On discrete least squares polynomial fit, linear spaces and data classification

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Abstract : The best discrete least squares polynomial fit to a data set is revisited. We point out some properties related to the best polynomial and precise the dimension of vector spaces encountered to solve the problem. Finally, we suggest a classification of data sets based on their increasing or decreasing trend, and on their convexity or concavity form.

Key Words : Polynomial data fitting, weighted least squares, orthogonal polynomials, linear spaces, data classification.

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1. Introduction

Let $\{(\omega_i, t_i, f_i)\}_{i=1}^m$ be a set of m data points where the t_i's represent the distinct values of the independent variable, the f_i's are the values of the measured function, and each ω_i is the weight associated to the data (t_i, f_i). The problem we consider is to find a polynomial p_n of degree at most n to fit the data. To measure how well the polynomial fit the data we use the weighted least squares deviation given by

$$F(p_n) = \sum_{i=1}^{m} \omega_i \big(f_i - p_n(t_i)\big)^2. \tag{1.1}$$

The best polynomial, called the weighted least squares estimate (WLSE), is given by

$$p_n^* = \mathrm{argmin}_{p_n \in \mathcal{P}_n} F(p_n). \tag{1.2}$$

where $\mathcal{P}_n$ is the set of polynomials of degree at most n.

The goal of this paper is to clarify the dimension of some vector spaces encountered in solving this problem, establish a property useful for proving the existence of a WLSE for exponential models [2], and suggest a way to classify data using the best polynomial fits. For a standard presentation of the theory related to best (polynomial) least squares fit see [1, 3, 6, 7].

The best polynomial fit problem can be solved by considering an orthogonal projection onto $\mathcal{P}_n$ or, equivalently, by considering an orthogonal projection onto a subspace of $\mathbb{R}^m$. In Section 2 we briefly review the solution of the problem in $\mathcal{P}_n$ and specify the dimension of subspaces of polynomials. In the first part of the

Section 3 we consider the subspaces of $\mathbb{R}^m$ that play a role in solving the problem in $\mathbb{R}^m$. In the second part of this Section 3 we solve the problem using a projection onto a subspace of $\mathbb{R}^m$. Finally in Section 4 we suggest a way to classify data which will be useful in the problem of finding existence results for weighted least squares estimator [2].

2. Polynomial weighted least squares fitting in $\mathcal{P}_n$

In the first part of this section we present the underlying subspaces of $\mathcal{P} = \mathrm{Lin}\{\, t^j \mid j = 0, 1, 2, \dots \,\}$ related to the polynomial weighted least squares problem. In the second part we solve the problem using a projection onto a subspace of $\mathcal{P}$.

2.1. **Vector spaces.** Let us recall that $\mathcal{P}_n = \mathrm{Lin}\{\, t^j \mid j = 0, 1, ..., n \,\}$. We consider also the following two other polynomial subspaces

$$\mathcal{PV}_k^+ = \mathrm{Lin}\{\, v_{k,i}^+(t) = (t + t_i)^k \mid i = 1, ..., m \,\} \subseteq \mathcal{P}_k, \tag{2.1}$$

$$\mathcal{PV}_k^- = \mathrm{Lin}\{\, v_{k,i}^-(t) = (t - t_i)^k \mid i = 1, ..., m \,\} \subseteq \mathcal{P}_k, \tag{2.2}$$

for any nonnegative integer $k = 0, 1, 2, \dots$ The next two results specify the dimension of these subspaces.

Theorem 2.1. *Let* $\mathcal{P}_n = \mathrm{Lin}\{t^j \mid j = 0, ..., n\,\} \subseteq \mathcal{P}$*, then* $\dim \mathcal{P}_n = n + 1$. □

Theorem 2.2. *Let k be any nonnegative integer and let $\mathcal{PV}_k^+$ and $\mathcal{PV}_k^-$ be defined by* (2.1) *and* (2.2)*.*
(a) *If $k \le m - 1$ then $\mathcal{PV}_k^+ = \mathcal{P}_k = \mathcal{PV}_k^-$, and $\dim \mathcal{PV}_k^+ = k + 1 = \dim \mathcal{PV}_k^-$.*
(b) *If $k \ge m$ then $\mathcal{PV}_k^+ \subsetneq \mathcal{P}_k$, $\mathcal{PV}_k^- \subsetneq \mathcal{P}_k$, and $\dim \mathcal{PV}_k^+ = m = \dim \mathcal{PV}_k^-$.*

Proof. We prove the result for $\mathcal{PV}_k^+$ only, the proof for $\mathcal{PV}_k^-$ is identical. Since

$$\sum_{i=1}^m \mu_i v_{k,i}^+(t) = \sum_{i=1}^m \mu_i \Big(\sum_{j=0}^k \left(\begin{array}{c} k \\ j \end{array} \right) t_i^j t^{k-j} \Big) = \sum_{j=0}^k \left(\begin{array}{c} k \\ j \end{array} \right) \Big(\sum_{i=1}^m \mu_i t_i^j \Big) t^{k-j},$$

then $\sum_{i=1}^m \mu_i v_{k,i}^+(t) = 0$ if and only if $\sum_{j=0}^k \left(\begin{array}{c} k \\ j \end{array} \right) \Big(\sum_{i=1}^m \mu_i t_i^j \Big) t^{k-j} = 0$. From Theorem 2.1, the set $\{t^j\}_{j=0}^k$ is linearly independent, it follows that $\sum_{i=1}^m \mu_i t_i^j = 0$ for $j = 0, ..., k$. The matrix associated to this system is a Vandermonde type matrix. The rank of this matrix in $\min\{k + 1, m\}$ and the result follows. □

2.2. **Polynomial weighted least squares fitting.** Under the condition that $n < m$, we introduce the scalar product on $\mathcal{P}_n$ defined by

$$\langle p, q \rangle = \sum_{i=1}^m \omega_i p(t_i) q(t_i)$$

for any pair of polynomials p and q in $\mathcal{P}_n$. In this case (1.1) becomes

$$F(p_n) = \|f - p_n\|^2$$

where $\|\cdot\|$ is the norm on $\mathcal{P}_n$ induced by the scalar product. For the f_i's we use the notation $f_i = f(t_i)$ $(i = 1, ..., m)$. It is well known that p_n^* is unique and is characterized by the normal equations

$$\langle f - p_n^*, p_n \rangle = 0$$

for all $p_n \in \mathcal{P}_n$.

In this setting, to simplify the computation of p_n^*, we can find a sequence of orthogonal polynomials by applying the Gram-Schmidt orthogonalization process to the standard basis $\{1, t, t^2, \cdots, t^n\}$ of $\mathcal{P}_n$. These orthogonal polynomials are given by

$$q_0(t) = 1, \quad q_1(t) = t - \alpha_1,$$

and for $j = 2, ..., n$,

$$q_j(t) = (t - \alpha_j) q_{j-1}(t) - \beta_j q_{j-2}(t)$$

where

$$\alpha_j = \frac{\langle t q_{j-1}, q_{j-1} \rangle}{\langle q_{j-1}, q_{j-1} \rangle} \qquad (j = 1, 2, ..., n),$$

and

$$\beta_j = \frac{\langle t q_{j-1}, q_{j-2} \rangle}{\langle q_{j-2}, q_{j-2} \rangle} \qquad (j = 2, 3, ..., n).$$

Hence the best n-degree least squares polynomial p_n^* can be written as

$$p_n^*(t) = \sum_{j=0}^{n} \gamma_j^* q_j(t) \tag{2.3}$$

where

$$\gamma_j^* = \frac{\langle f, q_j \rangle}{\langle q_j, q_j \rangle} \quad (j = 0, 1, \ldots, n).$$

The next two results will be useful for finding sufficient conditions for the existence of the WLSE for a 3-parametric exponential model [2].

Theorem 2.3. $\langle f - p_{n-1}^*, t^n \rangle = \gamma_n^* \|q_n\|^2$ *for* $n = 0, \ldots, m-1$.

Proof. For $n = 0$ it is obvious because $p_{n-1}^* = 0$. For $n > 0$, since $q_n(t) = t^n + p_{n-1}(t)$ where $p_{n-1}(t)$ is a polynomial of degree $\leq n-1$, and

$$p_n^*(t) = \gamma_n^* q_n(t) + p_{n-1}^*(t),$$

we have

$$\begin{aligned}
\gamma_n^* \|q_n\|^2 &= \langle \gamma_n^* q_n, q_n \rangle \\
&= \langle p_n^* - p_{n-1}^*, q_n \rangle \\
&= \langle p_n^* - f, q_n \rangle + \langle f - p_{n-1}^*, q_n \rangle \\
&= \langle f - p_{n-1}^*, t^n + p_{n-1} \rangle \\
&= \langle f - p_{n-1}^*, t^n \rangle. \qquad \square
\end{aligned}$$

Theorem 2.4. *If the* q_j *'s are the orthogonal polynomials associated to* $\{(\omega_i, t_i)\}_{i=1}^m$, *the orthogonal polynomials* $\tilde{q}_j$ *'s associated to* $\{(\omega_i, \tilde{t}_i = -t_i)\}_{i=1}^m$ *are given by*

$$\tilde{q}_j(t) = (-1)^j q_j(-t). \qquad \square$$

3. Polynomial weighted least squares fitting in $\mathbb{R}^m$

In the first part of this section we present the underlying subspaces of $\mathbb{R}^m$ related to the polynomial weighted least squares problem. In the second part we solve the problem using a projection onto a subspace of $\mathbb{R}^m$.

3.1. **Vector spaces.** Let $\{t_i\}_{i=1}^m$ be a set of m distinct real numbers. For any positive integer j let us define the vectors $\overrightarrow{t}^j \in \mathbb{R}^m$ by

$$\overrightarrow{t}^j = \begin{pmatrix} t_1^j \\ t_2^j \\ \vdots \\ t_m^j \end{pmatrix} \in \mathbb{R}^m.$$

For any positive integer k, we also define the vectors

$$\overrightarrow{v}^+_{k,i} = \left(\overrightarrow{t} + t_i \overrightarrow{1}\right)^k = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} t_i^j \overrightarrow{t}^{k-j} \qquad \text{for} \qquad i = 1, ..., m,$$

and

$$\overrightarrow{v}^-_{k,i} = \left(\overrightarrow{t} - t_i \overrightarrow{1}\right)^k = \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} t_i^j \overrightarrow{t}^{k-j} \qquad \text{for} \qquad i = 1, ..., m.$$

In this section we clarify the properties of the following vector spaces, in particular the dimension of the vector spaces,

$$\mathcal{T}^n = \text{Lin}\left\{ \overrightarrow{t}^j \,\middle|\, j = 0, ..., n \right\}, \tag{3.1}$$

$$\mathcal{V}^+_k = \text{Lin}\left\{ \overrightarrow{v}^+_{k,i} \,\middle|\, i = 1, ..., m \right\}, \tag{3.2}$$

$$\mathcal{V}^-_k = \text{Lin}\left\{ \overrightarrow{v}^-_{k,i} \,\middle|\, i = 1, ..., m \right\}, \tag{3.3}$$

for any integers n and k such that $n \geq 0$ and $0 \leq k \leq m - 1$.

Theorem 3.1. *Let* $\mathcal{T}^n = \text{Lin}\left\{ \overrightarrow{t}^j \mid j = 0, ..., n \right\} \subseteq \mathbb{R}^m$.
(a) *If* $n < m$, *the set* $\left\{ \overrightarrow{t}^j \right\}_{j=0}^n$ *is linearly independent and* $\dim \mathcal{T}^n = n + 1$.
(b) *If* $n \geq m$, *the set* $\left\{ \overrightarrow{t}^j \right\}_{j=0}^n$ *is linearly dependent and* $\dim \mathcal{T}^n = m$.

Proof. We consider $\sum_{j=0}^n \lambda_j \overrightarrow{t}^j = 0$. But the Vandermonde matrix

$$A_{m,n+1} = \begin{pmatrix} \overrightarrow{t}^0 & \overrightarrow{t}^1 & \cdots & \overrightarrow{t}^n \end{pmatrix}$$

is of rank $n + 1$ as long as $n < m$, and hence $\lambda_j = 0$ for $j = 0, ..., n$. If $n \geq m$ its rank is m and there exits non zero solutions to the system. Hence the result follows because $\mathcal{T}^n \subseteq \mathbb{R}^m$. □

Remark 3.2. *For any positive integer* l, *since* $\overrightarrow{t}^{m+l} \in \mathcal{T}^{m-1} = \mathbb{R}^m$, *we have*

$$\overrightarrow{t}^{m+l} = \sum_{j=0}^{m-1} \lambda_j(l) \overrightarrow{t}^j,$$

where

$$\overrightarrow{\lambda}(l) = \begin{pmatrix} \lambda_0(l) \\ \lambda_1(l) \\ \vdots \\ \lambda_{m-1}(l) \end{pmatrix} = A^{-1}_{m,m} \overrightarrow{t}^{m+l} = A^{-1}_{m,m} \text{diag}\left(\overrightarrow{t}^m\right) \overrightarrow{t}^l,$$

and

$$\operatorname{diag}(\overrightarrow{t}^{m}) = \begin{pmatrix} t_1^m & 0 & \cdots & 0 \\ 0 & t_2^m & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & t_m^m \end{pmatrix}. \qquad \square$$

Theorem 3.3. *Let k be any integer such that $0 \le k \le m-1$, and let $\mathcal{V}_k^+$ and $\mathcal{V}_k^-$ be defined by* (3.2) *and* (3.3)*, then*

$$\mathcal{V}_k^+ = \mathcal{T}^k = \mathcal{V}_k^-$$

and

$$\dim \mathcal{V}_k^+ = k+1 = \dim \mathcal{V}_k^-.$$

Proof. We prove the result for $\mathcal{V}_k^+$ only, the proof for $\mathcal{V}_k^-$ is identical. Since

$$\begin{aligned} \sum_{i=1}^{m} \mu_i \overrightarrow{v}_{k,i}^{+} &= \sum_{i=1}^{m} \mu_i \Big(\sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} t_i^j \overrightarrow{t}^{k-j} \Big) \\ &= \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \Big(\sum_{i=1}^{m} \mu_i t_i^j \Big) \overrightarrow{t}^{k-j}, \end{aligned}$$

then $\sum_{i=1}^{m} \mu_i \overrightarrow{v}_{k,i}^{+} = 0$ if and only if $\sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \Big(\sum_{i=1}^{m} \mu_i t_i^j \Big) \overrightarrow{t}^{k-j} = 0$. From Theorem 3.1, the set $\{\overrightarrow{t}^{k-j}\}_{j=0}^{k}$ is linearly independent for $k < m$, it follows that $\sum_{i=1}^{m} \mu_i t_i^j = 0$ for $j = 0, ..., k$. But this system of $k+1$ equations and m unknowns has a unique solution only for $k = m-1$. Moreover the matrix associated to this system is $A_{m,k+1}^T$ is of rank $k+1$ for $k < m$. Hence

$$\dim \mathcal{V}_k^+ = k+1. \qquad \square$$

For $k \ge m$ we have no clear result about the dimension of $\mathcal{V}_k^-$ and $\mathcal{V}_k^+$ as illustrated by the following example for $m = 3$.

Example 3.4. *Let $m = 3$.*

(a) *For $\mathcal{V}_k^-$, since we have*

$$\begin{aligned} \operatorname{Det}(\overrightarrow{v}_{k,1}^{-}, \overrightarrow{v}_{k,2}^{-}, \overrightarrow{v}_{k,3}^{-}) &= \begin{vmatrix} 0 & (t_1-t_2)^k & (t_1-t_3)^k \\ (t_2-t_1)^k & 0 & (t_2-t_3)^k \\ (t_3-t_1)^k & (t_3-t_2)^k & 0 \end{vmatrix} \\ &= [1+(-1)^k](t_1-t_2)^k(t_2-t_3)^k(t_3-t_1)^k \\ &= \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2(t_1-t_2)^k(t_2-t_3)^k(t_3-t_1)^k & \text{if } k \text{ is even,} \end{cases} \end{aligned}$$

it follows that

$$\dim \mathcal{V}_k^- = \begin{cases} 2 & \text{if } k \text{ is odd,} \\ 3 & \text{if } k \text{ is even.} \end{cases}$$

(b) *For $\mathcal{V}_k^+$, we have*

$$\mathrm{Det}(\overrightarrow{v}^{+}_{k,1}, \overrightarrow{v}^{+}_{k,2}, \overrightarrow{v}^{+}_{k,3}) = \begin{vmatrix} (2t_1)^k & (t_1+t_2)^k & (t_1+t_3)^k \\ (t_2+t_1)^k & (2t_2)^k & (t_2+t_3)^k \\ (t_3+t_1)^k & (t_3+t_2)^k & (2t_3)^k \end{vmatrix}$$

$$= (8t_1t_2t_3)^k + 2(t_1+t_2)^k(t_2+t_3)^k(t_3+t_1)^k - 2^k[t_1^k(t_2+t_3)^{2k} + t_2^k(t_3+t_1)^{2k} + t_3^k(t_1+t_2)^{2k}].$$

This determinant can be 0*. Indeed for* $t_1 + t_3 = 0$ *and* $t_2 = 0$ *the determinant is* 0 *for odd* k*. It follows that* $\dim \mathcal{V}_k^+$ *is* 2 *or* 3 *depending on the values of* t_1*,* t_2 *and* t_3*.* □

Remark 3.5. *In* [4] *it is asserted that* $\mathcal{V}_2^-$ *is of dimension* m *which is clearly false except for* $m = 3$*. As a consequence the proof given in* [4] *for the existence of a WLSE for a* 3*-parametric exponential function is not correct. There are also errors in the proof of the existence of a WLSE in* [5]*.* □

3.2. **Polynomial weighted least squares fitting.** We introduce the scalar product on $\mathbb{R}^m$ defined by

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = \sum_{i=1}^{m} \omega_i u_i v_i \ ,$$

for any pair of vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^m$

$$\overrightarrow{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \quad \text{and} \quad \overrightarrow{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}.$$

The norm on $\mathbb{R}^m$ induced by the scalar product is $\|\overrightarrow{u}\| = \langle \overrightarrow{u}, \overrightarrow{u} \rangle^{1/2}$. Then (1.1) becomes

$$F(p_n) = \|\overrightarrow{f} - \overrightarrow{p}_n\|^2,$$

where

$$\overrightarrow{p}_n = \sum_{j=0}^{n} \alpha_j \overrightarrow{t}^{\,j}, \qquad \overrightarrow{t}^{\,j} = \begin{pmatrix} t_1^j \\ t_2^j \\ \vdots \\ t_m^j \end{pmatrix}, \quad \text{and} \quad \overrightarrow{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}.$$

The problem is to find the orthogonal projection of $\overrightarrow{f}$ on $\mathcal{T}^n$. This projection is completely characterized by the normal equations

$$\langle \overrightarrow{f} - \overrightarrow{p}^{*}_n, \overrightarrow{p}_n \rangle = 0$$

for all $\overrightarrow{p}_n \in \mathcal{T}^n$.

Again, to simplify the computation of p_n^*, we can determine an orthogonal basis $\{\overrightarrow{q}_j\}_{j=0}^n$ for $\mathcal{T}^n$ by applying the Gram-Schmidt process to its basis $\{\overrightarrow{t}^{\,j}\}_{j=0}^n$. We obtain

$$\overrightarrow{q}_0 = \overrightarrow{1}, \quad \overrightarrow{q}_1 = \overrightarrow{t} - \alpha_1 \overrightarrow{1},$$

and for $j = 2, ..., n$,

$$\overrightarrow{q}_j = (\overrightarrow{t} - \alpha_j \overrightarrow{1}) \cdot \overrightarrow{q}_{j-1} - \beta_j \overrightarrow{q}_{j-2}$$

where

$$\alpha_j = \frac{\langle \overrightarrow{t} \cdot \overrightarrow{q}_{j-1}, \overrightarrow{q}_{j-1} \rangle}{\langle \overrightarrow{q}_{j-1}, \overrightarrow{q}_{j-1} \rangle} \quad (j = 1, 2, 3, \ldots),$$

and

$$\beta_j = \frac{\langle \overrightarrow{t} \cdot \overrightarrow{q}_{j-1}, \overrightarrow{q}_{j-2} \rangle}{\langle \overrightarrow{q}_{j-2}, \overrightarrow{q}_{j-2} \rangle} \quad (j = 2, 3, 4, \ldots).$$

In these identities, $\overrightarrow{u} \cdot \overrightarrow{v}$ is the coordinatewise multiplication of two vectors of $\mathbb{R}^m$ defined by

$$\overrightarrow{u} \cdot \overrightarrow{v} = \begin{pmatrix} u_1 v_1 \\ u_2 v_2 \\ \vdots \\ u_m v_m \end{pmatrix}.$$

Let us observe that $\overrightarrow{q}_j \in \mathcal{T}^j$ for $j = 0, ..., n$.

It follows that the projection is given by

$$\overrightarrow{p}_n^* = \sum_{j=0}^{n} \gamma_j^* \overrightarrow{q}_j \tag{3.4}$$

where

$$\gamma_j^* = \frac{\langle \overrightarrow{f}, \overrightarrow{q}_j \rangle}{\langle \overrightarrow{q}_j, \overrightarrow{q}_j \rangle} \qquad (j = 0, 1, \ldots, n).$$

The next theorem is equivalent to Theorem 2.3.

Theorem 3.6. $\langle \overrightarrow{f} - \overrightarrow{p}_{n-1}^*, \overrightarrow{t}^n \rangle = \gamma_n^* \|\overrightarrow{q}_n\|^2$ *for* $n = 0, \ldots, m-1$.

Proof. For $n = 0$ we have $\overrightarrow{p}_{n-1}^* = \overrightarrow{0}$ and the result follows. For $n > 0$, since $\overrightarrow{q}_n = \overrightarrow{t}^n + \overrightarrow{p}_{n-1}$ where $\overrightarrow{p}_{n-1}$ is a vector in $\mathcal{T}^{n-1}$, and

$$\overrightarrow{p}_n^* = \gamma_n^* \overrightarrow{q}_n + \overrightarrow{p}_{n-1}^*,$$

we have

$$\begin{aligned} \gamma_n^* \|\overrightarrow{q}_n\|^2 &= \langle \gamma_n^* \overrightarrow{q}_n, \overrightarrow{q}_n \rangle \\ &= \langle \overrightarrow{p}_n^* - \overrightarrow{p}_{n-1}^*, \overrightarrow{q}_n \rangle \\ &= \langle \overrightarrow{p}_n^* - \overrightarrow{f}, \overrightarrow{q}_n \rangle + \langle \overrightarrow{f} - \overrightarrow{p}_{n-1}^*, \overrightarrow{q}_n \rangle \\ &= \langle \overrightarrow{f} - \overrightarrow{p}_{n-1}^*, \overrightarrow{t}^n + \overrightarrow{p}_{n-1} \rangle \\ &= \langle \overrightarrow{f} - \overrightarrow{p}_{n-1}^*, \overrightarrow{t}^n \rangle. \end{aligned}$$

□

4. Classification of data

Let $\{(\omega_i, t_i, f_i)\}_{i=1}^m$ be a set of m data points. If we use a discrete least squares polynomial to fit the data with the orthogonal basis $\left\{q_j\right\}_{j=0}^n$, the coefficients of p_n^* with respect to its expansion (2.3) or (3.4) suggest the following classification of the data.

Definition 4.1. *The data* $\{(\omega_i, t_i, f_i)\}_{i=1}^m$ *are said to be :*
(i) *essentially stationary if* $\gamma_1^* = 0$*;*
(ii) *essentially increasing, respectively decreasing, if* $\gamma_1^* > 0$*, respectively* $\gamma_1^* < 0$*;*
(iii) *essentially linear if* $\gamma_2^* = 0$*;*
(iv) *essentially convex, respectively concave, if* $\gamma_2^* > 0$*, respectively* $\gamma_2^* < 0$. □

Let us note that we could continue the classification with the higher order coefficients γ_n^* for $n = 3, ..., m-1$.

Finally if we apply symmetric transformations to the data we obtain the following result.

Theorem 4.2. *Effect of symmetric transformations on the data.*
(a) *If the data* $\{(\omega_i, t_i, f_i)\}_{i=1}^m$ *are essentially increasing, resp. decreasing, then the data* $\{(\omega_i, -t_i, f_i)\}_{i=1}^m$ *are essentially decreasing, resp. increasing. The stationarity, linearity, and concavity or convexity properties are not modified by this transform.*
(b) *If the data* $\{(\omega_i, t_i, f_i)\}_{i=1}^m$ *are essentially increasing, resp. decreasing, and essentially convex, resp. concave, then the data* $\{(\omega_i, t_i, -f_i)\}_{i=1}^m$ *are essentially decreasing, resp. increasing, and essentially concave, resp. convex. The stationarity and linearity properties are not modified by this transform.* □

5. Conclusion

We have revisited the polynomial weighted least squares analysis. Doing so we have specified the dimension of three vector subspaces of $\mathcal{P}$ (Theorem 2.1 and Theorem 2.2) and of $\mathbb{R}^m$ (Theorem 3.1 and Theorem 3.3) used for solving this problem . We also have established a property (Theorem 2.3 and Theorem 3.6) and suggested a classification of data (Definition 4.1) which will play a role in finding sufficient conditions for the existence of a WLSE for a 3-parametric exponential model [2].

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References

[1] S.D. Conte and C. de Boor, 1980. Elementary Numerical Analysis: An Algorithmic Approach, McGraw-Hill, New York.
[2] F. Dubeau and Y. Mir, 2007. Existence of optimal weighted least squares estimate for 3-parametric exponential model, Technical Report, Mathematics Department, Université de Sherbrooke, Sherbrooke (Qc), Canada.
[3] G.E. Forsythe, 1956. Generation and use of orthogonal polynomials for data-fitting with digital computer, J. SIAM, 5, 74-78.
[4] D. Jukić and R. Scitovski, 2000. The best least squares approximation problem for a 3-parametric exponential regression model, ANZIAM J., 42, 254-266.
[5] D. Jukić, 2004 A necessary and sufficient criteria for the existence of the least squares estimate for a 3-parametric exponential function, Applied Mathematics and Computation, 147, 1-17.
[6] L.F. Shampine, 1975. Discrete least squares polynomial fits, Communications of the ACM, 18, 179-180.
[7] L.F. Shampine and R.C. Allen, 1973. Numerical Computing: An introduction, Saunders, Philadelphia.