# Prediction of $k$-records from a general class of distributions under balanced type loss functions

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**Abstract**

We study the problem of predicting future $k$-records based on $k$-record data for a large class of distributions, which includes several well-known distributions such as: Exponential, Weibull (one parameter), Pareto, Burr type XII, among others. With both Bayesian and non-Bayesian approaches being investigated here, we pay more attention to Bayesian predictors under balanced type loss functions as introduced by Jafari Jozani, Marchand and Parsian (2006b). The results are presented under the balanced versions of some well-known loss functions, namely squared error loss (SEL), Varian's linear-exponential (LINEX) loss and absolute error loss (AEL) or $L_1$ loss functions. Some of the previous results in the literatures such as Ahmadi et al. (2005), and Raqab et al. (2007) can be achieved as special cases of our results.

*Keywords:* Absolute value error loss; Balanced loss function; Bayes prediction; Conditional median prediction; Maximum likelihood prediction; LINEX loss; Record values.

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## 1 Introduction

Consider a sequence $\{X_i,\ i \geq 1\}$ of iid absolutely continuous random variables distributed according to the cumulative distribution function (cdf) $F(x;\theta)$ and probability density function (pdf) $f(x;\theta)$, where $\theta$ is an unknown parameter. An observation $X_j$ will be called an *upper record value* if its value exceeds all previous observations. Thus, $X_j$ is an *upper record* if $X_j > X_i$ for every $i < j$. Records of iid random variables and their properties have been extensively studied in the literature. Interest in records has increased steadily over the years since Chandler (1952) formulated the theory of records. See Arnold et al. (1998), Nevzorov (2001) and the references therein for more details on applications

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of records. An upper $k$-record process is defined in terms of the $k$-th largest $X$ yet seen. For a formal definition, we consider the definition in Arnold et al. (1998), page 43, for the continuous case. Let $T_{1,k} = k$, and for $n \geq 2$

$$T_{n,k} = \min\{j : j > T_{n-1,k}, X_j > X_{T_{n-1,k}-k+1:T_{n-1,k}}\},$$

where $X_{i:m}$ denotes the $i$-th order statistic in a sample of size $m$. The sequence of *upper $k$-records* are then defined by $R_{n(k)} = X_{T_{n,k}-k+1:T_{n,k}}$ for $n \geq 1$. For $k = 1$, note that the usual records are recovered. These sequences of $k$-records were introduced by Dziubdziela and Kopocinski (1976) and they have found acceptance in the literature. Using the joint density of usual records, the joint pdf of the first $n$, $k$-records $\mathbf{R}_{(k)} = (R_{1(k)}, ..., R_{n(k)})$ is given by

$$f_{1,...,n}(\mathbf{r}_{(k)}; \theta) = k^n [\bar{F}(r_{n(k)}; \theta)]^k \prod_{i=1}^{n} \frac{f(r_{i(k)}; \theta)}{\overline{F}(r_{i(k)}; \theta)}, \tag{1}$$

where $\mathbf{r}_{(k)} = (r_{1(k)}, ..., r_{n(k)})$ and $\bar{F} \equiv 1 - F$, see, Arnold et al. (1998).

The problem of prediction of future usual ($k$=1) records based on observed record data has been extensively studied by several statisticians in view of classical and Bayesian framework, see Ahsanullah (1980), Awad and Raqab (2000), Dunsmore (1983), Ali Mousa et al. (2000), Jaheen (2003, 2004), Madi and Raqab (2004), Ahmadi and Doostparast (2006) and Doostparast and Ahmadi (2006) among others. Ahmadi et al. (2005) developed Bayesian inference and prediction based on $k$-record values under a LINEX loss function.

The record statistics are of interest and importance, these statistics are applied in estimating strength of materials, predicting natural disasters, sports achievements, etc. Consider a technical systems or subsystems with $k$-out-of-$n$ structure. A $k$-out-of-$n$ system breaks down at the time of the $(n-k+1)$-th component failure. So, in reliability analysis, the life length of a $k$-out-of-$n$ system is the $(n-k+1)$-th order statistic in a sample of size $n$. Consequently, the $n$-th upper $k$-record value can be regarded as the life length of a $k$-out-of-$T_{n,k}$ system. Several application of $k$-record values can be found in the literature, for instance, see the examples cited in Kamps (1995) or Danielak and Raqab (2004b) in reliability theory. Suppose that a technical system is subject to shocks, e.g. peaks of voltage. These shocks may be modeled as realizations of records. If not record values themselves, but second or third largest values are of special interest, then the model of $k$-record values is adequate. When record values themselves are viewed as outliers, then second or third largest values are of special interest. Insurance claims in some non-life insurance can be used as an example. So predicting future $k$-record values is an important problem. Several statisticians have investigated upper bounds for the expectation of future k-records, see Klimczak (2006), Klimczak and Rychlik (2004), Raqab (2004), Danielak and Raqab (2004a, b) among others. One of the motivations here is the relative paucity of work concerning a Bayesian approach to predicting future $k$-records.

A second important feature of this paper is the proposed use of balanced type loss functions, which

to us represent an interesting tool for decision making, or specifically here for predictions. In decision theory, the loss function usually focusses on precision of estimation. However, goodness of fit is also a very important criterion. Zellner (1994), introduced the notion of a balanced loss function in the context of a general linear model to reflect both goodness of fit and precision of estimation. Often loss functions reflecting one or the other of these two criteria, but not both have been employed in statistical inferences. For example, least squares estimation reflects goodness of fit considerations whereas the use of quadratic or LINEX loss functions involve a sole emphasis on precision of estimation. Following Jafari Jozani, Marchand and Parsian (2006b) we introduce a general class of balanced type loss functions of the form of

$$L_{\rho,\omega,d_0}(y,d) = \omega\rho(d_0,d) + (1-\omega)\rho(y,d), \tag{2}$$

where $\rho(y,d)$ is an arbitrary loss function in predicting $y$ by $d$, while $d_0$ is a chosen *a priori* "target" predictor of $y$, obtained for instance from the criterion of maximum likelihood predictor, least square or unbiasedness among others. Our objective is to predict a future $k$-record based on a sample of observed $k$-record data from Bayesian point of view for a general class of distributions under balanced type loss functions as in (2). As first modelled by Jafari Jozani, Marchand and Parsian (2006a) for a squared error $\rho$ in (2), the choice of the target $d_0$ in (2) is arbitrary. This constitutes a particularly appealing feature with the Bayesian developments of Section 4 being applicable for any target predictor $d_0$, such as those presented in Section 3. In this paper, we shall use balanced squared error loss (BSEL), and balanced absolute error loss (BAEL) or balanced $L_1$ loss functions as symmetric BLFs and balanced LINEX loss function as an asymmetric BLF to derive the Bayes predictions of the future $k$-records in a general class of distributions based on a sample of $k$-record values observed.

## 2 Preliminaries

Let $X_1, X_2, \cdots$, be a sequence of iid random variables from the class $C_1$ of continuous distribution functions with

$$F(x;\theta) = 1 - [\bar{G}(x)]^{\alpha(\theta)}, \quad -\infty \le c < x < d \le \infty, \tag{3}$$

where $\alpha(\theta) > 0$ and $G \equiv 1 - \bar{G}$ is an arbitrary continuous distribution function, free of unknown parameters with $G(c) = 0$ and $G(d) = 1$. The family in $C_1$ is well-known in the lifetime experiments as proportional hazard models (see for example Lawless, 2003), which includes several well-known lifetime distributions such as: Exponential, Pareto, Lomax, Burr type XII, Resnick, Weibull (one parameter) among others. Also, the family in $C_1$ is a subclass of a regular one parameter exponential family of distributions.

Let $g(x) = \frac{d}{dx}G(x)$ be the pdf corresponding to $G(x)$, then

$$f(x;\theta) = \alpha(\theta)g(x)[\bar{G}(x)]^{\alpha(\theta)-1}, -\infty \le c < x < d \le \infty. \tag{4}$$

Suppose that $R_{1(k)}, R_{2(k)}, ...$ is the sequence of $k$-records coming from (3) with $\alpha(\theta) = \theta$, then the following can be obtained easily ( see for instance Arnold et al., 1998):

- The joint pdf of $\mathbf{R}_{(k)} = (R_{1(k)}, ..., R_{n(k)})$ is given by

$$f_{1,...,n}(r_{1(k)}, ..., r_{n(k)}; \theta) = (k\theta)^n \{\bar{G}(r_{n(k)})\}^{k\theta} \prod_{i=1}^{n} \frac{g(r_{i(k)})}{\bar{G}(r_{i(k)})}. \tag{5}$$

- The marginal pdf of the $n$th $k$-records, $R_{n(k)}$, is given by

$$f_n(r_{n(k)}; \theta) = \frac{(k\theta)^n}{(n-1)!} g(r_{n(k)}) \{\bar{G}(r_{n(k)})\}^{k\theta-1} \{-\log \bar{G}(r_{n(k)})\}^{n-1}. \tag{6}$$

- The conditional pdf of $R_{s(k)}$ given $R_{n(k)} = r_{n(k)}$, $s > n$ is

$$f_{s|n}(r_{s(k)}|r_{n(k)}, \theta) = \frac{(k\theta)^{s-n}}{(s-n-1)!} \left(\log \frac{\bar{G}(r_{n(k)})}{\bar{G}(r_{s(k)})}\right)^{s-n-1} \frac{g(r_{s(k)})}{\bar{G}(r_{s(k)})} \left(\frac{\bar{G}(r_{s(k)})}{\bar{G}(r_{n(k)})}\right)^{k\theta}. \tag{7}$$

The following results are an immediate consequences of (5) and (6):

- $R_{n(k)}$ is a complete sufficient statistic for $\theta$ among the first $n$ $k$-records.
- $-\log \bar{G}(R_{n(k)})$ has a Gamma distribution with parameters $n$ and $k\theta$.
- The maximum likelihood estimator (MLE) of $\theta$ is $\hat{\theta}^{mle} = \frac{-n}{k \log \bar{G}(R_{n(k)})}$.
- The UMVUE of $\theta$ is $\hat{\theta}_U = \frac{1-n}{k \log \bar{G}(R_{n(k)})}$.

Suppose that we observed the first $n$ upper $k$-record values $\mathbf{R}_{(k)} = \mathbf{r}_{(k)}$ from a distribution, with cdf and pdf given, respectively, by (3) and (4) with $\alpha(\theta) = \theta$. Let $\hat{R}_{\omega,s(k)} = \psi(\mathbf{R}_{(k)})$ be the point predictor of $R_{\omega,s(k)}$, then we consider the balanced type predictive loss function as follows:

$$L_{\rho,\omega,R^0_{s(k)}}(R_{s(k)}, \hat{R}_{\omega,s(k)}) = \omega\rho(R^0_{s(k)}, \hat{R}_{\omega,s(k)}) + (1-\omega)\rho(R_{s(k)}, \hat{R}_{\omega,s(k)}), \tag{8}$$

where $\omega \in [0, 1)$ and $\rho(R_{s(k)}, \hat{R}_{\omega,s(k)})$ is an arbitrary loss function in predicting $R_{s(k)}$ by $\hat{R}_{\omega,s(k)}$, while $R^0_{s(k)}$ is a chosen *a priori* "target" predictor of $R_{s(k)}$, obtained for instance from the criterion of maximum likelihood predictor, best linear unbiased predictor, best linear invariant predictor, or conditional median predictor among others. In the sequel, we first derive the target predictor $R^0_{s(k)}$ of $R_{s(k)}$ based on observed records via two methods namely, conditional median and maximum likelihood.

## 3 Non-Bayesian Prediction

There are several non-Bayesian methods for predicting future $k$-records. Here we present two schemes, maximum likelihood and conditional median prediction, when the parent distribution is as in (4) with $\alpha(\theta) = \theta$.

### 3.1 Conditional median prediction

The median of the distribution of $R_{s(k)}$ given $R_{n(k)} = r_{n(k)}$, whose density is given in (7), is called the conditional median predictor (CMP) (see, Raqab and Nagaraja, 1995). The CMP, given in (9) below, depends on $\theta$ but we can derive a plausible predictor by using a plug-in estimate of $\theta$. We have

$$\int_{r_{n(k)}}^{R^0_{s(k)}} f_{s|n}(r_{s(k)}|r_{n(k)},\theta)dr_{s(k)} = \int_{R^0_{s(k)}}^{d} f_{s|n}(r_{s(k)}|r_{n(k)},\theta)dr_{s(k)}.$$

Now, suppose that we observed $R_{n(k)} = r_{n(k)}$ then by taking $t = \log\left(\frac{\bar{G}(r_{n(k)})}{\bar{G}(r_{s(k)})}\right)$ the above identity can be rewritten as follows

$$\int_0^{\log\left(\frac{\bar{G}(r_{n(k)})}{\bar{G}(R^0_{s(k)})}\right)} t^{s-n-1}e^{-k\theta t}dt = \int_{\log\left(\frac{\bar{G}(r_{n(k)})}{\bar{G}(R^0_{s(k)})}\right)}^{\infty} t^{s-n-1}e^{-k\theta t}dt.$$

From the above identity we conclude that

$$R^0_{s(k)} = \bar{G}^{-1}\left(\bar{G}(r_{n(k)})e^{-\frac{Med(W)}{2k\theta}}\right), \tag{9}$$

where $W$ has chi-square distribution with $2(s-n)$ degrees of freedom and $Med(W)$ stands for median of $W$. Substituting the MLE and unbiased estimator of $\theta$ in (9) we get

$$R^0_{M,s(k)} = \bar{G}^{-1}\left([\bar{G}(r_{n(k)})]^{(1+\frac{Med(W)}{2n})}\right) \tag{10}$$

and

$$R^0_{U,s(k)} = \bar{G}^{-1}\left([\bar{G}(r_{n(k)})]^{(1+\frac{Med(W)}{2(n-1)})}\right), \tag{11}$$

respectively. For $s = n+1$, using the fact that $Med(\chi^2_{(2)}) = \log 4$, (10) and (11) reduce to

$$R^0_{M,n+1(k)} = \bar{G}^{-1}\left([\bar{G}(r_{n(k)})]^{(1+\frac{\log 4}{2n})}\right) \quad \text{and} \quad R^0_{U,n+1(k)} = \bar{G}^{-1}\left([\bar{G}(r_{n(k)})]^{(1+\frac{\log 4}{2(n-1)})}\right).$$

**Example 1** ***(i) (Exponential distribution):*** *Taking $\bar{G}(x) = e^{-x}, 0 < x < \infty$, in (3), $X$ has Exponential distribution, and using (10) and (11) the target predictor (CMP) of $R_{s(k)}$ on the basis of $R_{n(k)} = r_{n(k)}$ are*

$$R^0_{M,s(k)} = \left(1 + \frac{Med(W)}{2n}\right)R_{n(k)} \quad \textit{and} \quad R^0_{U,s(k)} = \left(1 + \frac{Med(W)}{2n-2}\right)R_{n(k)}.$$

*For the predicting next record, $s = n+1$, we get*

$$R^0_{M,n+1(k)} = (1 + \frac{\log 4}{2n})R_{n(k)} \quad \textit{and} \quad R^0_{U,n+1(k)} = (1 + \frac{\log 4}{2(n-1)})R_{n(k)}.$$

***(ii) (Pareto distribution):*** *Taking $\bar{G}(x) = \frac{\eta}{x}$, $x > \eta > 0$, with known $\eta$, in (3), $X$ has Pareto distribution, and using (10) and (11) the target predictor (CMP) of $R_{s(k)}$ on the basis of $R_{n(k)} = r_{n(k)}$ are*

$$R^0_{M,s(k)} = \eta\left(\frac{R_{n(k)}}{\eta}\right)^{-(1+\frac{Med(W)}{2n})}, \quad \textit{and} \quad R^0_{U,s(k)} = \eta\left(\frac{R_{n(k)}}{\eta}\right)^{-(1+\frac{Med(W)}{2(n-1)})}.$$

*For the predicting next record, $s = n+1$, we get*

$$R^0_{M,n+1(k)} = \eta\left(\frac{R_{n(k)}}{\eta}\right)^{-(1+\frac{\log 4}{2n})} \quad \textit{and} \quad R^0_{U,n+1(k)} = \eta\left(\frac{R_{n(k)}}{\eta}\right)^{-(1+\frac{\log 4}{2(n-1)})}.$$

*For $k=1$ the results of Raqab et al. (2007) are obtained as an special case.*

### 3.2 Maximum likelihood prediction

Assume that we observed $\mathbf{r}_{(k)} = (r_{1(k)}, ..., r_{n(k)})$ from a population with unknown parameter $\theta$. We intend to predict $R_{s(k)}, s > n$ based on $\mathbf{r}_{(k)}$. The predictive likelihood function of $Y = R_{s(k)}$, and $\theta$ is given by (see Basak and Balakrishnan, 2003)

$$\begin{aligned} L(\mathbf{r}_{(k)}, r_{s(k)}, \theta) &= f(\mathbf{r}_{(k)}|\theta)h(r_{s(k)}|\mathbf{r}_{(k)}, \theta) \\ &= f(\mathbf{r}_{(k)}|\theta)f_{s|n}(r_{s(k)}|r_{n(k)}, \theta), \end{aligned}$$

where the second equality is obtained by Markovian property of $k$-records. If there exists $R^0_{L,s(k)} = \hat{R}^{mle}_{s(k)}$ and $\hat{\theta}_L = \hat{\theta}^{mle}$ such that

$$L(\mathbf{r}_{(k)}, R^0_{L,s(k)}, \hat{\theta}_L) = \sup_{r_{s(k)},\theta} L(\mathbf{r}_{(k)}, r_{s(k)}, \theta),$$

then $\hat{\theta}_L$ and $R^0_{L,s(k)}$ are the predictive maximum likelihood estimator (PMLE) for $\theta$ and the maximum likelihood predictor (MLP) of $R_{s(k)}$, respectively. Now suppose that $k$-records are obtained from our proposed model (3) with $\alpha(\theta) = \theta$, then using (5) and (7) the predictive log-likelihood function is given by

$$l(\mathbf{r}_{(k)}, r_{s(k)}, \theta) \propto (s-n-1)\log\left(\log\left(\frac{\bar{G}(r_{s(k)})}{\bar{G}(r_{n(k)})}\right)\right) + \log g(r_{s(k)}) + (k\theta - 1)\log\bar{G}(r_{s(k)}) + s\log\theta.$$

Substituting $\theta$ by $\hat{\theta}^{mle}$ in $l(\mathbf{r}_{(k)}, r_{s(k)}, \theta)$ the MLP of $R_{s(k)}$ can be obtained from the following equation:

$$\frac{g(R^0_{L,s(k)})}{\bar{G}(R^0_{L,s(k)})}\left\{1 + \frac{s}{\log\bar{G}(R^0_{L,s(k)})} - \frac{s-n-1}{\log\bar{G}(R^0_{L,s(k)}) - \log\bar{G}(r_{n(k)})}\right\} = \frac{g'(R^0_{L,s(k)})}{g(R^0_{L,s(k)})}, \tag{12}$$

**Example 2** *(continued)* ***(i)(Exponential distribution:)*** *For the case of exponential distribution the equation* (12) *reduces to:*

$$1 + \frac{s}{R^0_{L,s(k)}} - \frac{s-n-1}{r_{n(k)} - R^0_{L,s(k)}} = 1.$$

*Hence the MLP of $R_{s(k)}$ is*

$$R^0_{L,s(k)} = \frac{s}{n+1}R_{n(k)}.$$

***(ii)*** ***(Pareto distribution)****: Taking $\bar{G}(x) = \frac{\eta}{x}$, $x > \eta > 0$, with known $\eta$, in* (3) *with $\alpha(\theta) = \theta$, the maximum likelihood predictor of $R_{s(k)}$ is given as a solution of the following equation in $R^0_{L,s(k)}$:*

$$\frac{s-n-1}{\log r_{n(k)} - \log R^0_{L,s(k)}} - \frac{s}{\log\eta - \log R^0_{L,s(k)}} = 3.$$

*For $k=1$ the results of Raqab et al. (2007) are obtained as a special case.*

## 4 Bayesian prediction

A Bayesian approach may be adopted in order to derive the necessary predictive distribution; see Dunsmore (1983). Assuming that the parameter $\theta$ is a realization of a random variable with prior pdf $\pi(\theta)$. We use $Gamma(\alpha, \beta)$-distribution as a conjugate prior for the parameter $\theta$ with the following density function,

$$\pi(\theta; \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \quad \theta > 0, \tag{13}$$

where $\alpha$ and $\beta$ are positive constant.

It follows from (5) and (13) that the posterior pdf of the parameter $\theta$ is given by

$$\pi(\theta | r_{1(k)}, ..., r_{n(k)}) = \frac{(\beta - k \log \bar{G}(r_{n(k)}))^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{n+\alpha-1} e^{-\theta(\beta - k \log \bar{G}(r_{n(k)})}, \quad \theta > 0, \tag{14}$$

where $\Gamma(.)$ is the complete gamma function.

Assume that we have observed the first $n$ upper $k$-records $\mathbf{r}_{(k)} = (r_{1(k)}, ..., r_{n(k)})$ from (3) with $\alpha(\theta) = \theta$, and that, based on such a sample, prediction, either point or interval, is needed for $s$-th upper $k$-record, $s > n$. Now, let $Y = R_{s(k)}$ be the $s$-th upper $k$-record value, $s > n$. The Bayes predictive density function of $Y$ given $\mathbf{r}_{(k)}$ is given by (see, Arnold *et al.* 1998, p. 162)

$$f(y|\mathbf{r}_{(k)}) = \int_{\Theta} f(y|\mathbf{r}_{(k)}, \theta) \pi(\theta|\mathbf{r}_{(k)}) d\theta. \tag{15}$$

Hence, we find the Bayes predictive density function as follows

$$\begin{aligned} f(r_{s(k)}|r_{n(k)}) &= \int_0^{\infty} f(r_{s(k)}|r_{n(k)}, \theta) \pi(\theta|r_{n(k)}) d\theta \\ &= \frac{\Gamma(s+\alpha)}{\Gamma(s-n)\Gamma(n+\alpha)} \left( \frac{\beta + k t_{n(k)}}{\beta + k t_{s(k)}} \right)^{n+\alpha} \frac{k^{s-n} (t_{s(k)} - t_{n(k)})^{s-n-1} g(r_{s(k)})}{(\beta + k t_{s(k)})^{s-n} \bar{G}(r_{s(k)})}, \end{aligned}$$

where $t_{i(k)} = -\log \bar{G}(r_{i(k)}), i = n, s$ and $r_{n(k)} < r_{s(k)} < d$.

Let $U = \frac{\beta - k \log \bar{G}(R_{n(k)})}{\beta - k \log \bar{G}(R_{s(k)})}$, then it is easy to verify that $U$ given $R_{n(k)} = r_{n(k)}$ has a $Beta(n+\alpha, s-n)$-distribution which is independent of $R_{s(k)}$. Thus we can use $U$ to construct a Bayesian prediction interval for $R_{s(k)}$. Let $b_{\gamma}$ be the $\gamma$th percentile of a $Beta(n+\alpha, s-n)$-distribution, then the $100(1-\gamma)\%$ prediction interval for $R_{s(k)}$ is given by $(L_1,\ L_2)$, where

$$L_1 = \bar{G}^{-1} \left\{ \exp \left[ \frac{1}{k} (\beta - \frac{\beta - k \log \bar{G}(r_{n(k)})}{b_{1-\frac{\gamma}{2}}(n+\alpha, s-n)}) \right] \right\}, \tag{16}$$

and

$$L_2 = \bar{G}^{-1} \left\{ \exp \left[ \frac{1}{k} (\beta - \frac{\beta - k \log \bar{G}(r_{n(k)})}{b_{\frac{\gamma}{2}}(n+\alpha, s-n)}) \right] \right\} \tag{17}$$

**Remark 1** *We get the results of Ahmadi et al. (2005) by taking $\bar{G}(x) = e^{-x}$ (Exponential distribution) in* (16)*, and* (17)*, and the results of Raqab et al. (2007) by choosing $\bar{G}(x) = \frac{\eta}{x},\ x > \eta > 0$, with known $\eta$ (Pareto distribution) and $k = 1$ in* (16)*, and* (17)*, as special cases, respectively.*

Here we consider the Bayesian point prediction of $R_{s(k)}$ under BLF, $L_{\rho,\omega,R^0_{s(k)}}$, as in (8). When $\omega = 0$, we simply use $L_0$ instead of $L_{\rho,0,R^0_{s(k)}}$ unless we want to emphasize the role of $\rho$. The following lemma and proof are quite similar to Lemma 1 in Jafari Jozani, Marchand and Parsian (2006b), and establishes a connection between Bayesian prediction under BLF (8) for the cases $\omega > 0$ and $\omega = 0$.

**Lemma 1** *For predicting $R_{s(k)}$ with target predictor $R^0_{s(k)}$ under balanced loss function $L_{\rho,\omega,R^0_{s(k)}}$ as in* (8) *and for a prior $\pi(\theta)$, the Bayes predictor $\hat{R}_{\omega,s(k)}$ corresponds to the Bayes solution $R^*_{s(k)}$ with respect to $f^*(r_{s(k)}|r_{n(k)})$ under $L_0$; where*

$$f^*(r_{s(k)}|r_{n(k)}) = \omega I_{\{r^0_{s(k)}\}}(r_{s(k)}) + (1-\omega) f(r_{s(k)}|r_{n(k)}),$$

*i.e., $f^*(r_{s(k)}|r_{n(k)})$ is a mixture of a point mass at $r^0_{s(k)}$ and the conditional distribution $R_{s(k)}$ given $R_{n(k)} = r_n(k)$.*

**Proof:** Let $\mu_n$ and $\nu_n$ be dominating measures of $f(r_{s(k)}|r_{n(k)})$ and $f^*(r_{s(k)}|r_{n(k)})$, respectively. With the definitions of $R^*_{s(k)}$, $\hat{R}_{\omega,s(k)}$ and $L_0$, and with $\mathcal{X}_{s(k)}$ standing for the sample space of $R_{s(k)}$, we have

$$\begin{aligned}
\hat{R}_{\omega,s(k)} &= argmin_{r_{\omega,s(k)}} \int_{\mathcal{X}_{s(k)}} L_{\rho,\omega,R^0_{s(k)}}(r_{s(k)}, r_{\omega,s(k)}) f(r_{s(k)}|r_{n(k)}) d\mu_n(r_{s(k)}) \\
&= argmin_{r_{\omega,s(k)}} \int_{\mathcal{X}_{s(k)} \cup \{r^0_{s(k)}\}} \rho(r_{s(k)}, r_{\omega,s(k)}) f^*(r_{s(k)}|r_{n(k)}) d\nu_n(r_{s(k)}) \\
&= argmin_{r_{\omega,s(k)}} \int_{\mathcal{X}_{s(k)} \cup \{r^0_{s(k)}\}} L_0(r_{s(k)}, r_{\omega,s(k)}) f^*(r_{s(k)}|r_{n(k)}) d\nu_n(r_{s(k)}) \\
&= R^*_{s(k)}.
\end{aligned}$$

□

## 4.1 Bayes predictor under BSEL function

Choosing $\rho(y,d) = \rho_1(y,d) = \tau(y)(y-d)^2$ with $\tau(\cdot) > 0$, the balanced predictive loss function in (8) becomes

$$L_{\rho_1,\omega,R^0_{s(k)}}(R_{s(k)}, \hat{R}_{\omega,s(k)}) = \omega\tau(R^0_{s(k)})(R^0_{s(k)} - \hat{R}_{\omega,s(k)})^2 + (1-\omega)\tau(R_{s(k)})(R_{s(k)} - \hat{R}_{\omega,s(k)})^2.$$

Under $L_0$ and the prior $\pi$, we have $\hat{R}_{0,s(k)} = \frac{E_F[R_{s(k)}\tau(R_{s(k)})|r_{n(k)}]}{E_F[\tau(R_{s(k)})|r_{n(k)}]}$ (subject to the finiteness conditions $E_F[R^i_{s(k)}\tau(R_{s(k)})|r_{n(k)}] < \infty;\ i = 0,1;$ for all $r_{n(k)}$). Thus, following Lemma 1, the Bayes point

prediction of $R_{s(k)}$ under $L_{\rho_1,\omega,R^0_{s(k)}}$ is given by

$$\begin{aligned}
\hat{R}_{\omega,s(k)} &= \frac{E_{F^*}[R_{s(k)}\tau(R_{s(k)})|r_{n(k)}]}{E_{F^*}[\tau(R_{s(k)})|r_{n(k)}]} \\
&= \frac{\int_{\mathcal{X}_{s(k)}\cup\{r^0_{s(k)}\}} r_{s(k)}\tau(r_{s(k)})f^*(r_{s(k)}|r_{n(k)})d\nu_n(r_{s(k)})}{\int_{\mathcal{X}_{s(k)}\cup\{r^0_{s(k)}\}} \tau(R_{s(k)})f^*(r_{s(k)}|r_{n(k)})d\nu_n(r_{s(k)})} \\
&= \frac{\omega r^0_{s(k)}\tau(r^0_{s(k)}) + (1-\omega)E_F[R_{s(k)}\tau(R_{s(k)})|r_{n(k)}]}{\omega\tau(r^0_{s(k)}) + (1-\omega)E_F[\tau(R_{s(k)})|r_{n(k)}]}.
\end{aligned}$$

Taking $\tau(\cdot) = 1$ the predictive loss function $L_{\rho_1,\omega,R^0_{s(k)}}$ reduces to the balanced predictive SEL function. In this case the Bayes predictor of $R_{s(k)}$ simplifies as

$$\begin{aligned}
\hat{R}_{\omega,s(k)} &= \omega R^0_{s(k)} + (1-\omega)E[R_{s(k)}|R_{n(k)}] \qquad (18) \\
&= \omega R^0_{s(k)} + (1-\omega)\hat{R}_{0,s(k)}.
\end{aligned}$$

**Remark 2** *It may be noted that the Bayes predictor of $R_{s(k)}$ under balanced predictive SEL function is a convex linear combination of the target predictor $R^0_{s(k)}$ and the Bayes predictor of $R_{s(k)}$ under the usual predictive SEL function. Also, it is observed that when $\omega = 1$, $\hat{R}_{\omega,s(k)}$ reduces to the classical predictor $R^0_{s(k)}$, whereas, for $w = 0$, $\hat{R}_{\omega,s(k)}$ reduces to $\hat{R}_{0,s(k)}$ which is the Bayes predictor for $R_{s(k)}$ under SEL function.*

When $F \in \mathcal{C}_1$, we have

$$\begin{aligned}
\hat{R}_{0,s(k)} &= E[R_{s(k)}|r_{n(k)}] \\
&= E_U\left[\bar{G}^{-1}\left(\bar{G}(r_{n(k)})^{\frac{1}{U}}e^{-\frac{\beta}{k}\frac{1-U}{U}}\right)\middle| r_{n(k)}\right],
\end{aligned}$$

where the random variable $U = \frac{\beta - k\log\bar{G}(R_{n(k)})}{\beta - k\log\bar{G}(R_{s(k)})}$ given $R_{n(k)} = r_{n(k)}$ has a $Beta(n+\alpha, s-n)$- distribution. Thus we may rewrite (18) as

$$\hat{R}_{\omega,s(k)} = \omega R^0_{s(k)} + (1-\omega)E_U\left[\bar{G}^{-1}\left(\bar{G}(r_{n(k)})^{\frac{1}{U}}e^{-\frac{\beta}{k}\frac{1-U}{U}}\right)\middle| R_{n(k)}\right]. \qquad (19)$$

**Example 3** *(continued) (**Exponential distribution:**) For the case of exponential distribution the equation* (19) *reduces to*

$$\begin{aligned}
\hat{R}_{\omega,s(k)} &= \omega R^0_{s(k)} + (1-\omega)\left\{\left(R_{n(k)} + \frac{\beta}{k}\right)E\left[\frac{1}{U}|R_{n(k)}\right] - \frac{\beta}{k}\right\} \\
&= \omega R^0_{s(k)} + (1-\omega)\left\{\frac{s+\alpha-1}{n+\alpha-1}\left(R_{n(k)} + \frac{\beta}{k}\right) - \frac{\beta}{k}\right\}.
\end{aligned}$$

*Taking $R^0_{s(k)} = R^0_{L,s(k)} = \frac{s}{n+1}R_{n(k)}$ as the MLP of $R_{s(k)}$, then for $s = n+1$ we get*

$$\hat{R}_{\omega,n+1(k)} = \omega R_{n(k)} + (1-\omega)\left\{\frac{n+\alpha}{n+\alpha-1}\left(R_{n(k)} + \frac{\beta}{k}\right) - \frac{\beta}{k}\right\}.$$

*When $k = 1$ and $\omega = 0$, we find the results of Jaheen (2004) as an special case.*

### 4.2 Bayes predictor under BAEL function

Following Jafari Jozani, Marchand and Parsian (2006b), taking $\rho(y,d) = \rho_2(y,d) = |y-d|$, we get balanced absolute error loss (BAEL) function as

$$L_{\rho_2,\omega,R^0_{s(k)}}(R_{s(k)}, \hat{R}_{\omega,s(k)}) = \omega|R^0_{s(k)} - \hat{R}_{\omega,s(k)}| + (1-\omega)|R_{s(k)} - \hat{R}_{\omega,s(k)}|.$$

Since the Bayes prediction of $R_{s(k)}$ under $L_{\rho_2,0,R^0_{s(k)}}$ is the median of the Bayes predictive density function $f(r_{s(k)}|r_{n(k)}) = \int f(r_{s(k)}|r_{n(k)},\theta)\pi(\theta)d\theta$, following Lemma 1, the Bayes prediction of $R_{s(k)}$ under BAEL, $L_{\rho_2,\omega,R^0_{s(k)}}$, is a median of

$$f^*(r_{s(k)}|r_{n(k)}) = \omega I_{\{r^0_{s(k)}\}}(r_{s(k)}) + (1-\omega)f(r_{s(k)}|r_{n(k)}),$$

for all $r_{n(k)}$. It is easy to realize that when $\omega \geq \frac{1}{2}$, $\hat{R}_{\omega,s(k)} = R^0_{s(k)}$, i.e. the point mass of $R^0_{s(k)}$ has large enough probability to force the median to be $R^0_{s(k)}$. For $\omega < \frac{1}{2}$, the cdf corresponding to $f^*(r_{s(k)}|r_{n(k)})$ is given by

$$F^*(r_{s(k)}|r_{n(k)}) = \begin{cases} (1-\omega)P(R_{s(k)} \leq r_{s(k)}|r_{n(k)}), & r_{s(k)} < r^0_{s(k)}, \\ \omega + (1-\omega)P(R_{s(k)} \leq r_{s(k)}|r_{n(k)}), & r_{s(k)} \geq r^0_{s(k)}. \end{cases}$$

Taking $\zeta(r_{n(k)}) = (1-\omega)P(R_{s(k)} \leq r_{s(k)}|r_{n(k)})$ and $F^{-1}_{s|n}$ as the inverse cumulative Bayes predictive density function, straightforward calculations lead to

$$\hat{R}_{\omega,s(k)} = \begin{cases} F^{-1}_{s|n}(\frac{1-2\omega}{2(1-\omega)}), & 0 \leq \zeta(r_{n(k)}) < \frac{1}{2} - \omega, \\ R^0_{s(k)}, & \frac{1}{2} - \omega < \zeta(r_{n(k)}) < \frac{1}{2}, \\ F^{-1}_{s|n}(\frac{1}{2(1-\omega)}), & \frac{1}{2} \leq \zeta(r_{n(k)}) < 1. \end{cases} \tag{20}$$

Note that the above result is general with respect to the choice of the target predictor, the model and the prior distribution. Also in predicting $R_{n+1(k)}$, when the target predictor is chosen to be the maximum likelihood predictor of $R_{n+1(k)}$, i.e. $R^0_{n+1(k)} = R_{n(k)}$, it is easy to see that $\zeta(r_{n(k)}) = 0$ and so $\hat{R}_{\omega,n+1(k)} = r_{n(k)}$. Let $B(\cdot, n+\alpha, s-n)$ be the cdf of a $Beta(n+\alpha, s-n)$-distribution. For $F \in \mathcal{C}_1$, we can easily show that

$$\zeta(r_{n(k)}) = 1 - B\left(\frac{\beta - \log g(r_n(k))}{\beta - k\log g(r^0_{s(k)})}, n+\alpha, s-n\right),$$

and

$$F(r_{s(k)}|r_{n(k)}) = 1 - B\left(\frac{\beta - \log g(r_n(k))}{\beta - k\log g(r_{s(k)})}, n+\alpha, s-n\right).$$

Choosing $B^{-1}(\cdot, n+\alpha, s-n)$ as the inverse cdf of a $Beta(n+\alpha, s-n)$-distribution, then (20) leads us to the following Bayes prediction of $R_{s(k)}$ under BAEL,

$$\hat{R}_{\omega,s(k)} = \begin{cases} B^{-1}(\frac{1}{2(1-\omega)}, n+\alpha, s-n), & \frac{1}{2} + \omega \leq B(\frac{\beta - \log g(r_n(k))}{\beta - k\log g(r^0_{s(k)})}, n+\alpha, s-n) < 1, \\ R^0_{s(k)}, & \frac{1}{2} \leq B(\frac{\beta - \log g(r_n(k))}{\beta - k\log g(r^0_{s(k)})}, n+\alpha, s-n) < \frac{1}{2} + \omega, \\ B^{-1}(\frac{1-2\omega}{2(1-\omega)}, n+\alpha, s-n), & 0 \leq B(\frac{\beta - \log g(r_n(k))}{\beta - k\log g(r^0_{s(k)})}, n+\alpha, s-n) < \frac{1}{2}. \end{cases}$$

### 4.3 Bayes predictor under balanced LINEX loss function

The choice of $\rho(y,d) = \rho_a(y,d) = e^{a(y-d)} - a(y-d) - 1$, $a \neq 0$, in (8), leads to balanced LINEX loss function $L_{\rho_a,\omega,R^0_{s(k)}}$ as follows

$$\omega\{e^{a(R^0_{s(k)}-\hat{R}_{\omega,s(k)})} - a(R^0_{s(k)} - \hat{R}_{\omega,s(k)}) - 1\} + (1-\omega)\{e^{a(R_{s(k)}-\hat{R}_{\omega,s(k)})} - a(R_{s(k)} - \hat{R}_{\omega,s(k)}) - 1\}.$$

The choice of such an asymmetric $\rho_a$ is especially appropriate when over-prediction and under-prediction of the same magnitude have different economic consequences. For more details on LINEX loss see the review paper of Parsian and Kirmani (2002). The unique Bayes prediction of $R_{s(k)}$ under $L_{\rho_a,0,R^0_{s(k)}}$ with respect to the prior distribution $\pi(\theta)$ is given by $\hat{R}_{0,s(k)} = -\frac{1}{a}\log E_F[e^{-aR_{s(k)}}|r_{n(k)}]$. Following Lemma 1, the unique Bayes predictor of $R_{s(k)}$ under $L_{\rho_a,\omega,R^0_{s(k)}}$ is given as follows

$$\begin{aligned}
\hat{R}_{s(k)} &= -\frac{1}{a}\log E_{F^*}[e^{-aR_{s(k)}}|R_{n(k)}] \\
&= -\frac{1}{a}\log(\omega e^{-aR^0_{s(k)}} + (1-\omega)E_F[e^{-aR_{s(k)}}|R_{n(k)}]) \\
&= -\frac{1}{a}\log(\omega e^{-aR^0_{s(k)}} + (1-\omega)e^{-a\hat{R}_{0,s(k)}})
\end{aligned}$$

and for $F \in \mathcal{C}_1$ reduces to

$$\hat{R}_{\omega,s(k)} = -\frac{1}{a}\log\left(\omega e^{-aR^0_{s(k)}} + (1-\omega)E\left[\exp\{\bar{G}^{-1}\left(\bar{G}(R_{n(k)})^{\frac{1}{U}}e^{-\frac{\beta}{k}\frac{1-U}{U}}\right)\}\middle|R_{n(k)}\right]\right), \tag{21}$$

where $U|r_{n(k)} \sim Beta(n+\alpha, s-n)$.

**Example 4** *(continued) (**Exponential distribution:**) For the case of exponential distribution, since*

$$\begin{aligned}
e^{-a\hat{R}_{0,s(k)}} &= E\left[e^{-a\left(\frac{r_{n(k)}}{U}+\frac{\beta}{k}\frac{1-U}{U}\right)}\middle|r_{n(k)}\right] \\
&= \frac{e^{-\frac{a\beta}{k}}}{B(n+\alpha,s-n)}\Psi_s(a,\alpha,\beta,k,r_{n(k)}),
\end{aligned}$$

*where*

$$B(n+\alpha,s-n) = \frac{\Gamma(n+\alpha)\Gamma(s-n)}{\Gamma(s+\alpha)},$$

*and*

$$\Psi_s(a,\alpha,\beta,k,r_{n(k)}) = \int_0^1 \exp\left(-a(r_{n(k)}+\frac{\beta}{k})\frac{1}{u}\right)u^{n+\alpha-1}(1-u)^{s-n-1}du,$$

*the Bayes prediction of $R_{s(k)}$ obtained from $F \in \mathcal{C}_1$ under balanced LINEX loss function is given by*

$$\hat{R}_{\omega,s(k)} = -\frac{1}{a}\log\left(\omega e^{-ar^0_{s(k)}} + \frac{(1-\omega)e^{-\frac{a\beta}{k}}}{B(n+\alpha,s-n)}\Psi_s(a,\alpha,\beta,k,r_{n(k)})\right).$$

*For $k=1$ and $\omega=0$ the results of Ahmadi et al. (2005) are obtained as an special case.*

## 5 Conclusion

• The balanced type loss function, $L_{\rho,\omega,R^0_{s(k)}}$ as in (8), is a mixture of two loss functions, $\rho(\hat{R}_{\omega,s(k)}, R^0_{s(k)})$ and $\rho(\hat{R}_{\omega,s(k)}, R_{s(k)})$. Therefore it may be more appropriate than either of them in predicting the future records. This loss, which depends on the observed value of $R^0_{s(k)}$, reflects a desire of closeness of $\hat{R}_{\omega,s(k)}$ both to $R_{s(k)}$ and $R^0_{s(k)}$. In this paper, under non-Bayesian and Bayesian frameworks, we have studied the prediction of future $k$-records based on observed records which come from a general class of continuous distributions. The Bayes predictors of the $s$-th future $k$-record are obtained under BLF. We presented the results for the balanced type version of some well-known loss functions namely SEL, $L_1$ and LINEX loss.

• Taking a sequence of iid random variables $X_1, X_2, \cdots$, from the class $C_2$ of continuous distribution function $F(x;\theta)$ with

$$F(x;\theta) = [G(x)]^\theta, \quad -\infty \leq c < x < d \leq \infty, \ \theta > 0$$

where $G(x)$ is an arbitrary continuous distribution function free of unknown parameter and $G(c) = 0$, $G(d) = 1$. It is easy to show that the results of this paper are also hold true for lower $k$-record data from $C_2$ with some modifications. The above family of distributions is well known in the lifetime experiments as proportional reverse hazard models (see Lawless, 2003) and also a subclass of one parameter exponential family.

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