast (2006) are obtained as special cases.

3.3 Bayes estimator of $\gamma_3(\theta)=h_\theta(t)$ under balanced SEL

For estimating the hazard rate function of a proportional hazard rate model as in (4) or (5) with $\alpha(\theta)=\theta$, we have $\gamma_3(\theta)=h_\theta(t)=\theta\frac{g(t)}{\bar{G}(t)}$. Thus, the problem of estimating $\gamma_3(\theta)$ is (essentially) equivalent to the one of estimating $\gamma_1(\theta)=\theta$. Namely, under $L_{\omega,\delta_0}(\gamma_3(\theta),\delta)$ and based on k-record data,

$$E[\gamma_3(\theta)|\mathbf{r}_{(k)}]=\frac{g(t)}{\bar{G}(t)}\frac{n+\alpha}{\beta-k\log\bar{G}(r_{n(k)})};$$

so that the Bayes estimator of $\gamma_3(\theta)$ under $L_{\omega,\delta_0}(\gamma_3(\theta),\delta)$ with respect to prior distribution (8) is given by

$$\delta_{\omega,\gamma_3}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) + (1-\omega)\frac{g(t)}{\bar{G}(t)}\frac{n+\alpha}{\beta - k\log\bar{G}(R_{n(k)})}.$$

Choosing the MLE of $\gamma_3(\theta)$ as the target estimator, i.e., $\delta_0(\mathbf{R}_{(k)}) = -\frac{n}{k\log\bar{G}(R_{n(k)})}\frac{g(t)}{\bar{G}(t)}$, the Bayes estimator $\delta_{\omega,\gamma_3}(\mathbf{R}_{(k)})$ reduces to

$$\delta^M_{\omega,\gamma_3}(\mathbf{R}_{(k)}) = \frac{g(t)}{\bar{G}(t)}\left\{-\frac{n\omega}{k\log\bar{G}(R_{n(k)})} + \frac{(n+\alpha)(1-\omega)}{\beta - k\log\bar{G}(R_{n(k)})}\right\}.$$

Alternatively, if we choose the target estimator as the UMVUE of $\gamma_3(\theta)$, the Bayes estimator $\delta_{\omega,\gamma_3}(\mathbf{R}_{(k)})$ reduces to

$$\delta^U_{\omega,\gamma_3}(\mathbf{R}_{(k)}) = \frac{g(t)}{\bar{G}(t)}\left\{-\frac{(n-1)\omega}{k\log\bar{G}(R_{n(k)})} + \frac{(n+\alpha)(1-\omega)}{\beta - k\log\bar{G}(R_{n(k)})}\right\}.$$

Notice that $\delta_{\omega,\gamma_3}(\mathbf{R}_{(k)})$ is a unique Bayes estimator, has finite Bayes risk and hence is admissible. With the above equivalence, we note that $R_{\omega,\delta_0}(\gamma_3(\theta),\delta_{\omega,\gamma_3}(\mathbf{R}_{(k)})) = \left\{\frac{g(t)}{\bar{G}(t)}\right\}^2 R_{\omega,\delta_0}(\gamma_1(\theta),\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)}))$, so that, the admissibility or inadmissibility of $\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ for estimating $\gamma_1(\theta)$ under $L_{\omega,\delta_0}(\gamma_1(\theta),\delta)$ is equivalent to the admissibility or inadmissibility of $\frac{g(t)}{\bar{G}(t)}\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ for estimating $\gamma_3(\theta)$ under $L_{\omega,\frac{g(t)}{\bar{G}(t)}\delta_0}(\gamma_3(\theta),\delta)$.

Example 3 *(Pareto distribution). Taking $\bar{G}(x) = \frac{\eta}{x}$, $x > \eta > 0$, with known η, in* (4)*, then X has Pareto distribution. With the target estimator δ_0 chosen as the MLE of $\gamma_3(\theta)$ (given by $n/(tk\log(\frac{R_{n(k)}}{\eta}))$), then the unique Bayes estimator of $\gamma_3(\theta)$ under BLF $L_{\omega,\delta_0}(\gamma_3(\theta),\delta)$ with respect to a Gamma prior in* (8) *is given by*

$$\delta^M_{\omega,\gamma_3}(\mathbf{R}_{(k)}) = \frac{1}{t}\left[\omega\frac{n}{k\log(\frac{R_{n(k)}}{\eta})} + (1-\omega)\frac{n+\alpha}{\beta + k\log(\frac{R_{n(k)}}{\eta})}\right].$$

3.4 Bayes estimator of $\gamma_4(\theta) = \bar{F}(t;\theta)$ under balanced SEL

For estimating the survival function (reliability function) of a distribution as in (4) with $\alpha(\theta)=\theta$, we have $\gamma_4(\theta) = [\bar{G}(t)]^\theta$. Thus, under $L_{\omega,\delta_0}(\gamma_4(\theta),\delta)$ based on k-record data

$$E[\bar{F}(t;\theta)|\mathbf{R}_{(k)}] = E\left[(\bar{G}(t))^\theta|\mathbf{R}_{(k)}\right] = \left(\frac{\beta + kU_{n(k)}}{\beta + kU_{n(k)} - \log\bar{G}(t)}\right)^{n+\alpha}.$$

Choosing $\delta_0 = \delta_{MLE} = [\bar{G}(t)]^{\frac{n}{k}U^{-1}_{n(k)}}$, the unique Bayes estimator of $\gamma_4(\theta)$ with respect to a Gamma prior in (8) under $L_{\omega,\delta_0}(\gamma_4(\theta),\delta)$ is obtained from (10) as

$$\delta_{\omega,\gamma_4}(\mathbf{R}_{(k)}) = \omega[\bar{G}(t)]^{\frac{n}{k}U^{-1}_{n(k)}} + (1-\omega)\left(\frac{\beta + kU_{n(k)}}{\beta + kU_{n(k)} - \log\bar{G}(t)}\right)^{n+\alpha}.$$

Example 4 *Exponential distribution (continued): Let $\bar{G}(x) = e^{-x}$, $x > 0$, in (4). The unique Bayes estimator of the survival function (reliability function), i.e., $\gamma_4(\theta) = e^{-\theta t}$, under $L_{\omega,\delta_0}(\gamma_4(\theta), \delta)$ with $\delta_0 = \delta_{MLE}$, based on k-record data with respect to prior distribution in (8) becomes*

$$\delta_{\omega,\gamma_4}(\mathbf{R}_{(k)}) = \omega e^{\frac{nt}{kR_{n(k)}}} + (1-\omega)\left(1 + \frac{t}{\beta + kR_{n(k)}}\right)^{-(n+\alpha)}.$$

Remark 3 *One can also develop the previous framework for weighted balanced squared error loss. With the Fisher information equal to $1/\theta^2$ for a distribution in (4) it is reasonable to choose the weight function $q(\theta)$ in (7) as $1/\theta^2$. Then, for example, the Bayes estimators of $\gamma_2(\theta)$ and $\gamma_4(\theta)$ under L^q_{ω,δ_0}, with $q(\theta) = 1/\theta^2$ and $\delta_0 = \delta_{MLE}$, based on n upper k-record data with respect to a Gamma prior as in (8) are given by*

$$\delta^q_{\omega,\gamma_2}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) + (1-\omega)\frac{\beta - k\log\bar{G}(R_{n(k)})}{n+\alpha-3},$$

and

$$\delta^q_{\omega,\gamma_4}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) + (1-\omega)\frac{n+\alpha}{\beta - k\log\bar{G}(R_{n(k)}) - \log\bar{G}(t)},$$

provided $n+\alpha-3 > 0$ and $\beta - k\log\bar{G}(R_{n(k)}) - \log\bar{G}(t) > 0$, respectively.

4 Bayesian estimation under balanced LINEX loss

It has long been recognized that the commonly used squared error loss, and hence balanced squared error loss are inappropriate in many practical situations especially when overestimation and underestimation of the same magnitude have different consequences. A useful alternative to the balanced SEL is the asymmetric balanced LINEX loss function with shape parameter a ($a \neq 0$), obtained with the choice of $\rho(\gamma(\theta), \delta) = e^{a(\delta - \gamma(\theta))} - a(\delta - \gamma(\theta)) - 1$, and $q(\theta) = 1$ in (3), and given by

$$L^*_{\omega,\delta_0}(\gamma(\theta), \delta) = \omega\{e^{a(\delta-\delta_0)} - a(\delta - \delta_0) - 1\} + (1-\omega)\{e^{a(\delta-\gamma(\theta))} - a(\delta - \gamma(\theta)) - 1\},$$

where $\gamma(\theta)$ is the parameter, δ_0 is a target estimator, and $\omega \in [0, 1)$. A review of LINEX loss functions and their properties is given by Parsian and Kirmani (2002). Subject to finite posterior risk, the Bayes estimator of $\gamma(\theta)$ under LINEX loss is given by $\delta^*_{0,\gamma}(\mathbf{R}_{(k)}) = -\frac{1}{a}\log(E[e^{-a\gamma(\theta)}|\mathbf{R}_{(k)}])$. Hence, from Jafari Jozani, Marchand and Parsian (2006b, Example 5) the unique Bayes estimator of $\gamma(\theta)$ under $L^*_{\omega,\delta_0}(\gamma(\theta), \delta)$ with respect to the prior distribution $\pi(\theta)$ is given by

$$\delta^*_{\omega,\gamma}(\mathbf{R}_{(k)}) = -\frac{1}{a}\log\left\{\omega e^{-a\delta_0(\mathbf{R}_{(k)})} + (1-\omega)e^{-a\delta^*_{0,\gamma}(\mathbf{R}_{(k)})}\right\}. \tag{11}$$

In this section, we pursue with the study of Bayesian estimation of $\gamma_1(\theta) = \theta$, $\gamma_2(\theta) = \frac{1}{\theta}$, and the survival function $\gamma_4(\theta) = \bar{F}(t;\theta)$, for models in (4) with $\alpha(\theta) = \theta$, based on n observed k-record data under $L^*_{\omega,\delta_0}(\gamma(\theta), \delta)$ with respect to a Gamma prior as in (8). We present a similar analysis for a scale invariant balanced LINEX loss function in subsection 4.4.

4.1 Bayes estimator of $\gamma_1(\theta) = \theta$

Under LINEX loss the Bayes estimator of $\gamma_1(\theta) = \theta$ with respect to the prior distribution (8) is given by

$$\delta^*_{0,\gamma_1}(\mathbf{R}_{(k)}) = -\frac{1}{a}\log\left(E[e^{-a\gamma_1(\theta)}|\mathbf{R}_{(k)}]\right) = \frac{n+\alpha}{a}\log\left(1+\frac{a}{\beta+kU_{n(k)}}\right),$$

provided $a > -\beta - kU_{n(k)}$. Thus, from (11), the Bayes estimator of $\gamma_1(\theta) = \theta$ under $L^*_{\omega,\delta_0}(\gamma_1(\theta), \delta)$ is given by

$$\delta^*_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = -\frac{1}{a}\log\left\{\omega e^{-a\delta_0(\mathbf{R}_{(k)})} + (1-\omega)\left(\frac{\beta+kU_{n(k)}}{\beta+kU_{n(k)}+a}\right)^{n+\alpha}\right\},$$

provided $a > -\beta - kU_{n(k)}$. Notice that:

- $\lim_{a\to 0}\delta^*_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = \omega\delta_0(\mathbf{R}_{(k)}) + (1-\omega)\frac{n+\alpha}{\beta+kU_{n(k)}} = \delta_{\omega,\gamma_1}(\mathbf{R}_{(k)})$, where $\delta_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ is the unique Bayes estimator of $\gamma_1(\theta)$ under the balanced SEL function.

- Since $e^{-a\gamma_1(\theta)}$ is convex in θ, Jensen's inequality leads to the inequality $\delta^*_{\omega,\gamma_1}(\mathbf{R}_{(k)}) \leq \delta_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ for $a > 0$. The relation is more general of course but here it tells us that the Bayes estimator of $\gamma_1(\theta) = \theta$ under $L^*_{\omega,\delta_0}(\gamma_1(\theta), \delta)$ for $a > 0$ is smaller (with probability one) than the corresponding estimator under balanced SEL.

- $\lim_{\alpha,\beta\to 0}\delta^*_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = -\frac{1}{a}\log\{\omega e^{-a\delta_0(\mathbf{R}_{(k)})} + (1-\omega)(1+\frac{a}{kU_{n(k)}})^{-n}\} = \delta^{*N}_{\omega,\gamma_1}(\mathbf{R}_{(k)})$, where $\delta^{*N}_{\omega,\gamma_1}(\mathbf{R}_{(k)})$ is the Bayes estimate of $\gamma_1(\theta)$ under the balanced LINEX loss function with respect to non informative prior $\pi(\theta) = \frac{1}{\theta}$ (provided $a > -kU_{n(k)}$).

- As in section 3.3, the results here apply for estimating $\gamma_3(\theta) = \theta\frac{g(t)}{G(t)}$ under balanced LINEX loss function, $L^*_{\omega,\delta_0}(\gamma_3(\theta), \delta)$.

Example 5 *Exponential distribution (continued): The unique Bayes estimator of $\gamma_1(\theta)$ under balanced LINEX loss with an arbitrary target estimator δ_0 with respect to a Gamma prior as in* (8) *is*

$$\delta^*_{\omega,\gamma_1}(\mathbf{R}_{(k)}) = -\frac{1}{a}\log\left\{\omega e^{-a\delta_0(\mathbf{R}_{(k)})} + (1-\omega)\left(\frac{\beta+kR_{n(k)}}{\beta+kR_{n(k)}+a}\right)^{n+\alpha}\right\}.$$

For $\omega = 0$, we obtain the results of Ahmadi, Doostparast, and Parsian (2005) as a special case.

4.2 Bayes estimator of $\gamma_2(\theta) = 1/\theta$

For $a > 0$, we have

$$\begin{aligned} E[e^{-a\gamma_2(\theta)}|\mathbf{r}_{(k)}] &= \frac{(\beta+ku_{n(k)})^{n+\alpha}}{\Gamma(n+\alpha)}\int_0^\infty e^{-(\frac{a}{\theta}+\theta(\beta+ku_{n(k)}))}\theta^{n+\alpha-1}d\theta \\ &= \frac{(\beta+ku_{n(k)})^{n+\alpha}}{\Gamma(n+\alpha)}\Phi_n(\alpha,\beta,a,k,u_{n(k)}), \end{aligned}$$

where

$$\Phi_n(\alpha,\beta,a,k,t_{n(k)}) = 2a^{n+\alpha}\sqrt{a(\beta+ku_{n(k)})}\;\mathcal{K}\left(n+\alpha, 2\sqrt{a(\beta+ku_{n(k)})}\right),$$

and $\mathcal{K}(\nu,\cdot)$ is the modified Bessel function of the third kind, see Gradshteyn and Ryzhik (2000). Thus, from (11), the Bayes estimator of $\gamma_2(\theta)=1/\theta$ under $L^*_{\omega,\delta_0}(\gamma_4(\theta),\delta)$ with respect to the prior distribution (8) is given by

$$\delta^*_{\omega,\gamma_2}(\mathbf{r}_{(k)}) = -\frac{1}{a}\log\left\{\omega e^{-a\delta_0(\mathbf{r}_{(k)})} + (1-\omega)\frac{(\beta+ku_{n(k)})^{n+\alpha}}{\Gamma(n+\alpha)}\Phi_n(\alpha,\beta,a,k,u_{n(k)})\right\}. \tag{12}$$

For $\omega=0$, we obtain

$$\delta^*_{0,\gamma_2}(\mathbf{r}_{(k)}) = \frac{1}{a}\log\Gamma(n+\alpha) - \frac{n+\alpha}{a}\log(\beta+ku_{n(k)}) - \frac{1}{a}\log\Phi_n(\alpha,\beta,a,k,u_{n(k)}). \tag{13}$$

One can use Lindley's method for obtaining an approximation of (12) or (13). Lindley (1980) derived the approximation:

$$\frac{\int R(\theta)e^{U(\theta)}d\theta}{\int e^{U(\theta)}d\theta} \approx R(\theta_0) - \frac{1}{2}\frac{1}{\partial^2U/\partial\theta^2}\left[\frac{\partial^2R}{\partial\theta^2} - \frac{(\frac{\partial R}{\partial U})(\frac{\partial U}{\partial\theta^3})}{\frac{\partial^2U}{\partial\theta^2}}\right]_{\theta=\theta_0},$$

where θ_0 is the root of the equation $\frac{\partial U(\theta)}{\partial\theta}=0$. To obtain an approximation of (12), notice that

$$\frac{\Gamma(n+\alpha)}{(\beta+ku_{n(k)})^{n+\alpha}} = \int_0^\infty \theta^{n+\alpha-1}e^{-\theta(\beta+ku_{n(k)})}d\theta,$$

and hence

$$\begin{aligned}
E[e^{-\frac{a}{\theta}}|\mathbf{r}_{(k)}] &= \frac{\int_0^\infty e^{-\frac{a}{\theta}}e^{(n+\alpha-1)\log\theta-\theta(\beta+ku_{n(k)})}d\theta}{\int_0^\infty e^{(n+\alpha-1)\log\theta-\theta(\beta+ku_{n(k)})}d\theta} \\
&\approx e^{-\frac{a}{\theta_0}} + \frac{\theta_0^2}{2(n+\alpha-1)}\left[\frac{a^2}{\theta_0^4} - \frac{2a}{\theta_0^3}\right]e^{-\frac{a}{\theta_0}},
\end{aligned}$$

where $\theta_0 = \frac{n+\alpha-1}{\beta+ku_{n(k)}}$. So, an approximation of the Bayes estimator of $\gamma_2(\theta)$ under balanced LINEX loss $L^*_{\omega,\delta_0}(\gamma_2(\theta),\delta)$ is given by

$$\delta^*_{\omega,\gamma_2}(\mathbf{r}_{(k)}) \approx -\frac{1}{a}\log\left(\omega e^{-a\delta_0(\mathbf{r}_{(k)})} + (1-\omega)e^{-\frac{a}{\theta_0}}\left\{1+\frac{\theta_0^2}{2(n+\alpha-1)}\left[\frac{a^2}{\theta_0^4}-\frac{2a}{\theta_0^3}\right]\right\}\right).$$

Similarly, we obtain

$$\delta^*_{0,\gamma_2}(\mathbf{r}_{(k)}) \approx -\frac{1}{\theta_0} - \frac{1}{a}\log\left(1+\frac{1}{2\theta_0(n+\alpha-1)}(\frac{a^2}{\theta_0}-2a)\right).$$

Example 6 *Exponential distribution (continued): The unique Bayes estimator of $\gamma_2(\theta)$ under balanced LINEX loss, with respect to prior distribution (8) and for δ_0 as the MLE of $\gamma_2(\theta)$, is*

$$\delta^*_{\omega,\gamma_2}(\mathbf{R}_{(k)}) = -\frac{1}{a}\log\{\omega e^{-\frac{ak}{n}R_{n(k)}} + (1-\omega)\frac{(\beta+kR_{n(k)})^{n+\alpha}}{\Gamma(n+\alpha)}\Phi_n(\alpha,\beta,a,k,R_{n(k)})\}.$$

For $\omega=0$ we obtain the results of Ahmadi, Doostparast and Parsian (2005) as a special case. Also Lindley's approximation yields

$$\delta^*_{\omega,\gamma_2}(\mathbf{r}_{(k)}) \approx -\frac{1}{a}\log\left(\omega e^{-\frac{ak}{n}R_{n(k)}} + (1-\omega)e^{-\frac{a(\beta+kR_{n(k)})}{n+\alpha-1}}\left\{1-\frac{a(\beta+kR_{n(k)})}{2(n+\alpha-1)^2}\left[2-\frac{a(\beta+kR_{n(k)})}{n+\alpha-1}\right]\right\}\right),$$

provided $n+\alpha-1>0$.

4.3 Bayes estimator of $\gamma_4(\theta) = \bar{F}(t;\theta)$

From (9), we have

$$\begin{aligned} E[e^{-a\gamma_4(\theta)}|\mathbf{r}_{(k)}] &= E[e^{-a[\bar{G}(t)]^\theta}|\mathbf{r}_{(k)}] \\ &= \int_0^\infty e^{-a[\bar{G}(t)]^\theta} \frac{(\beta + ku_{n(k)})^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{n+\alpha-1} e^{-\theta(\beta+ku_{n(k)})} d\theta. \end{aligned}$$

Following Lindley's method, it is easy to verify that

$$\begin{aligned} E[e^{-a\gamma_4(\theta)}|\mathbf{r}_{(k)}] &= \frac{\int_0^\infty e^{-a\gamma_4(\theta)} \theta^{n+\alpha-1} e^{-\theta(\beta+ku_{n(k)})} d\theta}{\int_0^\infty \theta^{n+\alpha-1} e^{-\theta(\beta+ku_{n(k)})} d\theta} \\ &\approx e^{-a(\bar{G}(t))^{\theta_0}} - \frac{\theta_0^2 a^2 \log^2 \bar{G}(t)}{2(n+\alpha-1)} e^{-a(\bar{G}(t))^{\theta_0}}, \end{aligned}$$

where $\theta_0 = \frac{n+\alpha-1}{\beta+ku_{n(k)}}$. Thus

$$\delta^*_{\omega,\gamma_4}(\mathbf{R}_{(k)}) \approx -\frac{1}{a} \log \left\{ \omega e^{-a\delta_0(\mathbf{R}_{(k)})} + (1-\omega) e^{-a[\bar{G}(t)]^{\theta_0}} \left(1 - \frac{\theta_0^2 a^2 \log^2 \bar{G}(t)}{2(n+\alpha-1)} \right) \right\} \tag{14}$$

is an approximation of the Bayes estimator of $\gamma_4(\theta)$ under balanced LINEX loss $L^*_{\omega,\delta_0}(\gamma_4(\theta), \delta)$ with respect to prior distribution (8), and for an arbitrary target estimator δ_0.

Example 7 *Exponential distribution (continued): Expression* (14) *becomes*

$$\delta^*_{\omega,\gamma_4}(\mathbf{r}_{(k)}) \approx -\frac{1}{a} \log \left\{ \omega e^{-a\delta_0(\mathbf{r}_{(k)})} + (1-\omega) e^{-ae^{(-\theta_0 R_{n(k)})}} \left(1 - \frac{\theta_0^2 a^2 R^2_{n(k)}}{2(n+\alpha-1)} \right) \right\},$$

where $\theta_0 = \frac{n+\alpha-1}{\beta+kR_{n(k)}}$, *provided that* $n+\alpha-1 > 0$.

4.4 Bayes estimation under modified balanced LINEX loss function

One may wish to use the following scale invariant version of balanced LINEX loss:

$$L^{SI}_{\omega,\delta_0}(\gamma(\theta), \delta) = \omega \left\{ e^{a(\frac{\delta}{\delta_0}-1)} - a(\frac{\delta}{\delta_0} - 1) - 1 \right\} + (1-\omega) \left\{ e^{a(\frac{\delta}{\gamma(\theta)}-1)} - a(\frac{\delta}{\gamma(\theta)} - 1) - 1 \right\},$$

with $a \neq 0$ and δ_0 being a target estimator of $\gamma(\theta)$. It can be shown that the Bayes estimator of $\gamma(\theta)$ under $L^{SI}_{0,\delta_0}(\gamma(\theta), \delta)$ based on a sample of n upper k-record values is given by the following equation (e.g., Parsian and Sanjari, 1993),

$$E\left[\frac{1}{\gamma(\theta)} e^{a\frac{\delta_{0,\gamma}(\mathbf{R}_{(k)})}{\gamma(\theta)}} \middle| \mathbf{R}_{(k)} \right] = e^a E[\frac{1}{\gamma(\theta)}|\mathbf{R}_{(k)}].$$

Using Lemma 1 of Jafari Jozani, Marchand and Parsian (2006b), the Bayes estimator of $\gamma(\theta)$ under modified balanced LINEX loss function $L^{SI}_{\omega,\delta_0}(\gamma(\theta), \delta)$ is given as a solution of the following equation with respect to $\delta_{\omega,\gamma}$,

$$\frac{\omega}{\delta_0(\mathbf{R}_{(k)})} e^{a\frac{\delta_{\omega,\gamma}(\mathbf{R}_{(k)})}{\delta_0(\mathbf{R}_{(k)})}} + (1-\omega) E\left[\frac{1}{\gamma(\theta)} e^{a\frac{\delta_{\omega,\gamma}(\mathbf{R}_{(k)})}{\gamma(\theta)}} \middle| \mathbf{R}_{(k)} \right] = \frac{\omega e^a}{\delta_0(\mathbf{R}_{(k)})} + (1-\omega) e^a E[\frac{1}{\gamma(\theta)}|\mathbf{R}_{(k)}].$$

Now, for estimating $\gamma_2(\theta) = \frac{1}{\theta}$ under $L^{SI}_{0,\delta_0}(\gamma(\theta), \delta)$, the Bayes estimator δ_{0,γ_2} satisfies the following equation,

$$E[\theta e^{\{a\theta\delta_{0,\gamma_2}(\mathbf{R}_{(k)})\}}|\mathbf{R}_{(k)}] = e^a E[\theta|\mathbf{R}_{(k)}],$$

which reduces to

$$\delta_{0,\gamma_2}(\mathbf{R}_{(k)}) = c(\alpha)U_{n(k)} + d(\alpha, \beta),$$

where $c(\alpha) = [1 - e^{-\frac{a}{n+\alpha+1}}]/a$ and $d(\alpha, \beta) = c(\alpha)\beta$. Note that $0 < c(\alpha) < c^* = \frac{1}{a}(1 - e^{-\frac{a}{n+1}})$. Now, it follows from Parsian and Sanjari (1993) that the linear estimator $A_0U_{n(k)} + B_0$ is inadmissible under $L^{SI}_{0,\delta_0}(\gamma(\theta), \delta)$ whenever **(a)** $A_0 < 0$ or $B_0 < 0$; or **(b)** $A_0 > c^*$, $B_0 \geq 0$; or **(c)** $0 \leq A_0 < c^*$, $B_0 = 0$; and is admissible if $A_0 \in [0, c^*]$, $B_0 \geq 0$. Unfortunately, for the case of $\omega > 0$, it seems difficult to obtain a closed form for the Bayes estimator of $\gamma_2(\theta)$ under $L^{SI}_{\omega,\delta_0}(\gamma(\theta), \delta)$ and one may have to rely on a numerical evaluation.

5 Concluding remarks

In this paper, we showed how to develop Bayes estimation in the context of upper k-record data from a general class of distributions under some balanced type loss functions. We applied results from Jafari Jozani, Marchand and Parsian (2006a, b), as well as from the literature on k-records. With respect to several criteria (choice of balanced type loss, choice of target estimator, underlying model and parameter of interest), the treatment is unified. Illustrations were given for the particular class of balanced squared error and LINEX losses. The admissibility or inadmissibility of some estimators was discussed. Lindley's approximation was illustrated in some cases. Finally, for a sequence of iid random variables $X_1, X_2, ...$ from the class of continuous distribution functions $F(\cdot; \theta)$ with

$$F(x; \theta) = [H(x)]^{\beta(\theta)}, \quad -\infty \leq c < x < d \leq \infty,\ \beta(\theta) > 0,$$

H an arbitrary continuous distribution function with $H(c) = 0$ and $H(d) = 1$, one can show that the results of this paper also hold true for lower k-record data with some minor modifications. The above family of distributions is well known in lifetime experiments, and referred to as proportional reversed hazard rate models (see Lawless, 2003).

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