

# MUTATION OF CLUSTER-TILTING OBJECTS AND POTENTIALS

ASLAK BAKKE BUAN, OSAMU IYAMA, IDUN REITEN, AND DAVID SMITH

ABSTRACT. We prove that mutation of cluster-tilting objects in triangulated 2-Calabi-Yau categories is closely connected with mutation of quivers with potentials. This gives a close connection between 2-CY-tilted algebras and Jacobian algebras associated with quivers with potentials.

We show that cluster-tilted algebras are Jacobian and also that they are determined by their quivers. There are similar results when dealing with tilting modules over 3-CY algebras. The nearly Morita equivalence for 2-CY-tilted algebras is shown to hold for the finite length modules over Jacobian algebras.

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## INTRODUCTION

The Fomin-Zelevinsky mutation of quivers plays an important role in the theory of cluster algebras initiated in [FZ]. There is, motivated by this theory via [MRZ], a mutation of cluster-tilting objects in cluster categories, and more generally Hom-finite triangulated 2-Calabi-Yau (2-CY for short) categories over an algebraically closed field  $K$  [BMRRT, IY]. This has turned out to give a categorical model for

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the quiver mutation in certain cases [BMR2, BIRS]. On the other hand there is the recent theory of mutation of quivers with potentials  $(Q, W)$ , initiated in [DWZ1].

Associated with cluster-tilting objects  $T$  in 2-CY categories  $\mathcal{C}$  are the endomorphism algebras  $\text{End}_{\mathcal{C}}(T)$ , called 2-CY-tilted algebras. And associated with a quiver with potential  $(Q, W)$  are algebras called Jacobian algebras in [DWZ1]. The mutation of cluster-tilting objects induces an operation on the associated 2-CY-tilted algebras and the mutation of quivers with potentials induces an operation on the associated Jacobian algebras. The theme of this paper is to investigate these classes of algebras and their relationships, in particular with respect to mutation. In addition to cluster-tilting objects in triangulated 2-CY categories, we also deal with mutation of tilting modules over 3-CY algebras, and their relationship to mutation of quivers with potentials. We now state the main results of this paper, referring to section 1 for definitions and background material.

(A) Starting with a cluster-tilting object  $T$  in a 2-CY category  $\mathcal{C}$  such that  $\text{End}_{\mathcal{C}}(T)$  is Jacobian, our main theorem states that the two mutations “coincide” (Theorem 5.1). The same type of result is proved for tilting modules over 3-CY algebras (Theorem 5.2). Our basic idea for the proof is to use the relationship between Jacobian algebras and 2-almost split sequences introduced in [I2, IY], which is a higher analogue of the classical relationship between mesh categories of translation quivers and almost split sequences [Rie, BG, IT1, IT2, I1].

(B) The results in (A) can be used to show that large classes of 2-CY-tilted algebras are Jacobian. In particular, a main result is that cluster-tilted algebras, which are an important subclass of 2-CY-tilted algebras, are Jacobian (Corollary 5.12). This is an easy consequence of Theorem 5.1.

More generally we show that a large class of 2-CY-tilted algebras coming from triangulated 2-CY categories associated with elements in Coxeter groups (see [BIRS]) are also Jacobian (Theorem 6.4).

(C) It is an open question whether there exists a (mutation-) operation on algebras with the following property:  $\text{End}_{\mathcal{C}}(T)$  should be sent to  $\text{End}_{\mathcal{C}}(T^*)$ , when  $T$  is a cluster-tilting object in a triangulated 2-CY category and  $T^*$  is a cluster-tilting object obtained from  $T$  by mutation.

Similarly, in [DWZ1] a question was posed, asking if there is a (mutation-) operation defined on algebras with the property: the Jacobian algebra  $\mathcal{P}(Q, W)$  should be sent to the Jacobian algebra  $\mathcal{P}(Q', W')$ , when  $(Q, W)$  is a quiver with potential and  $(Q', W')$  is a quiver with potential obtained from  $(Q, W)$  by mutation.

It is an easy consequence of (A) that both questions have an affirmative answer for 2-CY-tilted algebras which are Jacobian. By recent results in [A], Jacobian algebras of Jacobi-finite QP’s have this property (Theorem 5.13, Corollary 5.14).

(D) Another main result in this paper is that cluster-tilted algebras are determined by their quivers. This was shown for finite representation type in [BMR3]. We here give a short proof of this fact (Theorem 2.3). Alternatively, it is also a direct consequence of (A).

(E) One of the starting points of tilting theory is the reflection functors [BGP, APR], giving a *nearly Morita equivalence*, in the terminology of [Rin], meaning that  $\frac{\text{mod } \Lambda}{\text{add } S_k} \simeq \frac{\text{mod } \Lambda'}{\text{add } S'_k}$ , where  $S_k$  and  $S'_k$  are the simple modules associated with the vertex  $k$ , where  $k$  is a sink or a source in the quiver of the path algebra  $\Lambda$ , and  $\Lambda'$  is the path algebra obtained by changing direction of all arrows adjacent to  $k$ . An important property of cluster-tilted algebras (and more generally 2-CY-tilted

algebras), is that this nearly Morita equivalence generalizes in the sense that one replaces reflection at a source/sink with mutation at any vertex [BMR1, KR1]. A consequence of this is that the cluster-tilted algebras (in one mutation class) have the same “number” of indecomposable modules. In view of the close connection between 2-CY-tilted algebras and Jacobian algebras it is natural to investigate if a property known to hold for one of the classes also holds for the other one. In this spirit we show that nearly Morita equivalence holds also for two Jacobian algebras where one is obtained from the other by mutation of potentials (Theorem 7.1). We show this also for algebras which are not finite dimensional, dealing with the category of finite length modules. In view of our previous results, this gives a generalization of the above result for cluster-tilted algebras. For the proof we use a functorial approach to results and techniques of [DWZ1].

The organisation of the paper is as follows. In section 1 we collect some known results on the different kinds of mutation and on cluster-tilted algebras. In section 2 we prove that cluster-tilted algebras are determined by their quiver. The connection between cluster-tilting mutation and mutation of quivers with potential is given in sections 3,4 and 5. In section 6 we show that a class of 2-CY-tilted algebras associated with reduced expressions of elements of the Coxeter groups are given by QP’s, which are rigid in the sense of [DWZ1]. The result on nearly Morita equivalence is given in section 7.

The results in this paper have been presented at conferences in Oxford and Oberwolfach [Re, I3], That cluster-tilted algebras and some of the 2-CY-tilted algebras associated with elements in Coxeter groups are Jacobian is proved independently in [K2] using completely different methods (see also [A]). We have been informed by Zelevinsky that the result on nearly Morita equivalence is independently proved in [DWZ2].

**Conventions** All modules are left modules, and a composition  $ab$  of morphisms (respectively, arrows)  $a$  and  $b$  means first  $a$  and then  $b$ . We denote by  $s(a)$  the start vertex of an arrow or path  $a$  and  $e(a)$  denotes the end vertex. For an algebra  $\Lambda$  we denote by  $\text{mod } \Lambda$  the category of finitely generated  $\Lambda$ -modules, and by  $\text{f.l. } \Lambda$  the category of finite length  $\Lambda$ -modules. We denote by  $J_\Lambda$  the Jacobson radical of an algebra  $\Lambda$ , and by  $J_{\mathcal{C}}$  the Jacobson radical of an additive category  $\mathcal{C}$ . For an object  $T$  in an additive category  $\mathcal{C}$  we denote by  $Q_T$  the quiver of  $\text{End}_{\mathcal{C}}(T)$ .

## 1. PRELIMINARIES ON MUTATION

In this section we discuss different kinds of mutations. We first recall the Fomin-Zelevinsky mutation of quivers and the recent mutation of quivers with potentials from [DWZ1]. Then we consider the mutation of cluster-tilting objects in 2-Calabi-Yau (2-CY) categories, which often gives a categorical modelling of quiver mutation. A main theme of this paper is the comparison of the last two mutations. Since cluster-tilted algebras play a central role in this paper, we recall their definition and some of their basic properties. Finally, we consider mutation of tilting modules over 3-Calabi-Yau algebras, which will also be compared to mutation of quivers with potentials.

**1.1. Mutation of quivers.** Let  $Q$  be a finite quiver with vertices  $1, \dots, n$ , and having no loops or 2-cycles. For any vertex  $k$ , Fomin-Zelevinsky [FZ] defined a new quiver  $\mu_k(Q)$  as follows. Let  $b_{ij}$  and  $b'_{ij}$  denote the number of arrows from  $i$  to  $j$

minus the number of arrows from  $j$  to  $i$  in  $Q$  and  $\mu_k(Q)$ , respectively. Then we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{else.} \end{cases}$$

Clearly we have  $\mu_k(\mu_k(Q)) \simeq Q$ . This mutation is an essential ingredient in the theory of cluster algebras initiated in [FZ]. Note that if the vertex  $k$  is a sink or a source, then this mutation coincides with the Bernstein-Gelfand-Ponomarev reflection of quivers [BGP].

**1.2. Mutation of quivers with potentials.** Let  $Q$  be a finite connected quiver without loops, and with vertices  $1, \dots, n$ , and set of arrows  $Q_1$ . We denote by  $KQ_i$  the  $K$ -vector space with basis  $Q_i$  consisting of paths of length  $i$  in  $Q$ , and by  $KQ_{i,\text{cyc}}$  the subspace of  $KQ_i$  spanned by all cycles. Consider the complete path algebra

$$\widehat{KQ} = \prod_{i \geq 0} KQ_i$$

over an algebraically closed field  $K$ . A *quiver with potential* ( $QP$  for short) is a pair  $(Q, W)$  consisting of a quiver  $Q$  without loops and an element  $W \in \prod_{i \geq 2} KQ_{i,\text{cyc}}$  (called a *potential*). It is called *reduced* if  $W \in \prod_{i \geq 3} KQ_{i,\text{cyc}}$ . The cyclic derivative  $\partial_a W$  is defined by  $\partial_a(a_1 \cdots a_\ell) = \sum_{a_i=a} a_{i+1} \cdots a_\ell a_1 \cdots a_{i-1}$  and extended linearly and continuously. A QP gives rise to what has been called the associated *Jacobian algebra* [DWZ1]

$$\mathcal{P}(Q, W) = \widehat{KQ} / \mathcal{J}(W),$$

where  $\mathcal{J}(W) = \overline{\langle \partial_a W \mid a \in Q_1 \rangle}$  is the closure of the ideal generated by  $\partial_a W$  with respect to the  $J_{\widehat{KQ}}$ -adic topology.

Two potentials  $W$  and  $W'$  are called *cyclically equivalent* if  $W - W' \in \overline{[KQ, KQ]}$ , where  $[-, -]$  denote the vector space spanned by commutators. Two QP's  $(Q, W)$  and  $(Q', W')$  are called *right-equivalent* if  $Q_0 = Q'_0$  and there exists a  $K$ -algebra isomorphism  $\phi: \widehat{KQ} \rightarrow \widehat{KQ}'$  such that  $\phi|_{Q_0} = \text{id}$  and  $\phi(W)$  and  $W'$  are cyclically equivalent. In this case  $\phi$  induces an isomorphism  $\mathcal{P}(Q, W) \simeq \mathcal{P}(Q', W')$ . It is shown in [DWZ1] that for any QP  $(Q, W)$  there exists a reduced QP  $(Q_{\text{red}}, W_{\text{red}})$  such that  $\mathcal{P}(Q, W) \simeq \mathcal{P}(Q_{\text{red}}, W_{\text{red}})$ , which is uniquely determined up to right-equivalence. We call  $(Q_{\text{red}}, W_{\text{red}})$  a *reduced part* of  $(Q, W)$ . For example, a reduced part of the QP  $(Q, W)$  below is given by the QP  $(Q_{\text{red}}, W_{\text{red}})$  below.

$$(Q, W) = \left( \begin{array}{ccc} & \begin{array}{c} \swarrow d \\ \downarrow a \\ \searrow b \end{array} & \\ 1 & \xrightarrow{a} 2 & \xrightarrow{b} 3 \\ & \begin{array}{c} \nearrow c \end{array} & \end{array} , cd + abd \right) \quad (Q_{\text{red}}, W_{\text{red}}) = \left( 1 \xrightarrow{a} 2 \xrightarrow{b} 3 , 0 \right)$$

A QP  $(Q, W)$  is called *rigid* [DWZ1] if the deformation space  $\mathcal{P}(Q, W) / (KQ_0 + \overline{[\mathcal{P}(Q, W), \mathcal{P}(Q, W)]})$  is zero, or equivalently  $J_{\widehat{KQ}} = \mathcal{J}(W) + \overline{[KQ, KQ]}$  holds.

A mutation  $\mu_k(Q, W)$  of a QP  $(Q, W)$  is introduced in [DWZ1] for each vertex  $k$  in  $Q$  not lying on a 2-cycle. It is defined as a reduced part of  $(Q', W') = \widetilde{\mu}_k(Q, W)$ , the latter one being given as follows. Replacing  $W$  by a cyclically equivalent potential, we assume that no cycles in  $W$  start at  $k$ .

- (a)  $Q'$  is a quiver obtained from  $Q$  by the following changes.
  - (i) Replace the fixed vertex  $k$  in  $Q$  by a new vertex  $k^*$ . (Although  $k$  and  $k^*$  are identified in [DWZ1], we distinguish them to avoid any confusion.)



We say that a 2-CY category  $\mathcal{C}$  has *no loops or 2-cycles* if the quiver  $Q_T$  of  $\text{End}_{\mathcal{C}}(T)$  has no loops or 2-cycles for any cluster-tilting object  $T$  in  $\mathcal{C}$  (see [BIRS]). Under this assumption we have the following important connection with the Fomin-Zelevinsky quiver mutation. For a cluster-tilting object  $T$  we have

$$Q_{\mu_k(T)} \simeq \mu_k(Q_T)$$

by [BMR2, BIRS]. Hence  $\mathcal{C}$  has a cluster structure in the sense of [BIRS].

**1.4. Cluster-tilted algebras.** Cluster categories are by definition the orbit categories  $\mathcal{C}_Q = \mathcal{D}^b(KQ)/\tau^{-1}[1]$ , where  $Q$  is a finite connected acyclic quiver,  $\mathcal{D}^b(KQ)$  is the bounded derived category of the finite dimensional (left)  $KQ$ -modules, and  $\tau$  is the AR-translation in  $\mathcal{D}^b(KQ)$ , see [BMRRT]. These orbit categories are known to be triangulated by [K1], and are Hom-finite 2-CY [BMRRT]. The *cluster-tilted* algebras are by definition the 2-CY-tilted algebras coming from cluster categories [BMR1].

For cluster categories  $\mathcal{C}_Q$ , the cluster-tilting objects are induced by tilting  $KQ'$ -modules over path algebras  $KQ'$  derived equivalent to  $KQ$ . It is known [BMRRT] that given any two cluster-tilting objects in  $\mathcal{C}_Q$ , then one can be obtained from the other by a finite sequence of mutations of cluster-tilting objects, and hence there is only one mutation class of cluster-tilted algebras by [BMRRT], based on [HU].

By [ABS] ([BR] in the Dynkin case) the quiver of  $\text{End}_{\mathcal{C}_Q}(T)$  is obtained from the quiver of the tilted algebra  $\text{End}_{KQ'}(T)$  by adding an arrow in the opposite direction for each relation in a minimal set of relations. In particular, we have the following, which is also easily seen directly.

**Lemma 1.1.** *Let  $T$  be a tilting module in  $\text{mod } KQ$ . Then  $\text{End}_{\mathcal{C}_Q}(T) \simeq KQ'$  for a quiver  $Q'$  with no oriented cycles, if and only if  $\text{End}_{KQ}(T) \simeq KQ'$ .*

If the quiver  $Q_T$  of a 2-CY-tilted algebra coming from a 2-CY category  $\mathcal{C}$  is acyclic, then it follows from [KR1] that  $\text{End}_{\mathcal{C}}(T) \simeq KQ_T$ . Moreover if  $\mathcal{C}$  is a connected algebraic triangulated category (that is, the stable category of a Frobenius category), then  $\mathcal{C}$  is equivalent to the cluster category  $\mathcal{C}_{Q_T}$  [KR2].

**1.5. Mutation of tilting modules over 3-Calabi-Yau algebras.** Let  $R$  be a formal power series ring  $K[[x, y, z]]$  of three variables over an algebraically closed field  $K$ , and let  $\Lambda$  be an  $R$ -algebra which is a finitely generated  $R$ -module. We call  $\Lambda$  *3-Calabi-Yau* (*3-CY* for short) if the bounded derived category  $\mathcal{D}^b(\text{f.l. } \Lambda)$  is a 3-CY category [Boc, CR, G, IR]. It was shown in [IR] that  $\Lambda$  is 3-CY if and only if  $\Lambda$  is a Cohen-Macaulay  $R$ -module which is a symmetric  $R$ -algebra with  $\text{gl. dim } \Lambda = 3$ . Rickard proved that 3-CY algebras are closed under derived equivalences (see [IR]).

Let  $\Lambda$  be a 3-CY algebra. For a basic tilting  $\Lambda$ -module  $T$  of projective dimension at most one, we have another 3-CY algebra  $\Gamma = \text{End}_{\Lambda}(T)$ . We take an indecomposable decomposition  $T = T_1 \oplus \cdots \oplus T_n$ . Let  $e_k \in \Gamma$  be the idempotent corresponding to  $T_k$  for  $k = 1, \dots, n$ . We let  $I_k = \Gamma(1 - e_k)\Gamma$ . Let us recall the following result [IR, Th. 5.4, Th. 7.1].

**Proposition 1.2.** *If  $\Gamma/I_k \in \text{f.l. } \Gamma$ , then there exists a minimal projective resolution*

$$0 \rightarrow \Gamma e_k \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} \Gamma e_k \rightarrow \Gamma/I_k \rightarrow 0 \quad (1)$$

*with  $P_0, P_1 \in \text{add } \Gamma(1 - e_k)$  satisfying the following conditions.*

(a) Applying  $\text{Hom}_\Gamma(-, \Gamma)$  to (1), we have an exact sequence

$$0 \rightarrow e_k \Gamma \rightarrow \text{Hom}_\Gamma(P_0, \Gamma) \rightarrow \text{Hom}_\Gamma(P_1, \Gamma) \rightarrow e_k \Gamma \rightarrow \Gamma/I_k \rightarrow 0. \quad (2)$$

(b) Let  $T_k^* = \text{Ker}(T \otimes_\Gamma f_0)$  and  $\mu_k(T) = (T/T_k) \oplus T_k^*$ . Then  $\mu_k(T)$  is a basic tilting  $\Lambda$ -module of projective dimension at most one and  $T_k^* \not\cong T_k$ .

(c) If  $Q_T$  and  $Q_{\mu_k(T)}$  have no loops or 2-cycles, then  $Q_{\mu_k(T)} \simeq \mu_k(Q_T)$  holds.

Notice that we can not drop the assumption  $\Gamma/I_k \in \text{f.l.}\Gamma$ , which is automatically satisfied if  $Q_T$  has no loops at  $k$ . We need the following additional information for later application.

**Proposition 1.3.** *If  $\Gamma/I_k \in \text{f.l.}\Gamma$ , then the following assertions hold.*

(a) *There exist exact sequences (called exchange sequences)*

$$0 \rightarrow T_k^* \xrightarrow{g} U_k \xrightarrow{f} T_k \quad \text{and} \quad 0 \rightarrow T_k \xrightarrow{g'} U'_k \xrightarrow{f'} T_k^*$$

*such that  $f$  and  $f'$  are right  $\text{add}(T/T_k)$ -approximations, and  $g$  and  $g'$  are left  $\text{add}(T/T_k)$ -approximations.*

(b) *In the category of finitely generated  $\Lambda$ -modules with projective dimension at most one,  $g$  and  $g'$  are kernels of  $f$  and  $f'$  respectively, and  $f$  and  $f'$  are cokernels of  $g$  and  $g'$  respectively.*

(c) *The complex  $T_k \xrightarrow{g'} U'_k \xrightarrow{f'g} U_k \xrightarrow{f} T_k$  induces exact sequences*

$$\begin{aligned} 0 \rightarrow (T, T_k) &\xrightarrow{g'} (T, U'_k) \xrightarrow{f'g} (T, U_k) \xrightarrow{f} J_{\text{mod } \Lambda}(T, T_k), \\ 0 \rightarrow (T_k, T) &\xrightarrow{f} (U_k, T) \xrightarrow{f'g} (U'_k, T) \xrightarrow{g'} J_{\text{mod } \Lambda}(T_k, T), \end{aligned}$$

*where the right maps are surjective if  $Q_T$  has no loops at  $k$ .*

*Proof.* (a)(c) We have  $\text{Tor}_i^\Gamma(T, \Gamma/I_k) = 0$  for any  $i > 1$ . Applying  $T \otimes_\Gamma -$  to the sequence (1) and putting  $U_k = T \otimes_\Gamma P_0$  and  $U'_k = T \otimes_\Gamma P_1$ , we have the exact sequences in (a).

Since the functor  $T \otimes_\Gamma : \text{add}_\Gamma \Gamma \rightarrow \text{add}_\Lambda T$  is an equivalence and the sequence (1) is exact, we have that  $f$  and  $f'$  are right  $\text{add}(T/T_k)$ -approximations, and that the upper sequence in (c) is exact if  $Q_T$  has no loops at  $k$ . Since the functor  $\text{Hom}_{\Gamma^{\text{op}}}(\cdot, T) : \text{add}_\Gamma \Gamma \rightarrow \text{add}_\Lambda T$  is a duality and the sequence (2) is exact, we have that  $g$  and  $g'$  are left  $\text{add}(T/T_k)$ -approximations, and that the lower sequence in (c) is exact if  $Q_T$  has no loops at  $k$ .

(b) Clearly  $g$  and  $g'$  are kernels of  $f$  and  $f'$  respectively. Since  $\Lambda$  is 3-CY, we have  $\text{Ext}_\Lambda^i(\text{f.l. } \Lambda, \Lambda) = 0$  for  $0 \leq i \leq 2$  [IR]. Thus any  $\Gamma$ -module  $X$  with projective dimension at most one satisfies  $\text{Ext}_\Lambda^i(\text{f.l. } \Lambda, X) = 0$  for  $i = 0, 1$ . On the other hand, since  $\Gamma/I_k \in \text{f.l.}\Gamma$  by our assumption, we have that  $\text{Cok } f = T \otimes_\Gamma (\Gamma/I_k)$  and  $\text{Cok } f' = \text{Tor}_1^\Gamma(T, \Gamma/I_k)$  are in  $\text{f.l.}\Lambda$ . Thus we have  $\text{Ext}_\Lambda^i(\text{Cok } f, X) = 0 = \text{Ext}_\Lambda^i(\text{Cok } f', X)$  for  $i = 0, 1$ . This implies that  $f$  and  $f'$  are cokernels of  $g$  and  $g'$  respectively.  $\square$

## 2. CLUSTER-TILTED ALGEBRAS ARE DETERMINED BY THEIR QUIVERS

In this section we prove that cluster-tilted algebras are determined by their quivers. Alternative proofs using potentials will be given as a consequence of the main result in section 5.

First we prove the following.

**Lemma 2.1.** *Let  $T$  be a cluster-tilting object in some cluster category  $\mathcal{C}_Q$ , with  $Q_T$  an acyclic quiver. Then  $\mathcal{C}_{Q_T}$  is equivalent to the cluster category  $\mathcal{C}_Q$ .*

*Proof.* Although the assertion follows from the main result in [KR2], we give an elementary proof here.

We know that  $T$  is induced by some tilting  $KQ'$ -module  $U$ , where  $KQ'$  is derived equivalent to  $KQ$ . Since  $\text{End}_{\mathcal{C}_Q}(T) \simeq KQ_T$ , we have by Lemma 1.1 that  $\text{End}_{KQ'}(U) \simeq \text{End}_{\mathcal{C}_{Q'}}(T) \simeq \text{End}_{\mathcal{C}_Q}(T) \simeq KQ_T$ , so that  $KQ'$  and  $KQ_T$  are also derived equivalent. Hence  $KQ$  and  $KQ_T$  are derived equivalent, so that the cluster categories  $\mathcal{C}_Q$  and  $\mathcal{C}_{Q_T}$  are equivalent.  $\square$

**Lemma 2.2.** *Let  $Q$  be an acyclic quiver and  $\mathcal{C}_Q$  the associated cluster category. Let  $T$  be a cluster-tilting object in  $\mathcal{C}_Q$  such that the quiver  $Q_T$  of  $\text{End}_{\mathcal{C}}(T)$  is isomorphic to  $Q$ . Then there is an autoequivalence of triangulated categories  $F: \mathcal{C}_Q \rightarrow \mathcal{C}_Q$  with  $F(T) \simeq KQ$ .*

*Proof.* We have that  $T$  is induced by a tilting  $KQ'$ -module  $U$ , satisfying  $\text{End}_{KQ'}(U) \simeq KQ$  by Lemma 1.1. Then we have a commutative diagram

$$\begin{array}{ccccc}
 U & \in & \mathcal{D}^b(KQ') & \xrightarrow{\mathbf{R}\text{Hom}_{KQ'}(U, \cdot)} & \mathcal{D}^b(KQ) & \ni & KQ \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 T & \in & \mathcal{C}_Q & \xrightarrow{F} & \mathcal{C}_Q & \ni & KQ
 \end{array}$$

where  $\mathbf{R}\text{Hom}_{KQ'}(U, \cdot): \mathcal{D}^b(KQ') \rightarrow \mathcal{D}^b(KQ)$  is an equivalence of triangulated categories since  $U$  is a tilting module [H]. Hence  $\mathbf{R}\text{Hom}_{KQ'}(U, \cdot)$  commutes with  $\tau$  and [1], so that there is an equivalence  $F: \mathcal{C}_Q \rightarrow \mathcal{C}_Q$  of triangulated categories, with  $F(T) \simeq KQ$ .  $\square$

Using this we get the following

**Theorem 2.3.** *Let  $T_1$  and  $T_2$  be cluster-tilting objects in the cluster categories  $\mathcal{C}_{Q_1}$  and  $\mathcal{C}_{Q_2}$  respectively, and assume that the quivers  $Q_{T_1}$  and  $Q_{T_2}$  are isomorphic. Then there is an equivalence of triangulated categories  $F: \mathcal{C}_{Q_1} \rightarrow \mathcal{C}_{Q_2}$  such that  $F(T_1) \simeq T_2$ . In particular  $\text{End}_{\mathcal{C}_{Q_1}}(T_1) \simeq \text{End}_{\mathcal{C}_{Q_2}}(T_2)$  holds.*

*Proof.* We have a sequence of mutations  $\mu = \mu_{k_\ell} \circ \cdots \circ \mu_{k_1}$  such that  $\mu(KQ_1) \simeq T_1$  in  $\mathcal{C}_{Q_1}$ . Let  $T$  be a cluster-tilting object in  $\mathcal{C}_{Q_2}$  such that  $\mu(T) \simeq T_2$ . Since  $Q_{T_1} \simeq Q_{T_2}$ , we have  $Q_1 \simeq Q_2$ . Hence by Lemmas 2.1 and 2.2 there is an equivalence of triangulated categories  $F: \mathcal{C}_{Q_1} \rightarrow \mathcal{C}_{Q_2}$  such that  $F(KQ_1) \simeq T$ . Since cluster-tilting mutations commute with equivalences of triangulated categories, we have

$$F(T_1) \simeq F(\mu(KQ_1)) \simeq \mu(F(KQ_1)) \simeq \mu(T) \simeq T_2,$$

and consequently  $\text{End}_{\mathcal{C}_{Q_1}}(T_1) \simeq \text{End}_{\mathcal{C}_{Q_2}}(T_2)$ .  $\square$

We can strengthen the above theorem as follows.

**Corollary 2.4.** *Let  $T_1$  be a cluster-tilting object in a cluster category and  $T_2$  a cluster-tilting object in an algebraic 2-CY triangulated category  $\mathcal{C}$  without loops or 2-cycles. If  $Q_{T_1} \simeq Q_{T_2}$ , then  $\text{End}_{\mathcal{C}}(T_1) \simeq \text{End}_{\mathcal{C}}(T_2)$ .*



*Proof.* By the same argument as in the proof of Theorem 2.3, there exists a sequence  $\mu$  of mutations such that  $\mu(T) \simeq T_2$  and  $Q_T$  is acyclic. Since  $\mathcal{C}$  is assumed to be algebraic, it follows from [KR2] that  $\mathcal{C}$  is equivalent to  $\mathcal{C}_{Q_T}$ , so the result follows from Theorem 2.3.  $\square$

### 3. PRELIMINARIES ON PRESENTATION OF ALGEBRAS

In this section we consider presentations of algebras, which are important for the rest of this paper.

Let  $K$  as before be an algebraically closed field and  $\Gamma$  a  $K$ -algebra with Jacobson radical  $J_\Gamma$ . We regard  $\Gamma$  as a topological algebra via the  $J_\Gamma$ -adic topology with a basic system  $\{J_\Gamma^i\}_{i \geq 0}$  of open neighborhoods of 0. Thus the closure of a subset  $S$  of  $\Gamma$  is given by

$$\bar{S} = \bigcap_{\ell \geq 0} (S + J_\Gamma^\ell). \quad (3)$$

We assume that  $\Gamma$  satisfies the following conditions:

- (A1)  $\Gamma/J_\Gamma \simeq K^n$  for some  $n > 0$  and  $\dim_K(J_\Gamma/J_\Gamma^2) < \infty$ ,
- (A2) the  $J_\Gamma$ -adic topology on  $\Gamma$  is complete and separated, i.e.  $\Gamma \simeq \varprojlim_{\ell \geq 0} \Gamma/J_\Gamma^\ell$ .

For example, finite dimensional  $K$ -algebras, complete path algebras  $\widehat{KQ}$  of finite quivers  $Q$ , and algebras over  $R = K[[x_1, \dots, x_d]]$  which are finitely generated  $R$ -modules satisfy (A1) and (A2). If  $\phi: \Lambda \rightarrow \Gamma$  is a homomorphism of  $K$ -algebras satisfying (A1), then  $J_\Lambda \subset \phi^{-1}(J_\Gamma)$  holds. In particular  $\phi$  is continuous.

Let  $Q$  be a finite quiver. It is convenient to introduce the following concepts. For  $a \in Q_1$ , define a *right derivative*  $\partial_a^r: J_{\widehat{KQ}}^r \rightarrow \widehat{KQ}$  and a *left derivative*  $\partial_a^l: J_{\widehat{KQ}}^l \rightarrow \widehat{KQ}$  by

$$\begin{aligned} \partial_a^r(a_1 a_2 \cdots a_{m-1} a_m) &= \begin{cases} a_1 a_2 \cdots a_{m-1} & \text{if } a_m = a, \\ 0 & \text{otherwise,} \end{cases} \\ \partial_a^l(a_1 a_2 \cdots a_{m-1} a_m) &= \begin{cases} a_2 \cdots a_{m-1} a_m & \text{if } a_1 = a, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and extend to  $J_{\widehat{KQ}}$  linearly and continuously. Then  $\partial_a^r$  is a homomorphism of  $\widehat{KQ}$ -modules and  $\partial_a^l$  is a homomorphism of  $\widehat{KQ}^{\text{op}}$ -modules. Clearly for any  $p \in J_{\widehat{KQ}}$  we have

$$p = \sum_{a \in Q_1} (\partial_a^r p) a = \sum_{a \in Q_1} a (\partial_a^l p).$$

For simplicity we call  $p \in \widehat{KQ}$  *basic* if  $p$  is a formal linear sum of paths in  $Q$  with a common start  $i$  and a common end  $j$ . In this case we write  $s(p) = i$  and  $e(p) = j$ . Clearly any ideal of  $\widehat{KQ}$  is generated by a set of basic elements.

We start with the following basic fact.

**Proposition 3.1.** *Let  $\Gamma$  be a  $K$ -algebra satisfying (A1) and (A2), and let  $Q$  be a finite quiver. Assume that we have a  $K$ -algebra homomorphism  $\phi_0: KQ_0 \rightarrow \Gamma$  and a homomorphism  $\phi_1: KQ_1 \rightarrow J_\Gamma$  of  $(KQ_0, KQ_0)$ -modules. Then the following assertions hold.*

- (a)  $\phi_0$  and  $\phi_1$  extend uniquely to a  $K$ -algebra homomorphism  $\phi: \widehat{KQ} \rightarrow \Gamma$ . Moreover  $\text{Ker } \phi$  is a closed ideal of  $\widehat{KQ}$ .
- (b) The following conditions are equivalent.

- (i)  $\phi: \widehat{KQ} \rightarrow \Gamma$  is surjective.
- (ii)  $\phi_0$  and  $\phi_1$  induce surjections  $KQ_0 \rightarrow \Gamma/J_\Gamma$  and  $KQ_1 \rightarrow J_\Gamma/J_\Gamma^2$ .
- (iii)  $\phi_0$  induces a surjection  $KQ_0 \rightarrow \Gamma/J_\Gamma$ , and for any  $i \in Q_0$  we have an exact sequence

$$\bigoplus_{a \in Q_1, e(a)=i} \Gamma(\phi s(a)) \xrightarrow{(\phi a)_a} J_\Gamma(\phi i) \rightarrow 0.$$

- (iv)  $\phi_0$  induces a surjection  $KQ_0 \rightarrow \Gamma/J_\Gamma$ , and for any  $i \in Q_0$  we have an exact sequence

$$\bigoplus_{a \in Q_1, s(a)=i} (\phi s(a))\Gamma \xrightarrow{(\phi a)_a} (\phi i)J_\Gamma \rightarrow 0.$$

*Proof.* (a) The first assertion is clear. We observed that  $\phi$  is continuous. Thus  $\{0\}$  is a closed subset of  $\Gamma$  and its inverse image  $\text{Ker } \phi$  is a closed subset of  $\widehat{KQ}$ .

(b) (ii) $\Rightarrow$ (i)  $\phi$  induces a surjection  $\phi_\ell: KQ_\ell \rightarrow J_\Gamma^\ell/J_\Gamma^{\ell+1}$  for any  $\ell$ . For any  $x \in \Gamma$ , there exists  $x_\ell \in J_\Gamma^\ell$  and  $p_\ell \in KQ_\ell$  such that  $x_{\ell+1} = x_\ell - \phi_\ell(p_\ell)$ . Then we have  $\phi(\sum_{\ell \geq 0} p_\ell) = x$ .

(i) $\Rightarrow$ (iii) It is enough to show that  $\phi(p)$  is in the image of  $(\phi a)_a$  for any basic element  $p \in J_{\widehat{KQ}}$  with  $e(p) = i$ . This follows from the equality  $p = \sum_{a \in Q_1} (\partial_a^r p)a$  since this implies  $\phi p = (\phi \partial_a^r p)_a \cdot (\phi a)_a$ .

(iii) $\Rightarrow$ (ii) The sequence in (iii) induces a surjection  $\bigoplus_{a \in Q_1} \Gamma/J_\Gamma \xrightarrow{(\phi a)_a} J_\Gamma/J_\Gamma^2$ .

The rest follows similarly.  $\square$

We shall need the following easy fact.

**Lemma 3.2.** *Let  $Q$  be a finite quiver and  $\mathbf{S}$  a finite subset of  $J_{\widehat{KQ}}$ . For the ideal  $I = \langle \mathbf{S} \rangle$  of  $\widehat{KQ}$ , we have  $I = \sum_{v \in \mathbf{S}} (\widehat{KQ})v + I \cdot Q_1$  and  $\bar{I} = \sum_{v \in \mathbf{S}} (\widehat{KQ})v + \bar{I} \cdot Q_1$ .*

*Proof.* Since  $I$  is generated by  $\mathbf{S}$ , we have the first equality.

Let  $p \in \bar{I}$ . For any  $\ell > 0$  we can write  $p = r_\ell + r'_\ell$  for some  $r_\ell \in I$  and  $r'_\ell \in J_\Gamma^\ell$ . Then  $p_\ell = r_{\ell+1} - r_\ell = r'_\ell - r'_{\ell+1} \in I \cap J_\Gamma^\ell$  for  $\ell > 0$ . Letting  $p_0 = r_1$ , we have  $p_0 + p_1 + \dots + p_\ell = p - r'_{\ell+1}$ . Thus  $p = \sum_{\ell \geq 0} p_\ell$ . We write  $p_\ell = \sum_{v \in \mathbf{S}} p'_{\ell,v}v + \sum_{a \in Q_1} p''_{\ell,a}a$  for  $p'_{\ell,v} \in \widehat{KQ}$  and  $p''_{\ell,a} \in I$ . Then  $p = \sum_{v \in \mathbf{S}} (\sum_{\ell \geq 0} p'_{\ell,v})v + \sum_{a \in Q_1} (\sum_{\ell \geq 0} p''_{\ell,a})a$  holds. Thus we have the second equality.  $\square$

We now give a result on description of relations.

**Proposition 3.3.** *Assume that the conditions in Proposition 3.1(b) are satisfied. For a finite set  $\mathbf{S}$  of basic elements in  $J_{\widehat{KQ}}$ , the following conditions are equivalent.*

- (a)  $\text{Ker } \phi = \bar{I}$  holds for the ideal  $I = \langle \mathbf{S} \rangle$  of  $\widehat{KQ}$ .
- (b) The following sequence is exact for any  $i \in Q_0$ .

$$\bigoplus_{v \in \mathbf{S}, e(v)=i} \Gamma(\phi s(v)) \xrightarrow{(\phi \partial_a^r v)_{v,a}} \bigoplus_{a \in Q_1, e(a)=i} \Gamma(\phi s(a)) \xrightarrow{(\phi a)_a} J_\Gamma(\phi i) \rightarrow 0. \quad (4)$$

- (c) The following sequence is exact for any  $i \in Q_0$ .

$$\bigoplus_{v \in \mathbf{S}, s(v)=i} (\phi e(v))\Gamma \xrightarrow{(\phi \partial_a^l v)_{v,a}} \bigoplus_{a \in Q_1, s(a)=i} (\phi e(a))\Gamma \xrightarrow{(\phi a)_a} (\phi i)J_\Gamma \rightarrow 0.$$

*Proof.* (a) $\Rightarrow$ (b) Since

$$(\phi \partial_a^r v)_a \cdot (\phi a)_a = \phi \left( \sum_{a \in Q_1, e(a)=i} (\partial_a^r v)_a \right) = \phi v = 0,$$

the sequence is a complex. Now we assume that  $(p_a)_a \in \bigoplus_{\substack{a \in Q_1 \\ e(a)=i}} \widehat{KQ}(s(a))$  satisfies  $(\phi p_a)_a \cdot (\phi a)_a = 0$ . Since

$$\sum_{a \in Q_1, e(a)=i} p_a a \in \text{Ker } \phi = \bar{I},$$

there exists  $q_v \in \widehat{KQ}$  by Lemma 3.2 such that

$$\sum_{a \in Q_1, e(a)=i} p_a a - \sum_{v \in \mathbf{S}} q_v v \in \bar{I} \cdot Q_1.$$

Applying  $\partial_a^r$  on both sides, we have

$$p_a - \sum_{v \in \mathbf{S}} q_v \partial_a^r v \in \bar{I} = \text{Ker } \phi.$$

Thus  $(\phi q_v)_v \in \bigoplus_{v \in \mathbf{S}, e(v)=i} \Gamma(s(v))$  satisfies

$$(\phi p_a)_a = (\phi q_v)_v \cdot (\phi \partial_a^r v)_{v,a}.$$

(b) $\Rightarrow$ (a) We shall show that  $\text{Ker } \phi = \bar{I}$ . Take any  $v \in \mathbf{S}$  with  $e(v) = i$ . Since (4) is exact, we have

$$\phi v = \phi \left( \sum_{a \in Q_1, e(a)=i} (\partial_a^r v)_a \right) = (\phi \partial_a^r v)_a \cdot (\phi a)_a = 0.$$

This implies  $v \in \text{Ker } \phi$  and  $I \subset \text{Ker } \phi$ . Since  $\text{Ker } \phi$  is a closed ideal by Proposition 3.1, we have  $\bar{I} \subset \text{Ker } \phi$ .

To prove  $\text{Ker } \phi \subset \bar{I}$ , we will first show that, for any  $p \in \text{Ker } \phi$ , there exists  $p' \in I$  such that  $p - p' \in (\text{Ker } \phi) \cdot Q_1$ . Without loss of generality, we can assume that  $p$  is basic with  $e(p) = i$ . Since

$$(\phi \partial_a^r p)_a \cdot (\phi a)_a = \phi p = 0$$

holds, we have that  $(\phi \partial_a^r p)_a$  factors through the left map  $(\phi \partial_a^r v)_{v,a}$  in (4). Thus there exists  $q_v \in \widehat{KQ}$  such that  $(\phi \partial_a^r p)_a = (\phi q_v)_v \cdot (\phi \partial_a^r v)_{v,a}$ . Then

$$\partial_a^r p - \sum_{v \in \mathbf{S}, e(v)=i} q_v \partial_a^r v \in \text{Ker } \phi.$$

Hence

$$p - \sum_{v \in \mathbf{S}, e(v)=i} q_v v = \sum_{a \in Q_1, e(a)=i} (\partial_a^r p - \sum_{v \in \mathbf{S}, e(v)=i} q_v \partial_a^r v) a \in (\text{Ker } \phi) \cdot Q_1.$$

It follows that  $p' = \sum_{v \in \mathbf{S}, e(v)=i} q_v v \in I$  satisfies the desired condition.

Consequently we have  $\text{Ker } \phi \subset I + (\text{Ker } \phi) \cdot Q_1$ . This implies

$$\text{Ker } \phi \subset I + (I + (\text{Ker } \phi) \cdot Q_1) \cdot Q_1 = I + (\text{Ker } \phi) \cdot Q_1^2 \subset \cdots \subset I + (\text{Ker } \phi) \cdot Q_1^\ell$$

for any  $\ell$ . By (3), we have  $\text{Ker } \phi \subset \bar{I}$ .

(a) $\Leftrightarrow$ (c) can be shown dually.  $\square$

We give a relationship between the dimension of  $\text{Ext}^2$ -spaces and minimal sets of generators for relation ideals in complete path algebras (cf. [Bon, Y]). Note that the corresponding result is not true in general if we deal with ordinary path algebras  $KQ$  instead of  $\widehat{KQ}$ . See [BIKR, section 7] for an example. Note however that if  $KQ/\langle \mathbf{S} \rangle$  is finite dimensional, then the corresponding result is true.

**Proposition 3.4.** *Let  $Q$  be a finite quiver,  $J = J_{\widehat{KQ}}$ ,  $I$  a closed ideal of  $\widehat{KQ}$  contained in  $J^2$ , and  $\Gamma = \widehat{KQ}/I$ . Let  $\mathbf{S}$  be a finite set of basic elements in  $I$ .*

- (a)  $\mathbf{S}$  spans the  $K$ -vector space  $I/(IJ + JI)$  if and only if  $I = \overline{\langle \mathbf{S} \rangle}$ . Moreover  $\mathbf{S}$  is a basis of the  $K$ -vector space  $I/(IJ + JI)$  if and only if  $\mathbf{S}$  is a minimal set satisfying  $I = \overline{\langle \mathbf{S} \rangle}$ .
- (b) If  $\mathbf{S}$  is a minimal set satisfying  $I = \overline{\langle \mathbf{S} \rangle}$ , then we have

$$\dim_K \text{Ext}_\Gamma^2(S_i, S_j) = \#(i\widehat{KQ}j) \cap \mathbf{S} = \dim_K i(I/(IJ + JI))j$$

for any simple  $\Gamma$ -modules  $S_i$  and  $S_j$  associated with  $i, j \in Q_0$ .

*Proof.* (a) Assume first that  $\mathbf{S}$  spans  $I/(IJ + JI)$ , and let  $I' = \overline{\langle \mathbf{S} \rangle}$ . Then we have  $I = I' + IJ + JI$ . This implies

$$I = I' + (I' + IJ + JI)J + J(I' + IJ + JI) = I' + IJ^2 + JIJ + J^2I = \dots = I' + \sum_{0 \leq i \leq \ell} J^i I J^{\ell-i}$$

for any  $\ell > 0$ . Thus we have  $I \subset \bigcap_{\ell > 0} (I' + J^\ell) = I'$  and hence  $I = I' = \overline{\langle \mathbf{S} \rangle}$ .

Next we assume that  $I = \overline{\langle \mathbf{S} \rangle}$ . By Lemma 3.2, we have  $I = \langle \mathbf{S} \rangle + IJ$ . Thus  $I = \langle \mathbf{S} \rangle + IJ + JI = K\mathbf{S} + IJ + JI$  holds.

The second equivalence is an immediate consequence of the first assertion.

- (b) The minimality of  $\mathbf{S}$  implies that the projective resolution in Proposition 3.3(b) is minimal. Thus we have the left equality. The right equality follows from (a) since  $(i\widehat{KQ}j) \cap \mathbf{S}$  is a basis of the  $K$ -vector space  $i(I/(IJ + JI))j$ .  $\square$

Now let  $\mathcal{C}$  be a  $K$ -linear category satisfying the following conditions:

- (C1)  $\mathcal{C}$  is *Krull-Schmidt*, i.e. any object in  $\mathcal{C}$  is isomorphic to a finite direct sum of objects whose endomorphism rings are local.
- (C2)  $\text{End}_{\mathcal{C}}(X)$  satisfies (A1) and (A2) for any basic object  $X \in \mathcal{C}$ .

For a finite quiver  $Q$  and a field  $K$ , we denote by  $\widehat{KQ}$  also the category of finitely generated projective  $\widehat{KQ}$ -modules. We have the following observation from Proposition 3.1(a).

**Lemma 3.5.** *For a category  $\mathcal{C}$  satisfying (C1) and (C2) and a quiver  $Q$ , assume that an object  $\Phi_0 i \in \mathcal{C}$  for any  $i \in Q_0$  and a morphism  $\Phi_1 a \in \mathcal{C}(\Phi s(a), \Phi e(a))$  for any  $a \in Q_1$  are given. Then  $\Phi_0$  and  $\Phi_1$  extend uniquely to a  $K$ -linear functor  $\Phi: \widehat{KQ} \rightarrow \mathcal{C}$ .*

Restating Proposition 3.3, we have the following result.

**Proposition 3.6.** *Let  $\mathcal{C}$  be a category satisfying (C1) and (C2), and let  $Q$  be a finite quiver. Let  $\Phi: \widehat{KQ} \rightarrow \mathcal{C}$  be a  $K$ -linear functor, and let  $T = \bigoplus_{i \in Q_0} \Phi i$ . For a finite set  $\mathbf{S}$  of basic elements in  $J_{\widehat{KQ}}$ , the following conditions are equivalent.*

- (a)  $\Phi$  induces a surjection  $\phi: \widehat{KQ} \rightarrow \text{End}_{\mathcal{C}}(T)$  with  $\text{Ker } \phi = \overline{\langle \mathbf{S} \rangle}$ .

(b) For any  $i \in Q_0$ , we have a complex

$$\bigoplus_{v \in \mathbf{S}, e(v)=i} \Phi_S(v) \xrightarrow{(\Phi \partial_a^r v)_{v,a}} \bigoplus_{a \in Q_1, e(a)=i} \Phi_S(a) \xrightarrow{(\Phi a)_a} \Phi i$$

in  $\text{add } T$  which induces an exact sequence

$$\mathcal{C}(T, \bigoplus_{v \in \mathbf{S}, e(v)=i} \Phi_S(v)) \xrightarrow{(\Phi \partial_a^r v)_{v,a}} \mathcal{C}(T, \bigoplus_{a \in Q_1, e(a)=i} \Phi_S(a)) \xrightarrow{(\Phi a)_a} J_{\mathcal{C}}(T, \Phi i) \rightarrow 0.$$

(c) For any  $i \in Q_0$ , we have a complex

$$\Phi i \xrightarrow{(\Phi a)_a} \bigoplus_{a \in Q_1, s(a)=i} \Phi e(a) \xrightarrow{(\Phi \partial_a^l v)_{a,v}} \bigoplus_{v \in \mathbf{S}, s(v)=i} \Phi e(v)$$

in  $\text{add } T$  which induces an exact sequence

$$\mathcal{C}(\bigoplus_{v \in \mathbf{S}, s(v)=i} \Phi e(v), T) \xrightarrow{(\Phi \partial_a^l v)_{a,v}} \mathcal{C}(\bigoplus_{a \in Q_1, s(a)=i} \Phi e(a), T) \xrightarrow{(\Phi a)_a} J_{\mathcal{C}}(\Phi i, T) \rightarrow 0.$$

#### 4. JACOBIAN ALGEBRAS AND WEAK 2-ALMOST SPLIT SEQUENCES

In this section we study properties of Jacobian algebras of QP's. We also investigate a variation of 2-almost split sequences and AR 4-angles discussed in [I2, IY], which we call weak 2-almost split sequences. We show that there is a close relationship between Jacobian algebras and weak 2-almost split sequences, which will be used to prove the connection between mutation of QP's and of cluster-tilting objects in the next section.

Let  $(Q, W)$  be a QP. For a pair of arrows  $a$  and  $b$ , we define  $\partial_{(a,b)}W$  by

$$\partial_{(a,b)}(a_1 a_2 \cdots a_m) = \sum_{a_i=a, a_{i+1}=b} a_{i+2} \cdots a_m a_1 \cdots a_{i-1}$$

for any cycle  $a_1 \cdots a_m$  in  $W$  and extend linearly and continuously. Clearly  $e(a) \neq s(b)$  implies  $\partial_{(a,b)}W = 0$ .

The following easy observation is useful.

**Lemma 4.1.** (a)  $\sum_{a \in Q_1} (\partial_{(a,b)}W)a = \partial_b W = \sum_{c \in Q_1} c(\partial_{(b,c)}W)$  for any  $b \in Q_1$ .

(b)  $\partial_{(a,b)}W = \partial_a^r(\partial_b W) = \partial_b^l(\partial_a W)$ .

We have the following property of Jacobian algebras of QP's.

**Proposition 4.2.** Let  $(Q, W)$  be a QP and  $\Gamma = \mathcal{P}(Q, W)$ . Let  $\phi : \widehat{KQ} \rightarrow \Gamma$  be a natural surjection. Then there exist complexes

$$\begin{aligned} \Gamma(\phi i) \xrightarrow{(\phi b)_b} \bigoplus_{\substack{b \in Q_1 \\ s(b)=i}} \Gamma(\phi e(b)) \xrightarrow{(\phi \partial_{(a,b)}W)_{b,a}} \bigoplus_{\substack{a \in Q_1 \\ e(a)=i}} \Gamma(\phi s(a)) \xrightarrow{(\phi a)_a} J_{\Gamma}(\phi i) \rightarrow 0, \\ (\phi i)\Gamma \xrightarrow{(\phi a)_a} \bigoplus_{\substack{a \in Q_1 \\ e(a)=i}} (\phi s(a))\Gamma \xrightarrow{(\phi \partial_{(a,b)}W)_{b,a}} \bigoplus_{\substack{b \in Q_1 \\ s(b)=i}} (\phi e(b))\Gamma \xrightarrow{(\phi b)_b} (\phi i)J_{\Gamma} \rightarrow 0 \end{aligned}$$

which are exact except at the second left terms.

*Proof.* Since  $\partial_{(a,b)}W = \partial_a^r(\partial_b W) = \partial_b^l(\partial_a W)$  holds by Lemma 4.1, the assertion follows immediately from Proposition 3.3(a) $\Rightarrow$ (b)(c).  $\square$

As an immediate application, we have the following consequence.

**Proposition 4.3.** *Let  $(Q, W)$  be a QP and  $\Gamma = \mathcal{P}(Q, W)$ .*

- (a)  $\text{Ext}_\Gamma^2(S, \Gamma) = 0 = \text{Ext}_{\Gamma^{\text{op}}}^2(S', \Gamma)$  holds for any simple  $\Gamma$ -module  $S$  and any simple  $\Gamma^{\text{op}}$ -module  $S'$ .
- (b) If  $\Gamma$  is a finite dimensional  $K$ -algebra, then  $\text{id}_\Gamma \Gamma \leq 1$  and  $\text{id}_{\Gamma^{\text{op}}} \Gamma \leq 1$ .

*Proof.* (b) follows immediately from (a).

(a) Let  $S$  be a simple  $\Gamma$ -module. By Proposition 4.2, there exists a complex

$$P \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P$$

of projective  $\Gamma$ -modules, which is exact at  $P_0$ , such that  $\text{Cok } f_0 = S$  and

$$\text{Hom}_\Gamma(P_0, \Gamma) \xrightarrow{f_1} \text{Hom}_\Gamma(P_1, \Gamma) \xrightarrow{f_2} \text{Hom}_\Gamma(P, \Gamma) \quad (5)$$

is exact. We can take a morphism  $f'_2: P' \rightarrow P_1$  such that

$$P \oplus P' \xrightarrow{\begin{pmatrix} f_2 \\ f'_2 \end{pmatrix}} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P \rightarrow S \rightarrow 0$$

gives a projective resolution of  $S$ . Since (5) is exact, we have that

$$\text{Hom}_\Gamma(P_0, \Gamma) \xrightarrow{f_1} \text{Hom}_\Gamma(P_1, \Gamma) \xrightarrow{\begin{pmatrix} f_2 \\ f'_2 \end{pmatrix}} \text{Hom}_\Gamma(P \oplus P', \Gamma)$$

is exact. Consequently we have the desired properties.  $\square$

To study the property of Jacobian algebras of QP's categorically, we introduce the following concept.

**Definition 4.4.** *Let  $\mathcal{C}$  be a category satisfying (C1) and (C2), and  $T \in \mathcal{C}$  an object. We call a complex*

$$X \xrightarrow{f_2} U_1 \xrightarrow{f_1} U_0 \xrightarrow{f_0} X$$

*in  $\text{add } T$  a weak 2-almost split sequence if the following sequences are exact*

$$\begin{aligned} \mathcal{C}(T, U_1) &\xrightarrow{f_1} \mathcal{C}(T, U_0) \xrightarrow{f_0} J_{\mathcal{C}}(T, X) \rightarrow 0, \\ \mathcal{C}(U_0, T) &\xrightarrow{f_1} \mathcal{C}(U_1, T) \xrightarrow{f_2} J_{\mathcal{C}}(X, T) \rightarrow 0. \end{aligned}$$

*In other words,  $f_0$  and  $f_2$  are right almost split and left almost split in  $\text{add } T$  respectively, and  $f_1$  is a pseudo-kernel of  $f_0$  and a pseudo-cokernel of  $f_2$  in  $\text{add } T$ . Note that we do not assume that  $f_2$  (respectively,  $f_0$ ) is a pseudo-kernel (respectively, pseudo-cokernel) of  $f_1$ .*

**Remark** The definition given here is more general than 2-almost split sequences and AR 4-angles in [I2, IY]. One difference is that we assume neither right minimality of  $f_0$  nor left minimality of  $f_2$ . This point is necessary to deal with Jacobian algebras of non-reduced QP's. Another difference is that our complex is not assumed to be a gluing of two exact sequences (respectively, triangles). This point simplifies our argument below. But note that we do not have uniqueness of weak 2-almost split sequences.

The following observation shows that weak 2-almost split sequences appear naturally in cluster-tilting theory.

**Proposition 4.5.** *Let  $\mathcal{C}$  and  $T = T_1 \oplus \cdots \oplus T_n$  be one of the following.*

- (a)  $\mathcal{C}$  is a 2-CY triangulated category and  $T$  is a basic cluster-tilting object.

- (b)  $\mathcal{C} = \text{mod } \Lambda$  for a 3-CY algebra  $\Lambda$  and  $T$  is a basic tilting  $\Lambda$ -module of projective dimension at most one (section 1.5).

Assume that the quiver of  $\text{End}_{\mathcal{C}}(T)$  has no loops at  $k$ . Gluing exchange triangles/sequences  $T_k^* \xrightarrow{g} U_k \xrightarrow{f} T_k$  and  $T_k \xrightarrow{g'} U'_k \xrightarrow{f'} T_k^*$  given in sections 1.3 and 1.5, we have a weak 2-almost split sequence  $T_k \xrightarrow{g'} U'_k \xrightarrow{f'g} U_k \xrightarrow{f} T_k$  in  $\text{add } T$ .

*Proof.* The case (a) is shown in [IY], and the case (b) is shown in Proposition 1.3(c).  $\square$

Now we give a key observation which gives a relationship between weak 2-almost split sequences and Jacobian algebras of QP's.

**Theorem 4.6.** *Let  $\mathcal{C}$  be a category satisfying (C1) and (C2), and let  $(Q, W)$  be a QP. Let  $\Phi: \widehat{KQ} \rightarrow \mathcal{C}$  be a  $K$ -linear functor, and  $T = \bigoplus_{i \in Q_0} \Phi i$ . Then the following conditions are equivalent.*

- (a)  $\Phi$  induces an isomorphism  $\mathcal{P}(Q, W) \simeq \text{End}_{\mathcal{C}}(T)$ .  
 (b) Any vertex  $i$  in  $Q$  gives rise to the following weak 2-almost split sequence in  $\mathcal{C}$

$$\Phi i \xrightarrow{(\Phi b)_b} \bigoplus_{b \in Q_1, s(b)=i} \Phi e(b) \xrightarrow{(\Phi \partial_{(a,b)} W)_{b,a}} \bigoplus_{a \in Q_1, e(a)=i} \Phi s(a) \xrightarrow{(\Phi a)_a} \Phi i.$$

*Proof.* We shall apply Proposition 3.6 to  $\mathbf{S} = \{\partial_a W \mid a \in Q_1\}$ .

(b) $\Rightarrow$ (a) By Lemma 4.1(b), we have  $\partial_{(a,b)} W = \partial_a^r(\partial_b W)$ . Looking at the complex

$$\bigoplus_{b \in Q_1, s(b)=i} \Phi e(b) \xrightarrow{(\Phi \partial_{(a,b)} W)_{b,a}} \bigoplus_{a \in Q_1, e(a)=i} \Phi s(a) \xrightarrow{(\Phi a)_a} \Phi i,$$

we have the assertion by Proposition 3.6(b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) By Proposition 3.6(a) $\Rightarrow$ (b)(c), we have exact sequences

$$\begin{aligned} \mathcal{C}(T, \bigoplus_{b \in Q_1, s(b)=i} \Phi e(b)) &\xrightarrow{(\Phi \partial_a^r(\partial_b W))_{b,a}} \mathcal{C}(T, \bigoplus_{a \in Q_1, e(a)=i} \Phi s(a)) \xrightarrow{(a)_a} J_{\mathcal{C}}(T, \Phi i) \rightarrow 0, \\ \mathcal{C}(\bigoplus_{a \in Q_1, e(a)=i} \Phi s(a), T) &\xrightarrow{(\Phi \partial_b^l(\partial_a W))_{b,a}} \mathcal{C}(\bigoplus_{b \in Q_1, s(b)=i} \Phi e(b), T) \xrightarrow{(b)_b} J_{\mathcal{C}}(\Phi i, T) \rightarrow 0. \end{aligned}$$

Since  $\partial_{(a,b)} W = \partial_a^r(\partial_b W) = \partial_b^l(\partial_a W)$  holds by Lemma 4.1, we have the assertion.  $\square$

We end this section with the following sufficient condition for an algebra to be a Jacobian algebra of a QP, which we will apply in section 6. We call a cycle  $a_1 a_2 \cdots a_m$  in a quiver  $Q$  *full* if the vertices  $s(a_1), \dots, s(a_m)$  are distinct, and if there is an arrow  $b$  in  $Q$  with  $s(b)$  and  $e(b)$  in the cycle, then there is some arrow  $a_i$  with  $s(b) = s(a_i)$  and  $e(b) = e(a_i)$ .

**Proposition 4.7.** *Let  $\Gamma$  be an algebra satisfying (A1) and (A2), and let  $(Q, W)$  be a QP satisfying the following conditions.*

- (i) *There exists a surjective  $K$ -algebra homomorphism  $\phi: \mathcal{P}(Q, W) \rightarrow \Gamma$  such that  $\text{Ker } \phi$  is the closure of a finitely generated ideal.*  
 (ii) *Every cycle in  $W$  is full.*  
 (iii) *The elements  $\partial_a W$  for all arrows  $a \in Q_1$  contained in cycles of  $W$  are linearly independent over  $K$ .*

(iv)  $\dim_K \text{Ext}_\Gamma^2(S, S') \leq \dim_K \text{Ext}_\Gamma^1(S', S)$  for any simple  $\Gamma$ -modules  $S, S'$ .

Then  $\phi: \mathcal{P}(Q, W) \rightarrow \Gamma$  is an isomorphism.

*Proof.* Let  $I$  be the kernel of the ring homomorphism  $\widehat{KQ} \rightarrow \Gamma$  induced by  $\phi$  in (i). Then  $\partial_a W \in I$  for any  $a \in Q_1$ .

Let  $J = J_{\widehat{KQ}}$ . We show that the image of  $\partial_a W$  for all arrows  $a \in Q_1$  contained in cycles of  $W$  in  $I/(IJ + JI)$  are linearly independent over  $K$ . We assume  $z = \sum_{i=1}^m c_i \partial_{a_i} W$  is in  $IJ + JI$  for some  $c_1, \dots, c_m$  in  $K$  and different arrows  $a_1, \dots, a_m$  contained in cycles of  $W$ . Fix a minimal set  $\mathbf{S}'$  of basic elements satisfying  $I = \langle \mathbf{S}' \rangle$ , which is finite by the assumption (i). We have an equality  $z = \sum_j p_j q_j r_j$  (possibly an infinite sum) with  $q_j \in \mathbf{S}'$  and  $p_j, r_j \in \widehat{KQ}$  such that for each  $j$  one of  $p_j$  and  $r_j$  belongs to  $J$ . Assume  $z \neq 0$ . Then there exists a cycle  $C$  in  $W$  such that  $\partial_{a_i} C$  appears in  $z$  for some  $i = 1, \dots, m$ . We have  $\partial_{a_i} C = pqr$  for paths  $p, q$  and  $r$  appearing in  $p_j, q_j$  and  $r_j$  respectively for some  $j$ , and one of  $p$  and  $r$  is a nontrivial path. Let  $S$  and  $S'$  be simple  $\Gamma$ -modules corresponding to  $s(q)$  and  $e(q)$  respectively. Then we have  $\text{Ext}_\Gamma^2(S, S') \neq 0$  by applying Proposition 3.4(b) to  $\mathbf{S}'$ . By (iv), we have  $\text{Ext}_\Gamma^1(S', S) \neq 0$ , so there is an arrow in  $Q$  from  $e(q)$  to  $s(q)$ . Since  $p$  or  $r$  is a nontrivial path, it follows that the cycle  $C$  is not full, a contradiction to (ii). Consequently we have  $z = 0$ . By (iii), we have that all  $c_i$  are 0.

For each arrow  $i \xrightarrow{a} j$  lying on a cycle in  $W$ , we have a relation of the form  $\partial_a W$ , hence  $\dim_K \text{Ext}_\Gamma^1(S_i, S_j)$  such relations, where  $S_i$  and  $S_j$  denote the simple modules at the vertices  $i$  and  $j$ . We then have

$$\begin{aligned} \dim_K \text{Ext}_\Gamma^1(S_i, S_j) &\leq \dim_K i(I/(IJ + JI))j \\ &= \dim_K \text{Ext}_\Gamma^2(S_j, S_i) \leq \dim_K \text{Ext}_\Gamma^1(S_i, S_j), \end{aligned}$$

where the first inequality follows by the above, the equality follows by Proposition 3.4(b), and the last inequality holds by (iv). Hence we have  $I = \langle \partial_a W \mid a \in Q_1 \rangle$  by Proposition 3.4(a). It follows that  $\phi$  is an isomorphism.  $\square$

## 5. MUTATION OF QUIVERS WITH POTENTIALS AND CLUSTER-TILTING OBJECTS

In this section we use the results from section 4 to show that there is a close connection between mutation of quivers with potentials on one hand and mutation of cluster-tilting objects in 2-CY triangulated categories in section 1.3, or mutation of tilting modules over 3-CY algebras in section 1.5, on the other hand. More specifically we will prove the following main results in this section.

**Theorem 5.1.** *Let  $\mathcal{C}$  be a 2-CY triangulated category with a basic cluster-tilting object  $T$ . If  $\text{End}_{\mathcal{C}}(T) \simeq \mathcal{P}(Q, W)$  for a QP  $(Q, W)$  and no 2-cycles start in the vertex  $k$  of  $Q$ , then  $\text{End}_{\mathcal{C}}(\mu_k(T)) \simeq \mathcal{P}(\mu_k(Q, W))$ .*

**Theorem 5.2.** *Let  $\Lambda$  be a 3-CY algebra with a basic tilting module  $T$  of projective dimension at most one. If  $\text{End}_{\Lambda}(T) \simeq \mathcal{P}(Q, W)$  for a QP  $(Q, W)$  and no 2-cycles start in the vertex  $k$  of  $Q$ , then  $\text{End}_{\Lambda}(\mu_k(T)) \simeq \mathcal{P}(\mu_k(Q, W))$ .*

To prove these theorems at the same time, we start with the following general setup. Let  $\mathcal{C}$  be an additive category satisfying (C1) and (C2). Let  $T = T_1 \oplus \dots \oplus T_n \in \mathcal{C}$  be a basic object and  $k = 1, \dots, n$ . We assume that there exists an indecomposable object  $T_k^* \in \mathcal{C}$  which does not belong to  $\text{add } T$  and that the following conditions for  $\mu_k(T) = (T/T_k) \oplus T_k^*$  are satisfied.



(1) For any  $i = 1, \dots, n$ , there exists a weak 2-almost split sequence in  $\text{add } T$ :

$$T_i \xrightarrow{f_{i2}} U_{i1} \xrightarrow{f_{i1}} U_{i0} \xrightarrow{f_{i0}} T_i.$$

(2k) There exist complexes

$$T_k \xrightarrow{f_{k2}} U_{k1} \xrightarrow{g_{k0}} T_k^* \quad \text{and} \quad T_k^* \xrightarrow{g_{k2}} U_{k0} \xrightarrow{f_{k0}} T_k$$

in  $\mathcal{C}$  such that  $U_{k0}, U_{k1} \in \text{add}(T/T_k)$ ,  $f_{k1} = g_{k0}g_{k2}$  and

$$T_k^* \xrightarrow{g_{k2}} U_{k0} \xrightarrow{f_{k0}f_{k2}} U_{k1} \xrightarrow{g_{k0}} T_k^*$$

is a weak 2-almost split sequence in  $\text{add } \mu_k(T)$ . Moreover the following sequences are exact:

$$\begin{aligned} (T_k^*, T_k^*) &\xrightarrow{g_{k2}} (T_k^*, U_{k0}) \xrightarrow{f_{k0}} (T_k^*, T_k), \\ (T_k^*, T_k^*) &\xrightarrow{g_{k0}} (U_{k1}, T_k^*) \xrightarrow{f_{k2}} (T_k, T_k^*). \end{aligned}$$

(3k) For any  $i \neq k$ , we have that  $T_k \notin (\text{add } U_{i1}) \cap (\text{add } U_{i0})$  and the following sequences are exact:

$$\begin{aligned} (T_k^*, U_{i1}) &\xrightarrow{f_{i1}} (T_k^*, U_{i0}) \xrightarrow{f_{i0}} (T_k^*, T_i), \\ (U_{i0}, T_k^*) &\xrightarrow{f_{i1}} (U_{i1}, T_k^*) \xrightarrow{f_{i2}} (T_i, T_k^*). \end{aligned}$$

Our key result is the following.

**Theorem 5.3.** *Assume that the conditions (1), (2k) and (3k) are satisfied. If  $\text{End}_{\mathcal{C}}(T) \simeq \mathcal{P}(Q, W)$  for a QP  $(Q, W)$ , then  $\text{End}_{\mathcal{C}}(\mu_k(T)) \simeq \mathcal{P}(\mu_k(Q, W))$ .*

Before proving Theorem 5.3, we notice that our main Theorems 5.1 and 5.2 follow immediately from Theorem 5.3 and the following observation.

**Lemma 5.4.** *Let  $T$  be as in Theorem 5.1 or 5.2. Then the conditions (1), (2k) and (3k) are satisfied.*

*Proof.* Take exchange triangles/sequences  $T_i \xrightarrow{f_{i2}} U_{i1} \xrightarrow{g_{i0}} T_i^*$  and  $T_i^* \xrightarrow{g_{i2}} U_{i0} \xrightarrow{f_{i0}} T_i$  given in sections 1.3 and 1.5.

(1)(2k) By Proposition 4.5, we have weak 2-almost split sequences  $T_i \xrightarrow{f_{i2}} U_{i1} \xrightarrow{g_{i0}g_{i2}} U_{i0} \xrightarrow{f_{i0}} T_i$  in  $\text{add } T$  and  $T_i^* \xrightarrow{g_{i2}} U_{i0} \xrightarrow{f_{i0}f_{i2}} U_{i1} \xrightarrow{g_{i0}} T_i^*$  in  $\text{add } \mu_i(T)$ . The sequences  $(T_k^*, T_k^*) \xrightarrow{g_{k2}} (T_k^*, U_{k0}) \xrightarrow{f_{k0}} (T_k^*, T_k)$  and  $(T_k^*, T_k^*) \xrightarrow{g_{i0}} (U_{k1}, T_k^*) \xrightarrow{f_{k2}} (T_k, T_k^*)$  are clearly exact (use Proposition 1.3(b) for Theorem 5.2).

(3k) Since no 2-cycle starts at the vertex  $k$ , we have  $T_k \notin (\text{add } U_{i1}) \cap (\text{add } U_{i0})$ . We only show that the upper sequence  $(T_k^*, U_{i1}) \xrightarrow{g_{i0}g_{i2}} (T_k^*, U_{i0}) \xrightarrow{f_{i0}} (T_k^*, T_i)$  is exact since the proof for the lower sequence is dual. Since  $(T_k^*, T_i^*) \xrightarrow{g_{i2}} (T_k^*, U_{i0}) \xrightarrow{f_{i0}} (T_k^*, T_i)$  is exact, we only have to show that  $(T_k^*, U_{i1}) \xrightarrow{g_{i0}} (T_k^*, T_i^*)$  is surjective. This is clear for the case Theorem 5.1 since we have  $\text{Ext}_{\mathcal{C}}^1(T_k^*, T_i) = 0$ .

We consider the case Theorem 5.2. Fix any  $a \in \text{Hom}_{\Lambda}(T_k^*, T_i^*)$ . Since  $0 \rightarrow (T, T_i) \xrightarrow{f_{i2}} (T, U_{i1}) \xrightarrow{g_{i0}} (T, T_i^*) \rightarrow 0$  is exact by Proposition 1.3(c), there exists

$b \in \text{Hom}_\Lambda(U'_k, U'_i)$  and  $c \in \text{Hom}_\Lambda(T_k^*, T_i^*)$  which make the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_k & \xrightarrow{f_{k2}} & U_{k1} & \xrightarrow{g_{k0}} & T_k^* \\ & & \downarrow c & & \downarrow b & & \downarrow a \\ 0 & \longrightarrow & T_i & \xrightarrow{f_{i2}} & U_{i1} & \xrightarrow{g_{i0}} & T_i^* \end{array}$$

commutative. Since  $0 \rightarrow (T_k^*, T) \xrightarrow{g_{k0}} (U_{k1}, T) \xrightarrow{f_{k2}} J_{\text{mod } \Lambda}(T_k, T) \rightarrow 0$  is exact by Proposition 1.3(c), there exist  $s \in \text{Hom}_\Lambda(U_{k1}, T_i)$  such that  $c = f_{k2}s$  and  $t \in \text{Hom}_\Lambda(T_k^*, U_{i1})$  such that  $b = sf_{i2} + g_{k0}t$ . Then we have  $g_{k0}(a - tg_{i0}) = 0$ . Since  $g_{k0}(a - tg_{i0})g_{i2} = 0$ , we have  $(a - tg_{i0})g_{i2} = 0$ . Since  $g_{i2}$  is injective, we have  $a = tg_{i0}$ .  $\square$

Now we shall prove Theorem 5.3. The idea of the proof is to use Theorem 4.6 and construct weak 2-almost split sequences in  $\text{add } \mu_k(T)$  using the following result.

**Proposition 5.5.** *Assume that the conditions (1), (2k) and (3k) are satisfied for some  $k$ . Fix  $i \neq k$ . If we have a commutative diagram*

$$\begin{array}{ccccccc} T_i & \xrightarrow{f_{i2}} & U_{i1} & \xrightarrow{f_{i1} = \begin{pmatrix} f'_{i1} & f''_{i1} \end{pmatrix}} & U_{i0} = T_k^\ell \oplus U''_{i0} & \xrightarrow{f_{i0} = \begin{pmatrix} f'_{i0} \\ f''_{i0} \end{pmatrix}} & T_i \\ \parallel & & \uparrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \uparrow \begin{pmatrix} f_{k0}^\ell & 0 \\ 0 & 1 \end{pmatrix} & & \parallel \\ T_i & \xrightarrow{\begin{pmatrix} s & f_{i2} \end{pmatrix}} & T_k^{*\ell} \oplus U_{i1} & \xrightarrow{\begin{pmatrix} g_{k2}^\ell & 0 \\ t & f''_{i1} \end{pmatrix}} & U_{k0}^\ell \oplus U''_{i0} & \xrightarrow{\begin{pmatrix} f_{k0}^\ell & f'_{i0} \\ f''_{i0} & \end{pmatrix}} & T_i \end{array}$$

of complexes such that

- $U''_{i0}$  and  $U_{i1}$  belong to  $\text{add}(T/T_k)$ ,
- $s \in \mathcal{C}(T_i, T_k^{*\ell})$  induces a bijection  $s : (\mathcal{C}/J_{\mathcal{C}})(T_k^{*\ell}, T_k^*) \rightarrow (J_{\mathcal{C}}/J_{\text{add } \mu_k(T)}^2)(T_i, T_k^*)$ ,

then the lower sequence is a weak 2-almost split sequence in  $\text{add } \mu_k(T)$ .

*Proof.* By our assumptions (1), (2k) and (3k) we have exact sequences

$$(\mu_k(T), T_k^*) \xrightarrow{g_{k2}} (\mu_k(T), U_{k0}) \xrightarrow{f_{k0}} (\mu_k(T), T_k), \quad (6)$$

$$(\mu_k(T), U_{i1}) \xrightarrow{f_{i1}} (\mu_k(T), U_{i0}) \xrightarrow{f_{i0}} (\mu_k(T), T_i), \quad (7)$$

$$(U_{i0}, \mu_k(T)) \xrightarrow{f_{i1}} (U_{i1}, \mu_k(T)) \xrightarrow{f_{i2}} (T_i, \mu_k(T)). \quad (8)$$

(i) We will show that  $\begin{pmatrix} f_{k0}^\ell & f'_{i0} \\ f''_{i0} & \end{pmatrix}$  is right almost split in  $\text{add } \mu_k(T)$ .

First we will show that any morphism  $a \in J_{\mathcal{C}}(T/T_k, T_i)$  factors through  $\begin{pmatrix} f_{k0}^\ell & f'_{i0} \\ f''_{i0} & \end{pmatrix}$ . Since  $f_{i0} = \begin{pmatrix} f'_{i0} \\ f''_{i0} \end{pmatrix}$  is right almost split in  $\text{add } T$ , there exists  $(a_1 \ a_2) \in \mathcal{C}(T/T_k, T_k^\ell \oplus U''_{i0})$  such that  $a = a_1 f'_{i0} + a_2 f''_{i0}$ . Since  $f_{k0}$  is right almost split in  $\text{add } T$ , there exists  $b \in \mathcal{C}(T/T_k, U_{k0}^\ell)$  such that  $a_1 = b f_{k0}^\ell$ . Then we have  $a = (b \ a_2) \begin{pmatrix} f_{k0}^\ell & f'_{i0} \\ f''_{i0} & \end{pmatrix}$ .

Next we take any  $a \in J_{\mathcal{C}}(T_k^*, T_i)$ . Since  $g_{k2}$  is left almost split in  $\text{add } \mu_k(T)$ , there exists  $b \in \mathcal{C}(U_{k0}, T_i)$  such that  $a = g_{k2}b$ . Since  $T_k \notin \text{add } U_{i1}$ , we have  $T_i \notin \text{add } U_{k0}$ . Thus  $b \in J_{\mathcal{C}}(U_{k0}, T_i)$  holds, and the first case implies that  $b$  factors through  $\begin{pmatrix} f_{k0}^\ell & f'_{i0} \\ f''_{i0} & \end{pmatrix}$ .

Hence  $a$  factors through  $\begin{pmatrix} f_{k0}^\ell & f'_{i0} \\ f''_{i0} & \end{pmatrix}$ .

(ii) We will show that  $\begin{pmatrix} g_{k2}^\ell & 0 \\ t & f''_{i1} \end{pmatrix}$  is a pseudo-kernel of  $\begin{pmatrix} f_{k0}^\ell & f'_{i0} \\ f''_{i0} & \end{pmatrix}$  in  $\text{add } \mu_k(T)$ .

Assume  $(a_1 \ a_2) \in \mathcal{C}(\mu_k(T), U_{k0}^\ell \oplus U''_{i0})$  satisfies  $(a_1 \ a_2) \begin{pmatrix} f_{k0}^\ell & f'_{i0} \\ f''_{i0} & \end{pmatrix} = 0$ . Since  $(a_1 \ a_2) \begin{pmatrix} f_{k0}^\ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f'_{i0} \\ f''_{i0} \end{pmatrix} = 0$  holds and (7) is exact, there exists  $b \in \mathcal{C}(\mu_k(T), U_{i1})$  such

that  $b(f'_{i_1} \ f''_{i_1}) = (a_1 \ a_2) \begin{pmatrix} f'_{k_0} & 0 \\ 0 & 1 \end{pmatrix}$ . Hence we have  $bf'_{i_1} = a_1 f'_{k_0}$  and  $bf''_{i_1} = a_2$ . Since  $(a_1 - bt)f'_{k_0} = bf'_{i_1} - bf'_{i_1} = 0$  and (6) is exact, there exists  $c \in \mathcal{C}(\mu_k(T), T_k^{\ell})$  such that  $a_1 - bt = cg'_{k_2}$ . Then we have  $(a_1 \ a_2) = (c \ b) \begin{pmatrix} g'_{k_2} & 0 \\ t & f''_{i_1} \end{pmatrix}$ .

(iii) We will show that  $(s \ f_{i_2})$  is left almost split in  $\text{add } \mu_k(T)$ .

Since  $f_{i_2}$  is left almost split in  $\text{add } T$ , any morphism in  $J_{\mathcal{C}}(T_i, T/T_k)$  factors through  $(s \ f_{i_2})$ . Then take any  $a \in J_{\mathcal{C}}(T_i, T_k^*)$ . By our assumption on  $s$ , there exists  $a_1 \in \mathcal{C}(T_k^{\ell}, T_k^*)$  such that  $a - sa_1 \in J_{\text{add } \mu_k(T)}^2(T_i, T_k^*)$ . Since  $g_{k_0}$  is right almost split in  $\text{add } \mu_k(T)$ , there exists  $b \in J_{\mathcal{C}}(T_i, U_{k_1})$  such that  $a - sa_1 = bg_{k_0}$ . Since  $f_{i_2}$  is left almost split in  $\text{add } T$ , there exists  $c \in \mathcal{C}(U_{i_1}, U_{k_1})$  such that  $b = f_{i_2}c$ . Then we have  $a = sa_1 + f_{i_2}cg_{k_0} = (s \ f_{i_2}) \begin{pmatrix} a_1 \\ cg_{k_0} \end{pmatrix}$ .

(iv) We will show that  $\begin{pmatrix} g'_{k_2} & 0 \\ t & f''_{i_1} \end{pmatrix}$  is a pseudo-cokernel of  $(s \ f_{i_2})$  in  $\text{add } \mu_k(T)$ . Assume  $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathcal{C}(T_k^{\ell} \oplus U_{i_1}, T')$  with  $T' \in \text{add } \mu_k(T)$  satisfies  $(s \ f_{i_2}) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$ . We first show that there exists  $b \in \mathcal{C}(U_{k_0}^{\ell}, T')$  such that  $a_1 = g'_{k_2}b$ . We have to consider the cases  $T' = T/T_k$  and  $T' = T_k^*$ . If  $T' = T/T_k$ , then this is clear since  $g_{k_2}$  is left almost split in  $\text{add } \mu_k(T)$ . If  $T' = T_k^*$ , then we have  $sa_1 = -f_{i_2}a_2 \in J_{\text{add } \mu_k(T)}^2$ . Thus our assumption on  $s$  implies that  $a_1 \in J_{\mathcal{C}}(T_k^{\ell}, T_k^*)$ . Since  $g_{k_0}$  is right almost split in  $\text{add } \mu_k(T)$ , we have that  $a_1$  factors through  $g_{k_0} \in \mathcal{C}(U_{k_1}, T_k^*)$ . Since  $U_{k_1} \in \text{add}(T/T_k)$ , the assertion follows from the case  $T' = T/T_k$ .

Since  $f_{i_2}(a_2 - tb) = f_{i_2}a_2 + sg'_{k_2}b = f_{i_2}a_2 + sa_1 = 0$  and (8) is exact, there exists  $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathcal{C}(T_k^{\ell} \oplus U_{i_0}'', T')$  such that  $a_2 - tb = (f'_{i_1} \ f''_{i_1}) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ . Then  $a_2 = tb + f'_{i_1}b_1 + f''_{i_1}b_2 = s(b + f'_{k_0}b_1) + f''_{i_1}b_2$  holds. Thus we have  $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} g'_{k_2} & 0 \\ t & f''_{i_1} \end{pmatrix} \begin{pmatrix} b + f'_{k_0}b_1 \\ b_2 \end{pmatrix}$ .  $\square$

We need the following information about mutation of QP's.

**Lemma 5.6.** *Let  $(Q, W)$  be a QP and  $(Q', W') = \widetilde{\mu}_k(Q, W)$ . Let  $a$  and  $a'$  be arrows in  $Q$  with  $e(a) = e(a') = k$ , let  $b$  and  $b'$  be arrows in  $Q$  with  $s(b) = s(b') = k$ , and let  $c$  and  $c'$  be arrows in  $Q \cap Q'$ . Then we have the following equalities.*

- (a)  $\partial_{(c, c')}W' = \partial_{(c, c')}[W]$ .
- (b)  $\partial_{(a^*, c)}W' = 0$  and  $\partial_{(c, b^*)}W' = 0$ .
- (c)  $\partial_{(c, [ab])}W' = \partial_{(c, [ab])}[W]$  and  $\partial_{([ab], c)}W' = \partial_{([ab], c)}[W]$ .
- (d)  $\partial_{(a^*, [ab])}W' = b^*$  and  $\partial_{(a'^*, [ab])}W' = 0$  if  $a \neq a'$ .
- (e)  $\partial_{([ab], b^*)}W' = a^*$  and  $\partial_{([ab], b'^*)}W' = 0$  if  $b \neq b'$ .
- (f)  $\partial_{(b^*, a^*)}W' = [ab]$ .
- (g)  $\partial_{([ab], [a'b'])}W' = 0$ .
- (h) For other pairs  $d, d' \in Q'_1$ , we have  $\partial_{(d, d')}W' = 0$ .

*Proof.* Immediate from the definition  $W' = [W] + \Delta$ .  $\square$

*Proof of Theorem 5.3.* Let  $(Q', W') = \widetilde{\mu}_k(Q, W)$ . We have a surjective  $K$ -algebra homomorphism  $\phi: \widehat{KQ} \rightarrow \text{End}_{\mathcal{C}}(T)$  which induces an isomorphism  $\mathcal{P}(Q, W) \simeq \text{End}_{\mathcal{C}}(T)$ . Then  $\phi$  gives a  $K$ -linear functor  $\Phi: \widehat{KQ} \rightarrow \mathcal{C}$  such that  $T = \bigoplus_{i \in Q_0} \Phi i$ . By Lemma 3.5, we can define a  $K$ -linear functor  $\Phi': \widehat{KQ}' \rightarrow \mathcal{C}$  in the following way.

- (i)  $\Phi'$  coincides with  $\Phi$  on  $Q \cap Q'$ .
- (ii)  $\Phi'[ab] = \Phi a \Phi b$  for each pair of arrows  $a: i \rightarrow k$  and  $k \rightarrow j$  in  $Q$ .

(iii)  $\Phi'k^*$ ,  $\Phi'a^*$  ( $e(a) = k$ ) and  $\Phi'b^*$  ( $s(b) = k$ ) are defined by sequences

$$\begin{array}{ccc} \Phi k & \xrightarrow{(\Phi b)_b} & \bigoplus_{b \in Q_1, s(b)=k} \Phi e(b) \xrightarrow{(\Phi' b^*)_b} \Phi' k^*, \\ \Phi' k^* & \xrightarrow{(\Phi' a^*)_a} & \bigoplus_{a \in Q_1, e(a)=k} \Phi s(a) \xrightarrow{(\Phi a)_a} \Phi k \end{array}$$

which are given in (2k).

We simply denote  $\Phi p$  (respectively,  $\Phi' p$ ) by  $p$  for any path  $p$  in  $Q$  (respectively,  $Q'$ ).

Let us calculate weak 2-almost split sequences in  $\text{add } \mu_k(T)$  by dividing into 3 cases.

**Lemma 5.7.** *We have the following weak 2-almost split sequence in  $\text{add } \mu_k(T)$ .*

$$k^* \xrightarrow{(a^*)_a} \bigoplus_{a \in Q_1, e(a)=k} s(a) \xrightarrow{([ab]_{a,b})} \bigoplus_{b \in Q_1, s(b)=k} e(b) \xrightarrow{(b^*)_b} k^*.$$

*Proof.* Immediate from our assumption (2k) and the construction (iii).  $\square$

**Lemma 5.8.** *For a vertex  $i$  in  $Q'$  with  $i \neq k^*$ , assume that there is no arrow  $i \rightarrow k$  in  $Q$ . Then  $i$  has a weak 2-almost split sequence*

$$i \xrightarrow{((b^*)_b \ (d)_d)} \left( \bigoplus_{\substack{b \in Q_1 \\ b: k \rightarrow i}} k^* \right) \oplus \left( \bigoplus_{\substack{d \in Q_1 \\ s(d)=i}} e(d) \right) \xrightarrow{\begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix}} \left( \bigoplus_{\substack{a, b \in Q_1 \\ e(a)=k \\ b: k \rightarrow i}} s(a) \right) \oplus \left( \bigoplus_{\substack{c \in Q_1 \\ s(c) \neq k \\ e(c)=i}} s(c) \right) \xrightarrow{\begin{pmatrix} ([ab]_{(a,b)} \\ (c)_c \end{pmatrix}} i$$

in  $\text{add } \mu_k(T)$ , where

$$\begin{aligned} f_1 &= \bigoplus_{\substack{b \in Q_1 \\ b: k \rightarrow i}} (k^* \xrightarrow{(a^*)} \bigoplus_{\substack{a \in Q_1 \\ e(a)=k}} s(a)) \\ f_2 &= (\partial_{([ab]_{a,b})} [W])_{d, (a,b)}: \bigoplus_{\substack{d \in Q_1 \\ s(d)=i}} e(d) \rightarrow \bigoplus_{\substack{a, b \in Q_1 \\ e(a)=k \\ b: k \rightarrow i}} s(a) \\ f_3 &= (\partial_{(c,d)} [W])_{d,c}: \bigoplus_{\substack{d \in Q_1 \\ s(d)=i}} e(d) \rightarrow \bigoplus_{\substack{c \in Q_1 \\ s(c) \neq k \\ e(c)=i}} s(c) \end{aligned}$$

*Proof.* Since  $\text{End}_C(T) \simeq \mathcal{P}(Q, W)$ , we have a weak 2-almost split sequence

$$i \xrightarrow{(d)_d} \bigoplus_{\substack{d \in Q_1 \\ s(d)=i}} e(d) \xrightarrow{(g_1 \ g_2)} \left( \bigoplus_{\substack{b \in Q_1 \\ b: k \rightarrow i}} k \right) \oplus \left( \bigoplus_{\substack{c \in Q_1 \\ s(c) \neq k \\ e(c)=i}} s(c) \right) \xrightarrow{\begin{pmatrix} (b)_b \\ (c)_c \end{pmatrix}} i$$

in  $\text{add } T$  by Theorem 4.6, where

$$g_1 = (\partial_{(b,d)}W)_{d,b}: \bigoplus_{\substack{d \in Q_1 \\ s(d)=i}} e(d) \rightarrow \bigoplus_{\substack{b \in Q_1 \\ b: k \rightarrow i}} k,$$

$$g_2 = (\partial_{(c,d)}W)_{d,c}: \bigoplus_{\substack{d \in Q_1 \\ s(d)=i}} e(d) \rightarrow \bigoplus_{\substack{c \in Q_1 \\ s(c) \neq k \\ e(c)=i}} s(c).$$

Consider the following diagram of complexes in  $\mathcal{C}$ .

$$\begin{array}{ccccc}
i & \xrightarrow{(d)_d} & \bigoplus_{\substack{d \in Q_1 \\ s(d)=i}} e(d) & \xrightarrow{(g_1 \ g_2)} & \left( \bigoplus_{\substack{b \in Q_1 \\ b: k \rightarrow i}} k \right) \oplus \left( \bigoplus_{\substack{c \in Q_1 \\ s(c) \neq k \\ e(c)=i}} s(c) \right) & \xrightarrow{\binom{(b)_b}{(c)_c}} & i \\
\parallel & & \uparrow \binom{(0)}{(1)} & & \uparrow \binom{(a) \ 0}{0 \ 1} & & \parallel \\
i & \xrightarrow{\binom{(b^*)_b}{(d)_d}} & \left( \bigoplus_{\substack{b \in Q_1 \\ b: k \rightarrow i}} k^* \right) \oplus \left( \bigoplus_{\substack{d \in Q_1 \\ s(d)=i}} e(d) \right) & \xrightarrow{\binom{f_1 \ 0}{f_2 \ f_3}} & \left( \bigoplus_{\substack{a, b \in Q_1 \\ e(a)=k \\ b: k \rightarrow i}} s(a) \right) \oplus \left( \bigoplus_{\substack{c \in Q_1 \\ s(c) \neq k \\ e(c)=i}} s(c) \right) & \xrightarrow{\binom{([ab])_{(a,b)}}{(c)_c}} & i
\end{array}$$

This is commutative since we have  $f_1(a) = 0$  by  $\sum_{a \in Q_1, e(a)=k} a^*a = 0$ , and  $f_2(a) = g_1$  by

$$\sum_{a \in Q_1, e(a)=k} \partial_{([ab],d)}[W]a = \partial_{(b,d)}W,$$

and  $f_3 = g_2$  clearly.

By virtue of Proposition 5.5, we only have to show that

$$(b^*)_b: (\mathcal{C}/J_{\mathcal{C}})\left(\bigoplus_{b: k \rightarrow i} k^*, k^*\right) \rightarrow (J_{\mathcal{C}}/J_{\text{add } \mu_k(T)}^2)(i, k^*)$$

is bijective. By the assumption (C2), we have  $K = (\mathcal{C}/J_{\mathcal{C}})(k^*, k^*)$ . By Lemma 5.7 we have that  $(b^*)_b: \bigoplus_{b \in Q_1, s(b)=k} e(b) \rightarrow k^*$  is minimal right almost split in  $\text{add } \mu_k(T)$  since the middle morphism in the sequence in Lemma 5.7 belongs to  $J_{\mathcal{C}}$ . Thus we have that  $(J_{\mathcal{C}}/J_{\text{add } \mu_k(T)}^2)(i, k^*)$  is a  $K$ -vector space with the basis  $\{b^* \mid b: k \rightarrow i\}$ . Thus the above map is bijective.  $\square$

The following result is just a dual of Lemma 5.8.

**Lemma 5.9.** *For a vertex  $i$  in  $Q'$  with  $i \neq k^*$ , assume that there is no arrow  $k \rightarrow i$  in  $Q$ . Then  $i$  has a weak 2-almost split sequence*

$$i \xrightarrow{\binom{([ab])_{(a,b)}}{(d)_d}} \left( \bigoplus_{\substack{a, b \in Q_1 \\ s(b)=k \\ a: i \rightarrow k}} e(b) \right) \oplus \left( \bigoplus_{\substack{d \in Q_1 \\ s(d)=i \\ e(d) \neq k}} e(d) \right) \xrightarrow{\binom{f_1 \ f_2}{0 \ f_3}} \left( \bigoplus_{\substack{a \in Q_1 \\ a: i \rightarrow k}} k^* \right) \oplus \left( \bigoplus_{\substack{c \in Q_1 \\ e(c)=i}} s(c) \right) \xrightarrow{\binom{(a^*)_a}{(c)_c}} i$$

in  $\text{add } \mu_k(T)$ , where

$$\begin{aligned}
f_1 &= \bigoplus_{\substack{a \in Q_1 \\ a: i \rightarrow k}} \left( \bigoplus_{\substack{b \in Q_1 \\ s(b)=k}} e(b) \xrightarrow{(b^*)} k^* \right) \\
f_2 &= (\partial_{(c,[ab])}[W])_{(a,b),c}: \bigoplus_{\substack{a,b \in Q_1 \\ s(b)=k \\ a: i \rightarrow k}} e(b) \rightarrow \bigoplus_{\substack{c \in Q_1 \\ e(c)=i}} s(c) \\
f_3 &= (\partial_{(c,d)}[W])_{d,c}: \bigoplus_{\substack{d \in Q_1 \\ s(d)=i \\ e(d) \neq k}} e(d) \rightarrow \bigoplus_{\substack{c \in Q_1 \\ e(c)=i}} s(c)
\end{aligned}$$

Consequently we have the following observation.

**Proposition 5.10.** *Any vertex  $i$  in  $Q'$  has the following weak 2-almost split sequence in  $\text{add } \mu_k(T)$ .*

$$i \xrightarrow{(d)_d} \bigoplus_{\substack{d \in Q'_1, \\ s(d)=i}} e(d) \xrightarrow{(\partial_{(c,d)}W')_{d,c}} \bigoplus_{\substack{c \in Q'_1, \\ e(c)=i}} s(c) \xrightarrow{(c)_c} i.$$

*Proof.* In Lemma 5.6, we have calculated the  $\partial_{(c,d)}W'$ . They coincide with the middle morphism of weak 2-almost split sequences given in Lemmas 5.7, 5.8 and 5.9. Note that we assumed that  $k$  is not contained in 2-cycles, so Lemmas 5.7, 5.8 and 5.9 cover all possible cases.  $\square$

Now Theorem 4.6 and Proposition 5.10 imply  $\text{End}_{\mathcal{C}}(\mu_k(T)) \simeq \mathcal{P}(Q', W')$ . By [DWZ1, Theorem 4.6], we have  $\mathcal{P}(Q', W') \simeq \mathcal{P}(\mu_k(Q, W))$ . Thus we have proved Theorem 5.3.  $\square$

We have the following direct consequences of Theorems 5.1 and 5.2.

**Corollary 5.11.** *Let  $\mathcal{C}$  be one of the following categories.*

- (a)  $\mathcal{C}$  is a 2-CY triangulated category with a basic cluster-tilting object  $T$ .
- (b)  $\mathcal{C} = \text{mod } \Lambda$  for a 3-CY algebra  $\Lambda$  with a basic tilting  $\Lambda$ -module  $T$  of projective dimension at most one.

*If  $\text{End}_{\mathcal{C}}(T) \simeq \mathcal{P}(Q, W)$  for a QP  $(Q, W)$ , then for any sequence  $k_1, \dots, k_\ell$  of vertices in  $Q$  such that  $k_i$  does not lie on a 2-cycle in  $\mu_{k_{i-1}} \circ \dots \circ \mu_{k_1}(Q, W)$ , we have*

$$\text{End}_{\mathcal{C}}(\mu_{k_\ell} \circ \dots \circ \mu_{k_1}(T)) \simeq \mathcal{P}(\mu_{k_\ell} \circ \dots \circ \mu_{k_1}(Q, W)).$$

**Remark** If  $\mathcal{C}$  is a 2-CY triangulated category with cluster structure and  $\text{End}_{\mathcal{C}}(T)$  is Jacobian, then none of the Jacobian algebras in the same mutation class will have 2-cycles in their quivers. Hence they are all non-degenerate QP's in the sense of [DWZ1].

The next result is of special interest.

**Corollary 5.12.** *Any cluster-tilted algebra is isomorphic to a Jacobian algebra of a rigid QP.*

*Proof.* We here use Theorem 5.1 and the fact that any cluster-tilted algebra belongs to the same mutation class as a hereditary algebra, which is trivially given by a QP. That the QP is rigid follows from [DWZ1].  $\square$

It is not known in general whether the algebra  $\text{End}_{\mathcal{C}}(T^*)$  can be defined directly from the algebra  $\text{End}_{\mathcal{C}}(T)$  when  $T^*$  is a cluster-tilting object in the triangulated 2-CY category  $\mathcal{C}$  obtained from the cluster tilting object  $T$  by mutation. When the potential  $(Q', W')$  is obtained by mutation from the potential  $(Q, W)$ , it is also not known if the Jacobian algebra  $\mathcal{P}(Q', W')$  can be defined directly from the Jacobian algebra  $\mathcal{P}(Q, W)$ .

However, we can prove this for Jacobian algebras of Jacobi-finite QP's by the following result together with a result of Amiot [A]. We call a QP  $(Q, W)$  *Jacobi-finite* if the factor algebra  $KQ/\langle \partial_a W \mid a \in Q_1 \rangle$  of the ordinary path algebra  $KQ$  is finite dimensional [A]. This condition implies that  $\mathcal{P}(Q, W)$  is finite dimensional.

**Theorem 5.13.** *Let  $\Lambda$  be an algebra isomorphic to a Jacobian algebra  $\mathcal{P}(Q, W)$  for some QP  $(Q, W)$  and to a 2-CY-tilted algebra  $\text{End}_{\mathcal{C}}(T)$  for some cluster-tilting object  $T$  in a 2-CY category  $\mathcal{C}$ . Assume  $k$  is a vertex in  $Q$  not lying on any 2-cycle.*

- (a) *The Jacobian algebra  $\mathcal{P}(\mu_k(Q, W))$  is determined by the algebra  $\Lambda$  and does not depend on the choice of a QP  $(Q, W)$ .*
- (b) *The 2-CY-tilted algebra  $\text{End}_{\mathcal{C}}(\mu_k(T))$  is determined by the algebra  $\Lambda$  and does not depend on the choice of a 2-CY category  $\mathcal{C}$  and a cluster-tilting object  $T$ .*

*Proof.* This is a direct consequence of Theorem 5.1. □

The following gives a partial answer to Question 12.2 in [DWZ1].

**Corollary 5.14.** *Let  $\mathcal{P}(Q, W)$  be a Jacobian algebra of a Jacobi-finite QP. Assume  $k$  is a vertex in  $Q$  not lying on any 2-cycle. If  $\mathcal{P}(Q, W) \simeq \mathcal{P}(Q', W')$  for some QP  $(Q', W')$ , then  $Q_0 \simeq Q'_0$  and  $\mathcal{P}(\mu_k(Q, W)) \simeq \mathcal{P}(\mu_k(Q', W'))$ .*

*Proof.* Clearly we have  $Q_0 \simeq Q'_0$ . The second assertion follows from Theorem 5.13(a) and the fact that any Jacobian algebra of a Jacobi-finite QP is a 2-CY-tilted algebra [A]. □

We end this section with using the results in this section to give an alternative proof of the following.

**Corollary 5.15.** *Cluster-tilted algebras are determined by their quiver.*

*Proof.* If a cluster-tilted algebra  $\Lambda$  has an acyclic quiver  $Q$ , then  $\Lambda$  is isomorphic to the path algebra  $KQ$  by [ABS, KR1], in particular it is determined by its quiver.

Assume that  $T_1$  and  $T_2$  are cluster-tilting objects in the cluster categories  $\mathcal{C}_{Q_1}$  and  $\mathcal{C}_{Q_2}$  respectively, such that the associated quivers are isomorphic. Let  $\mu = \mu_{k_\ell} \circ \cdots \circ \mu_{k_1}$  be a sequence of mutations such that for the projective  $KQ_1$ -module  $KQ_1$  we have  $\mu(KQ_1) \simeq T_1$ , and hence  $\mu(Q_1) \simeq Q_{T_1}$ . Then let  $T$  be the cluster-tilting object in  $\mathcal{C}_{Q_2}$  such that  $\mu(T) \simeq T_2$  and hence  $\mu(Q_T) \simeq Q_{T_2}$ . Then  $Q_1 \simeq Q_T$ , and hence  $KQ_1 \simeq \text{End}_{\mathcal{C}_{Q_2}}(T)$ . Since cluster-tilted algebras are QP's by Corollary 5.12, it follows from Theorem 5.13(b) that  $\text{End}_{\mathcal{C}_1}(T_1) \simeq \text{End}_{\mathcal{C}_2}(T_2)$ . □

**Remark** It follows from the work in [DWZ1] that if  $\Lambda = \mathcal{P}(Q, W)$  is a Jacobian algebra given by a rigid potential  $(Q, W)$  where  $Q$  belongs to the mutation class of an acyclic quiver, then  $\Lambda$  is determined by its quiver. Since by Corollary 5.12 any cluster-tilted algebra has this property, this gives yet another way of seeing that cluster-tilted algebras are determined by their quiver.

## 6. 2-CY-TILTED ALGEBRAS ASSOCIATED WITH ELEMENTS IN COXETER GROUPS

In this section we show that a large class of 2-CY-tilted algebras, including the cluster-tilted algebras and the class of 2-CY-tilted algebras coming from stable categories of preprojective algebras of Dynkin type, are given by QP's, by finding an explicit description of the potentials for some of them. Hence by Theorem 5.1 we get that all 2-CY-tilted algebras in the mutation class are given by QP's. We prove that these QP's are rigid in the sense of [DWZ1].

These 2-CY-tilted algebras come from 2-CY triangulated categories constructed from elements in the Coxeter groups associated with connected quivers with no loops, as investigated in [IR, BIRS]. (See [GLS2] for an alternative approach to the construction of a subclass of these 2-CY categories). We start with recalling the relevant results from [BIRS], including a description of the quivers of some special 2-CY-tilted algebras.

Let  $Q$  be a finite connected quiver with no loops, with vertices  $1, \dots, n$  and set of arrows  $Q_1$ . The associated preprojective algebra over the algebraically closed field  $K$  is defined as follows. For each arrow  $a \in Q_1$  from  $i$  to  $j$ , we add a corresponding arrow  $a^*$  from  $j$  to  $i$  to get a new quiver  $\bar{Q}$ . Then the *preprojective algebra* is defined by

$$\Lambda = K\bar{Q}/\langle \sum_{a \in Q_1} (aa^* - a^*a) \rangle.$$

We shall write  $(a^*)^* = a$ , and  $\epsilon(a) = 1, \epsilon(a^*) = -1$ .

The *Coxeter group*  $W_Q$  is presented by generators  $s_1, \dots, s_n$  with relations  $s_i s_j = s_j s_i$  if there is no arrow in  $Q$  between  $i$  and  $j$  and  $s_i s_j s_i = s_j s_i s_j$  if there is precisely one arrow in  $Q$  between  $i$  and  $j$ . Let  $w$  be an element in the associated Coxeter group  $W_Q$ , and  $w = s_{u_1} \cdots s_{u_m}$  a reduced expression, where the  $u_1, \dots, u_m$  are integers in  $\{1, \dots, n\}$ . For each integer  $i$  in  $\{1, \dots, n\}$ , consider the ideal  $I_i = \Lambda(1 - e_i)\Lambda$  in  $\Lambda$ , where  $e_i$  denotes the trivial path at the vertex  $i$ . Let  $I_w = I_{u_1} \cdots I_{u_m}$ , and let  $\Lambda_w = \Lambda/I_w$ . Then  $I_w$  and  $\Lambda_w$  are independent of the choice of reduced expression for  $w$ , and  $\Lambda_w$  is a finite dimensional  $K$ -algebra. Denote by  $\text{Sub } \Lambda_w$  the full subcategory of  $\text{mod } \Lambda_w$  whose objects are the submodules of finitely generated projective  $\Lambda_w$ -modules. We have that the injective dimensions  $\text{id}_{\Lambda_w} \Lambda_w$  and  $\text{id}_{\Lambda_w} \Lambda_w$  are at most one, and hence the stable category  $\mathcal{C} = \underline{\text{Sub}} \Lambda_w$  is a Hom-finite triangulated 2-CY category [H]. Associated with each reduced expression  $s_{u_1} \cdots s_{u_m}$  of  $w$  is the object  $T = \Lambda/I_{u_1} \oplus \Lambda/I_{u_1} I_{u_2} \oplus \cdots \oplus \Lambda/I_{u_1} \cdots I_{u_m}$  in  $\text{Sub } \Lambda_w$ . Consider the algebra  $\text{End}_{\Lambda_w}(T)$  and the associated 2-CY-tilted algebra  $\text{End}_{\mathcal{C}}(T)$ . Alternatively we can write  $T$  as a sum of indecomposable objects

$$\begin{aligned} T &= \frac{T_1}{P_{u_1}} \oplus \frac{T_2}{(I_{u_1} I_{u_2}) P_{u_2}} \oplus \cdots \oplus \frac{T_m}{(I_{u_1} I_{u_2} \cdots I_{u_m}) P_{u_m}}, \\ &= \frac{T_1}{I_{u_1} P_{u_1}} \oplus \frac{T_2}{(I_{u_1} I_{u_2}) P_{u_2}} \oplus \cdots \oplus \frac{T_m}{(I_{u_1} I_{u_2} \cdots I_{u_m}) P_{u_m}}, \end{aligned}$$

where  $P_i$  is the indecomposable projective  $\Lambda$ -module associated with the vertex  $i$ . The structure of  $T_v$  is determined by the sequence of integers  $u_1, \dots, u_v$ . Let  $K_{v,1}$  be the smallest submodule of  $P_{u_v}$  such that  $P_{u_v}/K_{v,1}$  is a sum of copies of  $S_{u_v}$  (in this case only  $S_{u_v}$ ). Then let  $K_{v,2}$  be the smallest submodule of  $K_{v,1}$  such that  $K_{v,1}/K_{v,2}$  is a sum of copies of  $S_{u_{v-1}}$  etc. Then  $T_v = P_{u_v}/K_{v,v}$ .

We also recall [BIRS] the following description of the quiver  $Q' = Q(u_1, \dots, u_m)$  of  $\text{End}_{\Lambda}(T)$ . The vertices are  $1, 2, \dots, m$  ordered from left to right. We often denote the vertex  $v$  by  $i_r$  if it is the  $r$ -th vertex of type  $i$ . In this case we write  $|i_r| = v$ . There are two kinds of arrows in  $Q'$ .



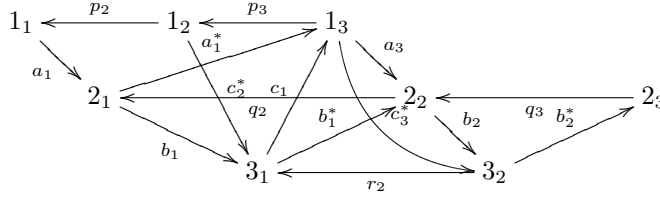


$i_r \xrightarrow{b} j_s$  in  $Q'$ , we let  $W_b = \epsilon(b)bb^*p$  if there is a (unique) arrow  $j_s \xrightarrow{b^*} i_t$  in  $Q'$ , where  $p$  denotes the path  $i_t \rightarrow i_{t-1} \rightarrow \cdots \rightarrow i_r$ . Otherwise we let  $W_b = 0$ . Then let

$$W = \sum_{b \in \underline{Q}'_1} W_b.$$

Our strategy is to show that all the relations  $\partial_a W$  are satisfied for  $\text{End}_{\mathcal{C}}(T)$ , and then apply Proposition 4.7 to show  $\text{End}_{\mathcal{C}}(T) \simeq \mathcal{P}(Q', W)$ .

**Example** Again we consider the above example. Deleting vertices  $1_4$ ,  $3_3$  and  $2_4$ , we have the quiver  $\underline{Q}(1, 2, 1, 3, 1, 2, 3, 1, 2, 3, 2)$  as follows.



The associated potential is given by

$$W = a_1 a_1^* p_3 p_2 - a_1^* a_3 q_2 + b_1 b_1^* q_2 - b_1^* b_2 r_2 + b_2 b_2^* q_3 - c_2^* c_1 p_3 + c_1 c_3^* r_2.$$

Let us start with the following information.

**Lemma 6.2.** *Let  $i_r \rightarrow i_{r-1}$  be an arrow in  $Q'$  going to the left. For any arrow  $i \xrightarrow{b} j$  in  $\overline{Q}$ , there exists a map  $T_{i_{r-1}} \xrightarrow{bb^*} T_{i_r}$  of  $\Lambda$ -modules given by multiplication with  $bb^*$ . Moreover precisely one of the following holds.*

- (a) *There do not exist vertices  $i_u$  and  $j_t$  in  $Q'$  satisfying  $|i_u| < |j_t| < |i_r|$ . In this case the map  $T_{i_{r-1}} \xrightarrow{bb^*} T_{i_r}$  is zero.*
- (b) *There exists a path  $i_u \xrightarrow{b} j_t \xrightarrow{b^*} i_v$  in  $Q'$  with  $u \leq r-1 < r \leq v$ . In this case such a path is unique, and the map  $T_{i_{r-1}} \xrightarrow{bb^*} T_{i_r}$  is equal to the composition  $T_{i_{r-1}} \xrightarrow{p} T_{i_u} \xrightarrow{b} T_{j_t} \xrightarrow{b^*} T_{i_v} \xrightarrow{q} T_{i_r}$  where  $p$  denotes a path  $i_{r-1} \rightarrow i_{r-2} \rightarrow \cdots \rightarrow i_u$  and  $q$  denotes a path  $i_v \rightarrow i_{v-1} \rightarrow \cdots \rightarrow i_r$ .*

*Proof.* We can write  $T_{i_{r-1}} = P_i / I P_i$  and  $T_{i_r} = P_i / (I I' I_i) P_i$ , where  $I'$  is a product of ideals  $I_1, \dots, I_n$  except  $I_i$ . Then we have  $bb^* \in I' I_i$ . Thus we have  $I b b^* \subset I I' I_i$ , and the map  $T_{i_{r-1}} \xrightarrow{bb^*} T_{i_r}$  is well-defined.

(a) Since  $bb^*$  is zero in  $T_{i_r}$ , we have the assertion.

(b) The uniqueness of the path is clear from the definition of  $Q'$ . The latter assertion is clear from Lemma 6.1.  $\square$

We now show that the 2-CY-tilted algebra  $\text{End}_{\mathcal{C}}(T)$  satisfies the relations for the Jacobian algebra  $\mathcal{P}(Q', W)$ .

**Proposition 6.3.**  *$\partial_a W$  belongs to the kernel of the surjection  $\widehat{KQ'} \rightarrow \text{End}_{\mathcal{C}}(T)$  for any arrow  $a$  in  $Q'$ .*

*Proof.* There are two cases to consider.

- (1) Consider an arrow  $i_r \xrightarrow{b} j_s$  going to the right in  $Q'$ . If there is an arrow  $j_s \xrightarrow{b^*} i_t$  in  $Q'$ , we have the cycle  $\epsilon(b)bb^*p$  as part of the potential  $W$ , where  $p$  is



There are no more arrows between the vertices of type  $i$  in  $C$ . Further  $i_r$  is the unique vertex of type  $i$  where there is an arrow from  $j_s$ , and  $i_u$  is the unique vertex of type  $i$  with an arrow to  $j_s$ . This shows that  $C$  is full.

For any cycle  $C$  of  $W$  and any arrow  $a$  in  $C$ , it is clear from the definition of  $W$  that the path  $\partial_a C$  determines  $a$ . Hence for any distinct arrows  $a$  and  $b$  contained in cycles of  $W$ , we have that  $\partial_a W$  and  $\partial_b W$  do not have common paths. Consequently the  $\partial_a W$  for all arrows  $a$  contained in cycles of  $W$  are linearly independent over  $K$ .

Since  $\Gamma = \text{End}_C(T)$  is 2-CY-tilted, it follows from [KR1] that  $\dim_K \text{Ext}_\Gamma^2(S, S') \leq \dim_K \text{Ext}_\Gamma^1(S', S)$  for simple  $\text{End}_C(T)$ -modules  $S$  and  $S'$ . It now follows from Proposition 4.7 that  $\text{End}_C(T)$  is isomorphic to the Jacobian algebra  $\mathcal{P}(\underline{Q}', W)$ .  $\square$

Since the 2-CY categories  $\text{Sub}\Lambda_w$  with the cluster-tilting objects determine a cluster structure [BIRS], we have the following consequence of Corollary 5.11.

**Corollary 6.5.** *With the previous notation, all the 2-CY-tilted algebras belonging to the mutation class of cluster-tilting objects associated with reduced expressions are given by QP's.*

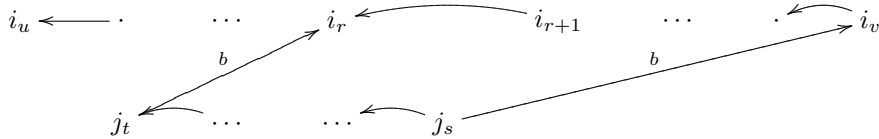
We now show that the potential  $W$  is rigid.

**Theorem 6.6.** *With the previous notation, the potential  $W$  on  $\underline{Q}'$  is rigid.*

*Proof.* Let  $C$  be any cycle in the quiver  $\underline{Q}'$ . We want to show that  $C$  belongs to  $\mathcal{J}(W)$  up to cyclic equivalences.

We say that a vertex  $v$  in  $C$  is a right turning point if an arrow in  $C$  going from left to right ends at  $v$ , and an arrow in  $C$  going from right to left starts at  $v$ . Then define  $r(C) = \sum_v 3^{|v|}$ , where we sum over all right turning points  $v$  of  $C$ . Consider a vertex  $i_u$  on  $C$ , with  $|i_u|$  minimal. Consider the last right turning point  $i_v$  preceding  $i_u$ , and let  $j_s \rightarrow i_v$  be the preceding arrow in  $C$ . Since  $|i_u| < |j_s| < |i_v|$ , we can choose  $i_r$  with  $|i_r| < |j_s| < |i_{r+1}|$ .

Assume first that there is some  $j_t$  with  $|j_t| < |i_r|$ . Then there is an arrow  $j_t \xrightarrow{b} i_r$  if  $j_t$  is chosen with  $|j_t|$  largest possible. We have the following subquiver of  $\underline{Q}'$ :



The composition  $j_s \xrightarrow{b} i_v \rightarrow i_{v-1} \rightarrow \cdots \rightarrow i_r$  coincides with the composition  $j_s \rightarrow j_{s-1} \rightarrow \cdots \rightarrow j_t \xrightarrow{b} i_r$  by Lemma 6.1. So we replace the first path by the second one to get a new cycle  $C'$  satisfying  $C - C' \in \mathcal{J}(W)$ . The right turning point  $i_v$  in  $C$  is replaced by at most two right turning points  $i_r$  and  $j_s$  in  $C'$ , where  $|i_r| < |i_v|$  and  $|j_s| < |i_v|$ , and hence  $3^{|i_r|} + 3^{|j_s|} < 3^{|i_v|}$ . This shows that  $r(C') < r(C)$ , and we are done by induction.

Assume now that there is no  $j_t$  with  $|j_t| < |i_r|$ . Then  $T_{i_r}$  has no composition factor  $S_j$ , so the composition  $T_{j_s} \rightarrow \cdots \rightarrow T_{i_r}$  must be 0. Hence  $C \in \mathcal{J}(W)$  in this case.  $\square$

Note that it follows from [DWZ1] that all the potentials in the mutation class of  $(\underline{Q}', W)$  are rigid, and hence we obtain a large class of rigid QP's. Some examples of rigid QP's where the quiver  $Q$  is not mutation equivalent to an acyclic quiver were given in [DWZ1].

## 7. NEARLY MORITA EQUIVALENCE FOR NEIGHBORING JACOBIAN ALGEBRAS

Let  $T$  be a cluster-tilting object in a triangulated 2-CY category  $\mathcal{C}$ , and  $T^* = \mu_k(T)$  another cluster-tilting object obtained by mutation. Then we have a nearly Morita equivalence between  $\Lambda = \text{End}_{\mathcal{C}}(T)$  and  $\Lambda' = \text{End}_{\mathcal{C}}(T^*)$ , that is an equivalence  $\frac{\text{mod } \Lambda}{[\text{add } S_k]} \rightarrow \frac{\text{mod } \Lambda'}{[\text{add } S'_k]}$ , where  $S_k$  and  $S'_k$  denote the simple modules at the vertex  $k$  [BMR1, KR1]. This generalizes the equivalence of [BGP] using reflection functors at sinks or sources. Since there are Jacobian algebras which are not 2-CY-tilted, it is natural to ask if we have a nearly Morita equivalence in general when performing mutations of potentials. It is the aim of this section to show that this is the case, when working with finite length modules.

Let  $(Q, W)$  be a quiver with potential and  $\Lambda = \mathcal{P}(Q, W)$  the associated Jacobian algebra. Denote by  $\tilde{\mu}_k(Q, W)$  the quiver with potential and  $\Lambda' = \mathcal{P}(\tilde{\mu}_k(Q, W))$  the Jacobian algebra obtained by mutation at the vertex  $k$ . We show that there is an equivalence of categories  $\frac{\text{f.l. } \Lambda}{[\text{add } S_k]} \rightarrow \frac{\text{f.l. } \Lambda'}{[\text{add } S'_k]}$ , where  $S_k$  and  $S'_k$  denote the simple modules at the vertex  $k$ .

Our starting point is the map  $G$  from objects in f.l.  $\Lambda$  to objects in f.l.  $\Lambda'$  used in [DWZ1], which we now recall.

Given an  $\Lambda$ -module  $M$  and a vertex  $k$ , we let  $a_1, \dots, a_s$  be all arrows in  $Q$  with  $e(a_p) = k$  and  $b_1, \dots, b_t$  be all arrows such with  $s(b_q) = k$ . We write

$$M_{\text{in}} = \bigoplus_{p=1}^s M_{e(a_p)}, \quad M_{\text{out}} = \bigoplus_{q=1}^t M_{s(b_q)}.$$

Let  $\alpha: M_{\text{in}} \rightarrow M_k$  and  $\beta: M_k \rightarrow M_{\text{out}}$  be the maps given in matrix form by

$$\alpha = (a_1, \dots, a_s), \quad \beta = \begin{pmatrix} b_1 \\ \vdots \\ b_t \end{pmatrix}.$$

Also, using the potential  $W$ , we define a map  $\gamma: M_{\text{out}} \rightarrow M_{\text{in}}$

$$\gamma = \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,t} \\ \vdots & \ddots & \vdots \\ \gamma_{s,1} & \cdots & \gamma_{s,t} \end{pmatrix}$$

where  $\gamma_{p,q} = \partial_{(a_p, b_q)} W$  in the notation of section 4. Thus, locally, we get a triangle

$$\begin{array}{ccc} & M_k & \\ \alpha \nearrow & & \searrow \beta \\ M_{\text{in}} & \xleftarrow{\gamma} & M_{\text{out}} \end{array}$$

Note that  $\gamma_{p,q}$  is a linear combination of paths in  $Q$  from  $e(b_q)$  to  $s(a_p)$ , and that there may be other paths from  $M_{\text{out}}$  to  $M_{\text{in}}$ .

Now, starting with  $M$ , we define a representation  $\tilde{M}$  of  $\Lambda' = \mathcal{P}(\tilde{\mu}_k(Q, W))$  as follows: first, we let  $i: \text{Ker } \gamma \rightarrow M_{\text{out}}$  be the inclusion and  $\varphi: \text{Ker } \alpha \rightarrow \frac{\text{Ker } \alpha}{\text{Im } \gamma}$  the natural surjection, and choose two  $K$ -linear splittings,

$$\begin{aligned} \rho: M_{\text{out}} &\rightarrow \text{Ker } \gamma \text{ such that } i\rho = \text{id}_{\text{Ker } \gamma}, \text{ and} \\ \sigma: \frac{\text{Ker } \alpha}{\text{Im } \gamma} &\rightarrow \text{Ker } \alpha \text{ such that } \sigma\varphi = \text{id}_{\frac{\text{Ker } \alpha}{\text{Im } \gamma}}. \end{aligned}$$

Then, locally, the representation  $\widetilde{M}$  is given by:

$$\begin{array}{ccc} & \widetilde{M}_k & \\ \tilde{\beta} \swarrow & & \nwarrow \tilde{\alpha} \\ M_{\text{in}} & \xrightleftharpoons[\gamma]{\alpha\beta} & M_{\text{out}} \end{array}$$

where

$$\widetilde{M}_k = \frac{\text{Ker } \gamma}{\text{Im } \beta} \oplus \text{Im } \gamma \oplus \frac{\text{Ker } \alpha}{\text{Im } \gamma}, \quad \tilde{\alpha} = (-\rho\pi, -\gamma, 0) \quad \text{and} \quad \tilde{\beta} = \begin{pmatrix} 0 \\ \iota \\ \sigma \end{pmatrix},$$

where  $\iota: \text{Im } \gamma \rightarrow \text{Ker } \alpha$  is the natural injection and  $\pi: \text{Ker } \gamma \rightarrow \frac{\text{Ker } \gamma}{\text{Im } \beta}$  is the natural projection.

By [DWZ1, Proposition 10.7], this makes  $\widetilde{M}$  a representation of  $\Lambda'$ . Moreover, by [DWZ1, Proposition 10.9], the isomorphism class of  $\widetilde{M}$  does not depend on the choice of the splittings  $\rho$  and  $\sigma$ .

We next want to define an associated map on morphisms. Assume that we have two representations  $M$  and  $M'$  of  $\Lambda$ , and a map  $f: M \rightarrow M'$ , that is locally a commutative diagram

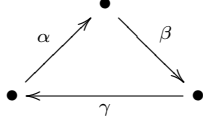
$$\begin{array}{ccc} & M_k & \\ & \alpha \nearrow & \searrow \beta \\ M & \xleftarrow{\gamma} & M_{\text{out}} \\ & \downarrow f_k & \\ & M'_k & \\ & \alpha' \nearrow & \searrow \beta' \\ M' & \xleftarrow{\gamma'} & M'_{\text{out}} \end{array} \quad (9)$$

We want to define a map  $\widetilde{M} \rightarrow \widetilde{M}'$ . To do so, we need to define a map  $\tilde{f}_k: \widetilde{M}_k \rightarrow \widetilde{M}'_k$  such that the squares of the following diagram commute:

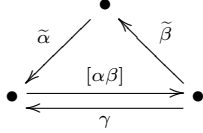
$$\begin{array}{ccc} & \widetilde{M}_k & \\ & \tilde{\beta} \swarrow & \nwarrow \tilde{\alpha} \\ \widetilde{M} & \xrightleftharpoons[\gamma]{\alpha\beta} & M_{\text{out}} \\ & \downarrow \tilde{f}_k & \\ & \widetilde{M}'_k & \\ & \tilde{\beta}' \swarrow & \nwarrow \tilde{\alpha}' \\ \widetilde{M}' & \xrightleftharpoons[\gamma']{\alpha'\beta'} & M'_{\text{out}} \end{array}$$

**Remark** From the first diagram we have induced maps  $\text{Ker } \gamma \rightarrow \text{Ker } \gamma'$ ,  $\text{Ker } \alpha \rightarrow \text{Ker } \alpha'$ ,  $\text{Im } \gamma \rightarrow \text{Im } \gamma'$ ,  $\text{Im } \beta \rightarrow \text{Im } \beta'$ ,  $\frac{\text{Ker } \gamma}{\text{Im } \beta} \rightarrow \frac{\text{Ker } \gamma'}{\text{Im } \beta'}$  and  $\frac{\text{Ker } \alpha}{\text{Im } \gamma} \rightarrow \frac{\text{Ker } \alpha'}{\text{Im } \gamma'}$ . This again induces componentwise a natural map  $\widetilde{M}_k \rightarrow \widetilde{M}'_k$ . However, this map turns out not to work for our purposes, as the following example shows.

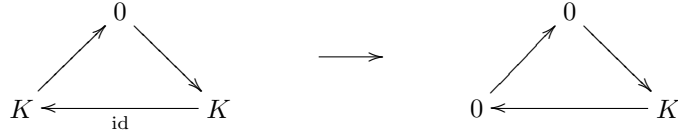
**Example** Consider the Jacobian algebra  $\mathcal{P}(Q, W)$  given by the quiver



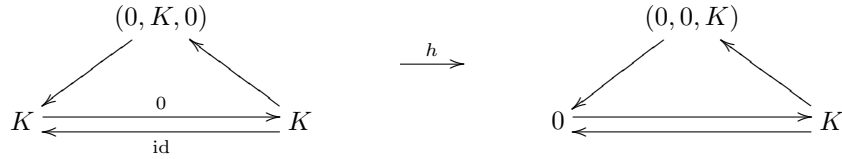
and the potential  $W = \alpha\beta\gamma$ . Then  $\widetilde{\mu}_k(\mathcal{P}(Q, W)) = (\widetilde{Q}, \widetilde{W})$ , where  $\widetilde{Q}$  is the quiver



and  $\widetilde{W}$  is the the potential  $[\alpha\beta]\gamma + [\alpha\beta]\widetilde{\alpha}\widetilde{\beta}$ . Consider a nonzero map between  $\mathcal{P}(Q, W)$ -modules



The image of this map in f.l.  $\mathcal{P}(\widetilde{Q}, \widetilde{W})$  would be



where  $h$  is the zero-map, which indicates that this is not the map we are looking for.

In view of the above, we now proceed to define a new map. Considering the expression

$$\frac{\text{Ker } \gamma}{\text{Im } \beta} \oplus \text{Im } \gamma \oplus \frac{\text{Ker } \alpha}{\text{Im } \gamma},$$

we observe that the two first terms extend to  $\text{Cok } \beta$ , while the two last terms extend to  $\text{Ker } \alpha$ . We have a natural induced map  $\text{Cok } \beta \rightarrow \text{Ker } \alpha$ . Taking these features into account we consider the knitting of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \frac{\text{Ker } \gamma}{\text{Im } \beta} & \xrightarrow{\widetilde{i}} & \text{Cok } \beta & \xrightarrow{\widetilde{\nu}} & \text{Ker } \alpha & \xrightarrow{\varphi} & \frac{\text{Ker } \alpha}{\text{Im } \gamma} & \longrightarrow & 0 \\
 & & & & \searrow \widetilde{\gamma} & & \nearrow \iota & & & & \\
 & & & & & \text{Im } \gamma & & & & & \\
 & & & & \nearrow & & \searrow & & & & \\
 & & & & 0 & & 0 & & & & 
 \end{array}$$

Moreover,  $\tilde{i}: \frac{\text{Ker } \gamma}{\text{Im } \beta} \rightarrow \text{Cok } \beta$  and  $\tilde{\gamma}: \text{Cok } \beta \rightarrow \text{Im } \gamma$  are the induced maps.

Considering the induced map  $\tilde{\rho}: \text{Cok } \beta \rightarrow \frac{\text{Ker } \gamma}{\text{Im } \beta}$ , we get a knitting of ( $K$ -split) short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \frac{\text{Ker } \gamma}{\text{Im } \beta} & \xrightleftharpoons[\tilde{\rho}]{\tilde{i}} & \text{Cok } \beta & \xrightarrow{\tilde{\gamma}\iota} & \text{Ker } \alpha & \xrightleftharpoons[\sigma]{\varphi} & \frac{\text{Ker } \alpha}{\text{Im } \gamma} & \longrightarrow & 0 \\
 & & & & \swarrow \tilde{\gamma} & & \swarrow \iota & & & & \\
 & & & & \text{Im } \gamma & & \searrow \varepsilon & & & & \\
 & & & & \uparrow j & & \uparrow \varepsilon & & & & \\
 & & & & 0 & & 0 & & & & 
 \end{array}$$

where the maps  $j$  and  $\varepsilon$  are induced from the splittings  $\tilde{\rho}$  and  $\sigma$  respectively, so that  $\text{id}_{\text{Cok } \beta} = \tilde{\rho}\tilde{i} + \tilde{\gamma}j$  and  $\text{id}_{\text{Ker } \alpha} = \varepsilon\iota + \varphi\sigma$ .

The idea is then to use the diagram

$$\begin{array}{ccccccc}
 \frac{\text{Ker } \alpha}{\text{Im } \gamma} & & & & & & \frac{\text{Ker } \alpha'}{\text{Im } \gamma'} \\
 \oplus & \xrightarrow{\sigma} & \text{Ker } \alpha & \xrightarrow{f_{\text{in}}} & \text{Ker } \alpha' & \xrightarrow{\varphi'} & \oplus \\
 \text{Im } \gamma & \xrightarrow{\iota} & \uparrow & & \uparrow & \xrightarrow{\varepsilon'} & \text{Im } \gamma' \\
 \oplus & \xrightarrow{j} & \text{Cok } \beta & \xrightarrow{f_{\text{out}}} & \text{Cok } \beta' & \xrightarrow{\tilde{\gamma}'} & \oplus \\
 \frac{\text{Ker } \gamma}{\text{Im } \beta} & \xrightarrow{\tilde{i}} & & & & & \frac{\text{Ker } \gamma'}{\text{Im } \beta'} \\
 & & & & & & \tilde{\rho}'
 \end{array}$$

where  $\widetilde{f_{\text{out}}}$  is the induced quotient map and where the middle part commutes, that is

$$\iota \circ f_{\text{in}} \circ \varepsilon' = j \circ \widetilde{f_{\text{out}}} \circ \tilde{\gamma}'.$$

We let

$$\begin{aligned}
 \tilde{f}_k &= \begin{pmatrix} \tilde{i} \circ \widetilde{f_{\text{out}}} \circ \tilde{\rho}' & \tilde{i} \circ \widetilde{f_{\text{out}}} \circ \tilde{\gamma}' & 0 \\ j \circ \widetilde{f_{\text{out}}} \circ \tilde{\rho}' & \iota \circ f_{\text{in}} \circ \varepsilon' & \iota \circ f_{\text{in}} \circ \varphi' \\ 0 & \sigma \circ f_{\text{in}} \circ \varepsilon' & \sigma \circ f_{\text{in}} \circ \varphi' \end{pmatrix} = \\
 & \begin{pmatrix} \tilde{i} \circ \widetilde{f_{\text{out}}} \circ \tilde{\rho}' & 0 & 0 \\ j \circ \widetilde{f_{\text{out}}} \circ \tilde{\rho}' & \iota \circ f_{\text{in}} \circ \varepsilon' & 0 \\ 0 & \sigma \circ f_{\text{in}} \circ \varepsilon' & \sigma \circ f_{\text{in}} \circ \varphi' \end{pmatrix}
 \end{aligned}$$

since  $\iota \circ f_{\text{in}} \circ \varphi' = 0$  and  $\tilde{i} \circ \widetilde{f_{\text{out}}} \circ \tilde{\gamma}' = 0$ . It is however often convenient to use the first expression of  $\tilde{f}_k$ .

We need to verify that  $\tilde{\beta} \circ f_{\text{in}} = \tilde{f}_k \circ \tilde{\beta}'$  and  $\tilde{\alpha} \circ \tilde{f}_k = f_{\text{out}} \circ \tilde{\alpha}'$ .



First,

$$\begin{aligned}
\widetilde{f}_k \circ \widetilde{\beta}' &= \begin{pmatrix} \widetilde{i} \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' & \widetilde{i} \circ \widetilde{f}_{\text{out}} \circ \widetilde{\gamma}' & 0 \\ j \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' & \iota \circ \widetilde{f}_{\text{in}} \circ \varepsilon' & \iota \circ \widetilde{f}_{\text{in}} \circ \varphi' \\ 0 & \sigma \circ \widetilde{f}_{\text{in}} \circ \varepsilon' & \sigma \circ \widetilde{f}_{\text{in}} \circ \varphi' \end{pmatrix} \begin{pmatrix} 0 \\ \iota' \\ \sigma' \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \iota \circ \widetilde{f}_{\text{in}} \circ (\varepsilon' \circ \iota' + \varphi' \circ \sigma') \\ \sigma \circ \widetilde{f}_{\text{in}} \circ (\varepsilon' \circ \iota' + \varphi' \circ \sigma') \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \iota \circ \widetilde{f}_{\text{in}} \\ \sigma \circ \widetilde{f}_{\text{in}} \end{pmatrix} \\
&= \widetilde{\beta}' \circ \widetilde{f}_{\text{in}}
\end{aligned}$$

On one hand, we have

$$\widetilde{f}_{\text{out}} \circ \widetilde{\alpha}' = (-\widetilde{f}_{\text{out}} \circ \rho' \circ \pi', -\widetilde{f}_{\text{out}} \circ \gamma', 0).$$

On the other hand,

$$\begin{aligned}
\widetilde{\alpha} \circ \widetilde{f}_k &= (-\rho\pi, -\gamma, 0) \begin{pmatrix} \widetilde{i} \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' & 0 & 0 \\ j \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' & \iota \circ \widetilde{f}_{\text{in}} \circ \varepsilon' & 0 \\ 0 & \sigma \circ \widetilde{f}_{\text{in}} \circ \varepsilon' & \sigma \circ \widetilde{f}_{\text{in}} \circ \varphi' \end{pmatrix} \\
&= (-\rho \circ \pi \circ \widetilde{i} \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' - \gamma \circ j \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}', -\gamma \circ \iota \circ \widetilde{f}_{\text{in}} \circ \varepsilon', 0)
\end{aligned}$$

Since  $-\gamma \circ \iota \circ \widetilde{f}_{\text{in}} \circ \varepsilon' = -\gamma \circ \widetilde{f}_{\text{in}} \circ \iota' \circ \varepsilon' = -\gamma \circ \widetilde{f}_{\text{in}} = -\widetilde{f}_{\text{out}} \circ \gamma'$ , it suffices (to get  $\widetilde{\alpha} \circ \widetilde{f}_k = \widetilde{f}_{\text{out}} \circ \widetilde{\alpha}'$ ) to show that

$$-\rho \circ \pi \circ \widetilde{i} \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' - \gamma \circ j \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' = -\widetilde{f}_{\text{out}} \circ \rho' \circ \pi'.$$

But, by the definition of  $\widetilde{\rho}$  and  $\widetilde{\gamma}$ , the diagrams

$$\begin{array}{ccc}
M_{\text{out}} & \xrightarrow{\rho} & \text{Ker } \gamma \\
\pi \downarrow & & \downarrow \pi \\
\text{Cok } \beta & \xrightarrow{\widetilde{\rho}} & \frac{\text{Ker } \gamma}{\text{Im } \beta}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M_{\text{out}} & \xrightarrow{\gamma} & \text{Im } \gamma \\
\pi \downarrow & \nearrow \widetilde{\gamma} & \\
\text{Cok } \beta & & 
\end{array}$$

are commutative (where, by abuse of notation, we also denote by  $\pi$  the natural projection  $M_{\text{out}} \rightarrow \text{Cok } \beta$ ). Therefore,

$$\begin{aligned}
-\rho \circ \pi \circ \widetilde{i} \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' - \gamma \circ j \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' &= -(\rho \circ \pi \circ \widetilde{i} + \gamma \circ j) \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' \\
&= -(\pi \circ \widetilde{\rho} \circ \widetilde{i} + \pi \circ \widetilde{\gamma} \circ j) \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' \\
&= -\pi \circ (\widetilde{\rho} \circ \widetilde{i} + \widetilde{\gamma} \circ j) \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' \\
&= -\pi \circ \text{id}_{\text{Cok } \beta} \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' \\
&= -\pi \circ \widetilde{f}_{\text{out}} \circ \widetilde{\rho}' \\
&= -\widetilde{f}_{\text{out}} \circ \rho' \circ \pi'
\end{aligned}$$

where the last equality follows from the commutativity of

$$\begin{array}{ccccc}
M_{\text{out}} & \xrightarrow{f_{\text{out}}} & M'_{\text{out}} & \xrightarrow{\rho'} & \text{Ker } \gamma' \\
\pi \downarrow & & \downarrow \pi' & & \downarrow \pi' \\
\text{Cok } \beta & \xrightarrow{f_{\text{out}}} & \text{Cok } \beta' & \xrightarrow{\rho'} & \frac{\text{Ker } \gamma'}{\text{Im } \beta'}
\end{array}$$

Note that the image  $\widetilde{f}$  apparently depends on the choice of the choice of splittings  $\rho, \sigma, \rho'$  and  $\sigma'$ . However, we only aim to show that we obtain a functor  $\text{f.l. } \Lambda \rightarrow \frac{\text{f.l. } \Lambda'}{[\text{add } S'_k]}$ .

In [DWZ1] it is proved that different choice of splittings will give isomorphic objects, and for each vertex  $i \neq k$ , the isomorphism is given by  $\text{id}: M_i \rightarrow \widetilde{M}_i$ . Therefore, different choices of splittings will give maps which are equal in  $\frac{\text{f.l. } \Lambda'}{[\text{add } S'_k]}$ .

So, we get a well defined map of morphisms from  $\text{f.l. } \Lambda$  to  $\frac{\text{f.l. } \Lambda'}{[\text{add } S'_k]}$ .

We next want to show that the definition of  $G$  on objects and morphisms from  $\text{f.l. } \Lambda$  to  $\frac{\text{f.l. } \Lambda'}{[\text{add } S'_k]}$  actually gives rise to a functor, and that we get an induced equivalence

$$\frac{\text{f.l. } \Lambda}{[\text{add } S_k]} \rightarrow \frac{\text{f.l. } \Lambda'}{[\text{add } S'_k]}.$$

We first observe that for  $M$  in  $\text{f.l. } \Lambda$  we have  $F(\text{id}_M) = \text{id}_{FM}$ . This follows from  $\sigma\varphi = \text{id}_{\frac{\text{Ker } \alpha}{\text{Im } \gamma}}$ ,  $\widetilde{i}\widetilde{\rho} = \text{id}_{\frac{\text{Ker } \gamma}{\text{Im } \beta}}$  and  $\iota\varepsilon = \text{id}_{\text{Im } \gamma}$ . Then, let  $f: M \rightarrow M'$  and  $f': M' \rightarrow M''$  be two maps in  $\text{f.l. } \Lambda$ . We will compare  $\widetilde{f \circ f'}$  and  $\widetilde{f} \circ \widetilde{f}'$ . By construction, these maps are obtained from the following diagrams:

$$\begin{array}{ccc} \widetilde{f \circ f'} & & \widetilde{f} \circ \widetilde{f}' \\ \\ \begin{array}{c} \frac{\text{Ker } \gamma}{\text{Im } \beta} \oplus \text{Im } \gamma \oplus \frac{\text{Ker } \alpha}{\text{Im } \gamma} \\ \downarrow \begin{array}{l} \widetilde{i} \quad j \quad \iota \quad \sigma \\ \text{Cok } \beta \quad \text{Ker } \alpha \end{array} \\ \text{Cok } \beta \quad \text{Ker } \alpha \\ \downarrow \begin{array}{l} \widetilde{f_{\text{out}}} \quad \widetilde{f_{\text{in}}} \\ \text{Cok } \beta' \quad \text{Ker } \alpha' \end{array} \\ \begin{array}{c} \frac{\text{Ker } \gamma'}{\text{Im } \beta'} \oplus \text{Im } \gamma' \oplus \frac{\text{Ker } \alpha'}{\text{Im } \gamma'} \\ \downarrow \begin{array}{l} \widetilde{i}' \quad j' \quad \iota' \quad \sigma' \\ \text{Cok } \beta' \quad \text{Ker } \alpha' \end{array} \\ \text{Cok } \beta' \quad \text{Ker } \alpha' \\ \downarrow \begin{array}{l} \widetilde{f_{\text{out}}'} \quad \widetilde{f_{\text{in}}}' \\ \text{Cok } \beta'' \quad \text{Ker } \alpha' \end{array} \\ \begin{array}{c} \frac{\text{Ker } \gamma''}{\text{Im } \beta''} \oplus \text{Im } \gamma'' \oplus \frac{\text{Ker } \alpha''}{\text{Im } \gamma''} \\ \downarrow \begin{array}{l} \widetilde{\rho}'' \quad \widetilde{\gamma}'' \quad \varepsilon'' \quad \varphi'' \\ \text{Cok } \beta'' \quad \text{Ker } \alpha' \end{array} \\ \text{Cok } \beta'' \quad \text{Ker } \alpha' \\ \downarrow \begin{array}{l} \widetilde{\rho}'' \quad \widetilde{\gamma}'' \quad \varepsilon'' \quad \varphi'' \\ \frac{\text{Ker } \gamma''}{\text{Im } \beta''} \oplus \text{Im } \gamma'' \oplus \frac{\text{Ker } \alpha''}{\text{Im } \gamma''} \end{array} \\ \frac{\text{Ker } \gamma''}{\text{Im } \beta''} \oplus \text{Im } \gamma'' \oplus \frac{\text{Ker } \alpha''}{\text{Im } \gamma''} \end{array} \end{array} \end{array}$$

and easy computations show that, locally, the morphism  $\widetilde{f} \circ \widetilde{f}' - \widetilde{f} \circ \widetilde{f}'$  is given by

$$\begin{array}{ccc}
 & \begin{array}{c} \text{Ker } \gamma \\ \text{Im } \beta \end{array} \oplus \text{Im } \gamma \oplus \begin{array}{c} \text{Ker } \alpha \\ \text{Im } \gamma \end{array} & \\
 & \begin{array}{c} \xrightarrow{\widetilde{\beta}} \\ \xleftarrow{\widetilde{\alpha}} \end{array} & \\
 M_{\text{in}} & \begin{array}{c} \xrightarrow{\alpha\beta} \\ \xleftarrow{\gamma} \end{array} & M_{\text{out}} \\
 & \downarrow F_k & \\
 & \begin{array}{c} \text{Ker } \gamma'' \\ \text{Im } \beta'' \end{array} \oplus \text{Im } \gamma'' \oplus \begin{array}{c} \text{Ker } \alpha'' \\ \text{Im } \gamma'' \end{array} & \\
 & \begin{array}{c} \xrightarrow{\beta''} \\ \xleftarrow{\widetilde{\alpha}''} \end{array} & \\
 M''_{\text{in}} & \begin{array}{c} \xrightarrow{\alpha''\beta''} \\ \xleftarrow{\gamma''} \end{array} & M''_{\text{out}} \\
 & \downarrow 0 & \\
 0 & & 0
 \end{array}$$

where  $F_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma \circ f_{\text{in}} \circ \varepsilon' \circ j' \circ f_{\text{out}}' \circ \rho'' & 0 & 0 \end{pmatrix}$ . Therefore, the image of  $\widetilde{f} \circ \widetilde{f}' - \widetilde{f} \circ \widetilde{f}'$  lies in  $\text{add } S'_k$ , showing that we have a functor  $\text{f.l. } \Lambda \rightarrow \frac{\text{f.l. } \Lambda'}{[\text{add } S'_k]}$ .

We have  $F(S_k) = 0$ , hence there is a functor

$$\frac{\text{f.l. } \Lambda}{[\text{add } S_k]} \rightarrow \frac{\text{f.l. } \Lambda'}{[\text{add } S'_k]}.$$

We now verify that this functor is an equivalence. By (the proof of) [DWZ1, Theorem 10.13], this functor is a bijection at the level of representations: in particular it is dense. To show that it is a bijection at the level of the morphisms, suppose that we have a morphism of representations as in diagram (9). By construction  $\widetilde{f}$  is of the form:

$$\begin{array}{ccc}
 & \widetilde{M}_k & \\
 & \begin{array}{c} \xrightarrow{\alpha=\widetilde{\alpha}} \\ \xleftarrow{\beta=\widetilde{\beta}} \end{array} & \\
 & \begin{array}{c} \xrightarrow{\widetilde{f}_k} \\ \xleftarrow{\alpha\beta} \end{array} & \\
 M_{\text{in}} & \begin{array}{c} \xrightarrow{\gamma} \\ \xleftarrow{\widetilde{\alpha}\widetilde{\beta}} \end{array} & M_{\text{out}} \\
 & \downarrow \widetilde{f}_k & \\
 & \widetilde{M}'_k & \\
 & \begin{array}{c} \xrightarrow{\alpha'=\widetilde{\alpha}'} \\ \xleftarrow{\beta'=\widetilde{\beta}'} \end{array} & \\
 M'_{\text{in}} & \begin{array}{c} \xrightarrow{\alpha'\beta'} \\ \xleftarrow{\widetilde{\alpha}'\widetilde{\beta}'} \end{array} & M'_{\text{out}} \\
 & \downarrow f_{\text{in}} & \downarrow f_{\text{out}} \\
 \widetilde{M} & & M' \\
 & \downarrow \widetilde{f} &
 \end{array}$$

By [DWZ1],  $M$  is isomorphic to the reduced form  $\mu_k(M)$  of  $\widetilde{M}$ . To obtain this reduced form, one applies three steps; see [DWZ1, Theorem 5.7, Steps 1-3]. It turns out that only the first step has an impact on  $\alpha, \beta$  and  $\gamma$ , and this modification consists of changing the map  $\beta$  to  $-\beta$ . So, applying these steps does not change

the commutativity of the above diagram, saying that after the reduction,  $\tilde{f}$  is still a morphism for the reduced representation.

Now, since  $f - \tilde{f}$  is (possibly) not zero only at position  $k$ , then  $f$  and  $\tilde{f}$  coincide in  $\frac{\text{f.l. } \Lambda}{\text{add } S_k}$ , showing the bijection at the level of morphisms.

Hence we have proved the following.

**Theorem 7.1.** *The Jacobian algebras  $\Lambda = \mathcal{P}(Q, W)$  and  $\Lambda' = \mathcal{P}(\mu_k(Q', W'))$  are nearly Morita equivalent.*

For Jacobi-finite QP's this result can be obtained with a very different approach by combining the result of [A] saying that these algebras are 2-CY-tilted with the corresponding result for 2-CY-tilted algebras [BMR1, KR1].

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INSTITUTT FOR MATEMATISKE FAG, NORGES TEKNISK-NATURVITENSKAPELIGE UNIVERSITET, N-7491 TRONDHEIM, NORWAY  
*E-mail address:* `aslab@math.ntnu.no`

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, CHIKUSA-KU, NAGOYA, 464-8602, JAPAN  
*E-mail address:* `iyama@math.nagoya-u.ac.jp`

INSTITUTT FOR MATEMATISKE FAG, NORGES TEKNISK-NATURVITENSKAPELIGE UNIVERSITET, N-7491 TRONDHEIM, NORWAY  
*E-mail address:* `idunr@math.ntnu.no`

DEPARTMENT OF MATHEMATICS, BISHOP'S UNIVERSITY, SHERBROOKE, QC, CANADA, J1M0C8  
*E-mail address:* `dsmith@ubishops.ca`