Newton's method and high-order algorithms for the n-th root computation

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Abstract. We analyze two modifications of the Newton's method to accelerate the convergence of the n-th root computation of a strictly positive real number. From these modifications, we can define new algorithms with prefixed order of convergence $p \in \mathbb{N}$, $p \geq 2$. Moreover an affine combination of the two modified methods, depending on one parameter, leads to a family of methods of order p, but of order $p+1$ for a specific value of the parameter.

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1. Introduction

The computation of the n-th root $r^{1/n}$ of a strictly positive real number r is an old problem [1, 14]. Recently several authors have suggested high-order methods for the computation of $r^{1/2}$. In [10] and [18], continued fraction expansions are used to derive such methods. In [12], methods similar to those presented in [18] are obtained as a special case of a determinantal family of root-finding methods [11]. Also, methods based on a modification of the Newton's method are obtained in [9]. For the computation of the n-th root $r^{1/n}$, general high-order methods can be derived from the application of the Newton's method to an appropriate modified function [3]. Also third and fourth order methods are presented in [8]. Finally, using combinations of basic functions identified for methods proposed in [3] and [9], new high-order methods are derived for the computation of $r^{1/2}$ in [15] .

In this paper we consider extensions of the Newton's method applied to the function

$$f(x) = x^n - r \tag{1.1}$$

to find the n-th root of r. In order to accelerate the convergence, we consider the following two possibilities. As suggested in [6], [4], the first possibility use a modification of the function $f(x)$ and then apply the Newton's method on the modified function. In fact we consider $F(x) = g(x)f(x)$ and then consider the iteration

$$x_{k+1} = \Phi_0(x_k) = x_k - \frac{F(x_k)}{F^{(1)}(x_k)}. \tag{1.2}$$

This approach, used in [3], is summarized in Section 2. The second possibility is to modify the Newton's method by changing the step size of the correction when we apply it to $f(x)$. Hence we look for a good choice of $G(x)$ and we consider the iteration

$$x_{k+1} = \Phi_1(x_k) = x_k - G(x_k)\frac{f(x_k)}{f^{(1)}(x_k)}. \tag{1.3}$$

This approach was used in [9] for the square-root computation and is extended in Section 3 to the computation of the n-th root. Finally in Section 4 we consider combinations of the p-th order methods obtained in Sections 2 and 3 of the form

$$x_{k+1} = \Phi_\lambda(x_k) = (1-\lambda)\Phi_0(x_k) + \lambda\Phi_1(x_k). \tag{1.4}$$

We obtain new p-order methods for any values of the parameter λ except for a specific value for which the new method is of order $p+1$.

2. Newton's method on $F(x) = g(x)f(x)$.

To get high order method for finding $r^{1/n}$, we try to find a function $g_p(x)$ such that

$$F_p(x) = g_p(x)f(x) = g_p(x)\left(x^n - r\right)$$

satisfies the assumptions of following result about Newton's method.

Theorem 2.1. [2],[6],[17] *Let p be an integer ≥ 2 and let $F_p(x)$ be a regular function such that $F_p(\alpha) = 0$, $F_p^{(1)}(\alpha) \neq 0$, $F_p^{(j)}(\alpha) = 0$ for $j = 2, \ldots, p-1$, and $F_p^{(p)}(\alpha) \neq 0$. Then the Newton's method applied to the equation $F_p(x) = 0$ generates a sequence $\{x_k\}_{k=0}^{+\infty}$ where*

$$x_{k+1} = \Phi_{0,p}(x_k) = x_k - \frac{F_p(x_k)}{F_p^{(1)}(x_k)} \qquad (k = 0, 1, 2, \ldots)$$

which converges to α for a given x_0 sufficiently close to α. Moreover, the convergence is of order p, and the asymptotic constant is

$$K_{0,p}(\alpha) = \lim_{k\to+\infty} \frac{x_{k+1}-\alpha}{(x_k-\alpha)^p} = \frac{p-1}{p!}\frac{F_p^{(p)}(\alpha)}{F_p^{(1)}(\alpha)}. \qquad \square$$

One such function $g_p(x)$ suggested in [3] is

$$g_p(x) = \sum_{i=1}^{p-1} a_i (x^n - r)^{i-1} = \sum_{i=1}^{p-1} \binom{1/n}{i} \frac{(x^n-r)^{i-1}}{r^i}. \tag{2.1}$$

where

$$\binom{1/n}{i} = \begin{cases} 1 & \text{for} \quad i = 0, \\ \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\cdots\left(\frac{1}{n}-(i-1)\right)}{i!} & \text{for} \quad i \geq 1. \end{cases} \tag{2.2}$$

We have the following result.

Theorem 2.2. [3] *Let n be an integer ≥ 2, p be an integer ≥ 2, and*

$$F_p(x) = \sum_{i=1}^{p-1} a_i (x^n - r)^i \tag{2.3}$$

where $a_i = \frac{1}{r^i}\binom{1/n}{i}$ *for* $i = 1, ..., p-1$. *Then*

$$
\begin{aligned}
&(2.4) & F_p(r^{1/n}) &= 0,\\
&(2.5) & F_p^{(1)}(r^{1/n}) &= r^{-1/n},\\
&(2.6) & F_p^{(j)}(r^{1/n}) &= 0 \qquad \textit{for} \quad j = 2, \ldots, p-1,\\
&(2.7) & F_p^{(p)}(r^{1/n}) &= -n^p p! \binom{1/n}{p} r^{-\frac{p}{n}},\\
&(2.8) & F_p^{(p+1)}(r^{1/n}) &= (p+2-np)\frac{p-1}{2} n^p p! \binom{1/n}{p} r^{-\frac{p+1}{n}}.
\end{aligned}
$$

Proof. We use the identity

$$Q_p(y) = F_p((y+r)^{1/n}) = \sum_{i=1}^{p-1} a_i y^i \tag{2.9}$$

to find $F_p^{(j)}(r^{1/n})$ for $j = 0, ..., p+1$. It is clear that $F_p(r^{1/n}) = 0$. For $j \geq 1$, we have

$$F_p^{(j)}((y+r)^{1/n}) = \sum_{k=1}^{j} F_p^{(k)}((y+r)^{1/n}) G_{k,j}(y),$$

where the $G_{k,j}(y)$ are regular functions of y near 0. More precisely, we have

$$
\begin{aligned}
G_{1,j}(y) &= \frac{d^j}{dy^j}(y+r)^{1/n},\\
G_{j-1,j}(y) &= \gamma_j \left[\frac{d}{dy}(y+r)^{1/n}\right]^{j-2} \frac{d^2}{dy^2}(y+r)^{1/n},\\
G_{j,j}(y) &= \left[\frac{d}{dy}(y+r)^{1/n}\right]^{j},
\end{aligned}
$$

where $\gamma_1 = 0$ and $\gamma_{j+1} = \gamma_j + j = \frac{j(j+1)}{2}$ for $j \geq 2$. Since

$$Q_p^{(j)}(y) = \begin{cases} \sum_{i=j}^{p-1} a_i \frac{i!}{(i-j)!} y^{i-j} & \text{for } j = 1, ..., p-1 \\ 0 & \text{for } j \geq p, \end{cases}$$

We obtain the result by setting $y = 0$ recursively for $j = 1, ..., p+1$. □

It follows that the sequence $\{x_k\}_{k=0}^{+\infty}$ generated by : x_0 given sufficiently close to $r^{1/n}$, and

$$
\begin{aligned}
&(2.10) & x_{k+1} &= \Phi_{0,p}(x_k)\\
&(2.11) & &= x_k - \frac{F_p(x_k)}{F_p^{(1)}(x_k)}\\
&(2.12) & &= x_k - \frac{(x_k^n - r)\sum_{i=1}^{p-1}\binom{1/n}{i}\left(\frac{x_k^n - r}{r}\right)^{i-1}}{n x_k^{n-1}\sum_{i=1}^{p-1} i \binom{1/n}{i}\left(\frac{x_k^n - r}{r}\right)^{i-1}}
\end{aligned}
$$

for $k = 0, 1, 2, \ldots,$ converges to $r^{1/n}$. Moreover, the convergence is of order p, and the asymptotic constant is

$$K_{0,p}(r^{1/n}) = \lim_{k\to+\infty} \frac{x_{k+1} - r^{1/n}}{(x_k - r^{1/n})^p} = -(p-1)n^p \binom{1/n}{p} r^{-\frac{p-1}{n}}. \tag{2.13}$$

3. Modified step size for the Newton's method applied on $f(x)$.

To get high order method for finding $r^{1/n}$, we try to find a function $G(x)$ such that the modified Newton's method given by

$$x_{k+1} = x_k - G(x_k)\frac{f(x_k)}{f^{(1)}(x_k)}$$

satisfies the assumptions of following result about fixed-point method.

Theorem 3.1. [13], [16] *Let p be an integer ≥ 2 and let $\Phi(x)$ be a regular function such that $\Phi(\alpha) = \alpha$, $\Phi^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-1$, and $\Phi^{(p)}(\alpha) \neq 0$. Then the sequence $\{x_k\}_{k=0}^{+\infty}$ generated by $x_{k+1} = \Phi(x_k)$ for $k = 0, 1, 2, \ldots,$ converges to α for a given x_0 sufficiently close to α. Moreover, the convergence is of order p, and the asymptotic constant is*

$$K_p(\alpha) = \lim_{k\to+\infty} \frac{x_{k+1} - \alpha}{(x_k - \alpha)^p} = \frac{\Phi^{(p)}(\alpha)}{p!}. \qquad \square$$

To determine $G(x)$ we start from the Taylor's expansion of $f(x) = x^n - r$ around x evaluated at $r^{1/n}$

$$0 = f(r^{1/n}) = f(x) + \sum_{i=1}^{n} \frac{f^{(i)}(x)}{i!}(r^{1/n} - x)^i. \tag{3.1}$$

For any x we look for a correction Δx such that $x + \Delta x = r^{1/n}$ of the form

$$\Delta x = r^{1/n} - x = -G(x)\frac{f(x)}{f^{(1)}(x)}. \tag{3.2}$$

Using (3.1) and (3.2), we get

$$0 = f(x) + \sum_{i=1}^{n} \binom{n}{i} x^{n-i}\Big(-G(x)\frac{f(x)}{f^{(1)}(x)}\Big)^i$$

because

$$f^{(i)}(x) = \frac{n!}{(n-i)!}x^{n-i}$$

for $i = 1, 2, \ldots, n$. Following [5], we define the two functions

$$u(x) = \frac{f(x)}{f^{(1)}(x)} \tag{3.3}$$

and

$$t(x) = \frac{f(x)f^{(2)}(x)}{(f^{(1)}(x))^2}. \tag{3.4}$$

These functions verify $u^{(1)}(x) = 1 - t(x)$. Also because the special form of $f(x)$, we also have

$$u(x) = \frac{x}{n-1}t(x), \tag{3.5}$$

$$t(x) = \frac{n-1}{n}\Big(1 - \frac{r}{x^n}\Big), \tag{3.6}$$

and

$$t^{(1)}(x) = \frac{n-1}{x}\Big(1 - \frac{n}{n-1}t(x)\Big). \tag{3.7}$$

Using (3.5), we get

$$\begin{aligned} 0 &= -r + x^n \sum_{i=0}^{n} \binom{n}{i} \Big(- G(x)\frac{t(x)}{n-1}\Big)^i . \\ 0 &= -r + x^n\Big(1 - G(x)\frac{t(x)}{n-1}\Big)^n . \end{aligned}$$

It follows from (3.6) that

$$\Big(1 - G(x)\frac{t(x)}{n-1}\Big)^n = \frac{r}{x^n} = 1 - \frac{n}{n-1}t(x),$$

and

$$G(x) = \frac{1 - \Big(1 - \frac{n}{n-1}t(x)\Big)^{1/n}}{\frac{1}{n-1}t(x)}.$$

Obviously it is a theoretical result because it requires the computation of the n-th root of $\Big(1 - \frac{n}{n-1}t(x)\Big)$. We can verify that with this $G(x)$, if we start with any x value we get α in one step because

$$\begin{aligned} x - G(x)u(x) &= x - \frac{1 - \Big(1 - \frac{n}{n-1}t(x)\Big)^{1/n}}{\frac{1}{n-1}t(x)}u(x) \\ &= x - x\Big(1 - \Big(1 - \frac{n}{n-1}t(x)\Big)^{1/n}\Big) \\ &= x\Big(1 - \frac{n}{n-1}t(x)\Big)^{1/n} \\ &= x\Big(\frac{r}{x^n}\Big)^{1/n} \\ &= r^{1/n}. \end{aligned}$$

To obtain a numerical method we use the MacLaurin's expansion of $(1+x)^{1/n}$

$$(1+x)^{1/n} = \sum_{i=0}^{p-1} \binom{1/n}{i} x^i + \binom{1/n}{p} (1 + \theta(x)x)^{\frac{1}{n}-p} x^p$$

where $\theta(x) \in (0,1)$. For $p \geq 2$ we take

$$G_p(x) = \frac{1 - \sum_{i=0}^{p-1} \binom{1/n}{i} \big(- \frac{n}{n-1}t(x)\big)^i}{\frac{1}{n-1}t(x)}, \tag{3.8}$$

and define

$$\Phi_{1,p}(x) = x - G_p(x)u(x) = x - G_p(x)\frac{x}{n-1}t(x). \tag{3.9}$$

Then we have the following result.

Theorem 3.2. *Let $f(x)$ and $t(x)$ given by (1.1) and (3.4). Let n be an integer ≥ 2, p be an integer ≥ 2, and let $\Phi_{1,p}(x)$ be defined by*

$$\Phi_{1,p}(x) = x \sum_{i=0}^{p-1} \binom{1/n}{i} \Big(-\frac{n}{n-1} t(x) \Big)^i. \tag{3.10}$$

Then

$$\Phi_{1,p}(r^{1/n}) = r^{1/n}, \tag{3.11}$$

$$\Phi_{1,p}^{(j)}(r^{1/n}) = 0 \quad \text{for} \quad j = 1, ..., p-1, \tag{3.12}$$

$$\Phi_{1,p}^{(p)}(r^{1/n}) = (-1)^p n^p p! \binom{1/n}{p} r^{-\frac{p-1}{n}}, \tag{3.13}$$

$$\Phi_{1,p+1}^{(p+1)}(r^{1/n}) = (-1)^p \frac{p(p-1)}{2} n^p (n+1) p! \binom{1/n}{p} r^{-p/n}. \tag{3.14}$$

Proof. Since $t(r^{1/n}) = 0$, we have $\Phi_{1,p}(r^{1/n}) = r^{1/n}$. Also, using (3.7) and the identity

$$\binom{1/n}{i+1} = \binom{1/n}{i} \frac{\frac{1}{n} - i}{i+1}, \tag{3.15}$$

we get

$$\begin{aligned}
\Phi_{1,p}^{(1)}(x) &= \sum_{i=0}^{p-1} \binom{1/n}{i} \Big(-\frac{n}{n-1} t(x) \Big)^i \\
&\quad + x \sum_{i=1}^{p-1} \binom{1/n}{i} i \Big(-\frac{n}{n-1} t(x) \Big)^{i-1} \Big(-\frac{n}{n-1} t^{(1)}(x) \Big) \\
&= \sum_{i=1}^{p-2} \Big[\binom{1/n}{i} (1 - ni) - \binom{1/n}{i+1} n(i+1) \Big] \Big(-\frac{n}{n-1} t(x) \Big)^i \\
&\quad + \binom{1/n}{p-1} (1 - n(p-1)) \Big(-\frac{n}{n-1} t(x) \Big)^{p-1} \\
&= np \binom{1/n}{p} \Big(-\frac{n}{n-1} t(x) \Big)^{p-1}.
\end{aligned}$$

Hence, because $t(r^{1/n}) = 0$ and $p \geq 2$, it follows that $\Phi_{1,p}^{(1)}(r^{1/n}) = 0$. For $j = 2, ..., p$, we have

$$\begin{aligned}
\Phi_{1,p}^{(j)}(x) &= \Big(t(x) \Big)^{p-(j-2)} g_j(x) \\
&+ (-1)^{p-1} \frac{(j-1)(j-2)}{2} \frac{n^p}{(n-1)^{p-1}} \frac{p!}{(p-(j-1))!} \binom{1/n}{p} \Big(t(x) \Big)^{p-(j-1)} \Big(t^{(1)}(x) \Big)^{j-3} t^{(2)}(x) \\
&+ (-1)^{p-1} \frac{n^p}{(n-1)^{p-1}} \frac{p!}{(p-j)!} \binom{1/n}{p} \Big(t(x) \Big)^{p-j} \Big(t^{(1)}(x) \Big)^{j-1}
\end{aligned} \tag{3.16}$$

where the $g_j(x)$ are regular functions of x near $r^{1/n}$. It follows that $\Phi_{1p}^{(j)}(r^{1/n}) = 0$ for $j = 2, ..., p-1$, and

$$\Phi_{1,p}^{(p)}(r^{1/n}) = (-1)^{p-1} n^p p! \binom{1/n}{p} r^{-\frac{p-1}{n}}. \tag{3.17}$$

Finally, taking the derivative of (3.16) for $j = p$, we obtain

$$\Phi_{1,p}^{(p+1)}(x) = t(x)g_{p+1}(x) + (-1)^{p-1}\frac{p(p-1)}{2}\frac{n^p}{(n-1)^{p-1}}p!\binom{1/n}{p}\Big(t^{(1)}(x)\Big)^{p-2}t^{(2)}(x)$$

where the $g_{p+1}(x)$ is regular functions of x near $r^{1/n}$. The result follows for $\Phi_{1,p}^{(p+1)}(r^{1/n})$ because $t^{(1)}(r^{1/n}) = (n-1)r^{-1/n}$ and $t^{(2)}(r^{1/n}) = -(n-1)(n+1)r^{-2/n}$. □

Then the sequence $\{x_k\}_{k=0}^{+\infty}$ generated by : x_0 given sufficiently close to $r^{1/n}$, and

$$x_{k+1} = \Phi_{1,p}(x_k) \tag{3.18}$$

$$= x_k \sum_{i=0}^{p-1}\binom{1/n}{i}\Big(-\frac{n}{n-1}t(x_k)\Big)^i \tag{3.19}$$

for $k = 0, 1, 2, \ldots$, converges to $r^{1/n}$. Moreover, the convergence is of order p, and the asymptotic constant is

$$K_{1,p}(r^{1/n}) = \frac{\Phi_{1,p}^{(p)}(\alpha)}{p!} = (-1)^{p-1}n^p\binom{1/n}{p}r^{-\frac{p-1}{n}}. \tag{3.20}$$

Moreover if $x_k > r^{1/n}$ we have

$$\begin{aligned} x_{k+1} - r^{1/n} &= \Phi_{1,p}(x_k) - \Phi_{1p}(r^{1/n}) \\ &= \Phi_{1,p}^{(1)}(\xi) \\ &= (-1)^{p-1}\frac{n^p}{(n-1)^{p-1}}\binom{1/n}{p}\Big(t(\xi)\Big)^{p-1} \\ &> 0, \end{aligned}$$

since $\xi \in (r^{1/n}, x_k)$, $t(\xi) = \frac{n-1}{\xi}u(\xi) > 0$ and $(-1)^{p-1}\binom{1/n}{p} > 0$. Also

$$\begin{aligned} x_{k+1} - x_k &= x_k\sum_{i=1}^{p-1}\binom{1/n}{i}\Big(-\frac{n}{n-1}t(x_k)\Big)^i \\ &= x_k\sum_{i=1}^{p-1}(-1)^i\Big(\frac{n}{n-1}\Big)^i\binom{1/n}{i}\Big(t(x_k)\Big)^i \\ &< 0, \end{aligned}$$

since $t(x_k) = \frac{n-1}{x_k}u(x_k) > 0$ and $(-1)^i\binom{1/n}{i} < 0$. Then for any $x_0 > r^{1/n}$, the sequence $\{x_k\}_{k=0}^{+\infty}$ monotonically decreases and converges to $r^{1/n}$.

4. Higher order methods.

In the preceding two sections we have obtained two families of methods for the computation of $r^{1/n}$. In this section we combine the two families to get new higher order methods.

We start with two p-order methods

$$x_{k+1} = \Phi_{0,p}(x_k) = x_k - \frac{(x_k^n - r)\sum_{i=1}^{p-1}\binom{1/n}{i}\left(\frac{x_k^n - r}{r}\right)^{i-1}}{n x_k^{n-1}\sum_{i=1}^{p-1} i\binom{1/n}{i}\left(\frac{x_k^n - r}{r}\right)^{i-1}},$$

and

$$x_{k+1} = \Phi_{1,p}(x_k) = x_k \sum_{i=0}^{p-1}\binom{1/n}{i}\left(-\frac{n}{n-1}t(x_k)\right)^i,$$

and consider an affine combination

$$\Phi_{\lambda,p}(x) = (1-\lambda)\Phi_{0,p}(x) + \lambda\Phi_{1,p}(x).$$

This method converges to $r^{1/n}$ if x_0 is given sufficiently close to $r^{1/n}$ because $\Phi_{\lambda,p}(r^{1/n}) = r^{1/n}$ and

$$\Phi^{(1)}_{\lambda,p}(r^{1/n}) = (1-\lambda)\Phi^{(1)}_{0,p}(r^{1/n}) + \lambda\Phi^{(1)}_{1,p}(r^{1/n}) = 0.$$

We obtain a method of order p with an asymptotic constant given by

$$K_{\lambda,p}(r^{1/n}) = (1-\lambda)K_{0,p}(r^{1/n}) + \lambda K_{1,p}(r^{1/n}).$$

From (2.13) and (3.20), we get

$$K_{\lambda,p}(r^{1/n}) = \Big[(-1)^p(p-1) + \lambda(1-(-1)^p(p-1))\Big](-1)^{p-1}n^p\binom{1/n}{p-1}r^{-\frac{p-1}{n}}.$$

This asymptotic constant can be made arbitrary small for $p > 2$, and is 0 for

$$\lambda = \lambda_p = \frac{(-1)^{p+1}(p-1)}{1+(-1)^{p+1}(p-1)}.$$

Using Taylor expansions for $F_p(x_{k+1})$, $F_p^{(1)}(x_{k+1})$, and $\Phi^{(1)}_{1,p}(x_{k+1})$ around $r^{1/n}$, we have

$$\begin{aligned}
x_{k+1} - r^{1/n} &= \Phi_{\lambda,p}(x_k) - \Phi_{\lambda,p}(r^{1/n}) \\
&= (1-\lambda)\Big[(x_k - r^{1/n}) - \frac{F_p(x_k)}{F_p^{(1)}(x_k)}\Big] + \lambda\Big[\Phi_{1,p}(x_k) - \Phi_{1,p}(r^{1/n})\Big] \\
&= \Big[(1-\lambda)\frac{p-1}{p!}\frac{F_p^{(p)}(r^{1/n})}{F_p^{(1)}(r^{1/n})} + \lambda\frac{\Phi^{(p)}_{1,p}(r^{1/n})}{p!}\Big](x_k - r^{1/n})^p \\
&\quad + (1-\lambda)F_p^{(p)}(r^{1/n})\frac{p-1}{p!}\Big[\frac{1}{F_p^{(1)}(x_k)} - \frac{1}{F_p^{(1)}(r^{1/n})}\Big](x_k - r^{1/n})^p \\
&\quad + \Big[(1-\lambda)\frac{(p+1)F_p^{(p+1)}(\xi') - F_p^{(p+1)}(\xi'')}{(p+1)!F_p^{(1)}(x_k)} + \lambda\frac{\Phi^{(p+1)}_{1,p}(\xi)}{(p+1)!}\Big](x_k - r^{1/n})^{p+1} \\
&= \Big[(1-\lambda)K_{0,p}(r^{1/n}) + \lambda K_{1,p}(r^{1/n})\Big](x_k - r^{1/n})^p \\
&\quad + (1-\lambda)F_p^{(p)}(r^{1/n})\frac{p-1}{p!}\Big[\frac{1}{F_p^{(1)}(x_k)} - \frac{1}{F_p^{(1)}(r^{1/n})}\Big](x_k - r^{1/n})^p \\
&\quad + \Big[(1-\lambda)\frac{(p+1)F_p^{(p+1)}(\xi') - F_p^{(p+1)}(\xi'')}{(p+1)!F_p^{(1)}(x_k)} + \lambda\frac{\Phi^{(p+1)}_{1,p}(\xi)}{(p+1)!}\Big](x_k - r^{1/n})^{p+1}
\end{aligned}$$

where ξ, ξ' and ξ'' are between x_k and $r^{1/n}$.

For $\lambda = \lambda_p$, it follows that the method is of order $p+1$ and its asymptotic constant is

$$K_{\lambda_p,p+1}(r^{1/n}) = \Big[(1-\lambda_p)\frac{pF_p^{(p+1)}(r^{1/n})}{(p+1)!F_p^{(1)}(r^{1/n})} + \lambda_p \frac{\Phi_{1,p}^{(p+1)}(r^{1/n})}{(p+1)!}\Big].$$

From (2.5), (2.8) and (3.14), we obtain

$$K_{\lambda_p,p+1}(r^{1/n}) = \Big[\frac{(3+n-2np)}{1+(-1)^{p+1}(p-1)}\Big]\frac{p(p-1)}{2(p+1)}n^p \binom{1/n}{p} r^{-p/n}.$$

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References

[1] D.F. Bateman, A historical survey of solution by functional iteration, *Math. Mag.*, 62 (1989), 155-166.
[2] J.E. Dennis and R.B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1983.
[3] F. Dubeau, Algorithms for n-th root approximation, *Computing*, 57 (1996), 365-369.
[4] W.F. Ford and J.A. Pennline, Accelerated convergence in Newton's method, *SIAM Review*, 38 (1996), 658-659.
[5] W. Gander, On Halley's iteration method, *Amer. Math. Month.*, 92 (1985), 131-134.
[6] J. Gerlach, Accelerated convergence in Newton's method, *SIAM Review*, 36 (1994), 272-276.
[7] A.A. Goldstein, *Constructive Real Analysis*, Harper and Row, N.Y., 1967.
[8] J.M. Gutierrez, M.A. Hernandez, and M.A. Solanova, Calculus of nth roots and third order iterative methods, *Nonlinear Analysis*, 47 (2001), 2875-2880.
[9] M.A. Hernandez and N. Romero, Accelerated convergence in Newton's method for approximating square roots, *Journal of Computational and Applied Mathematics*, 177 (2005), 225-229.
[10] K.R. Johnson, An iterative method for approximating square roots, *Math. Mag.*, 62 (1989), 253-259.
[11] B. Kalantari, On the order of convergence of determinantal family of root-finding methods, *BIT*, 39 (1999), 96-109.
[12] B. Kalantari and I. Kalantari, High order methods for approximating square roots, *BIT*, 36 (1996), 395-399.
[13] D. Kincaid and W. Cheney, *Numerical Analysis*, Brooks/Cole Pub. Co., California, 1991.
[14] R.J. Knill, A modified Babylonian algorithm, *Amer. Math. Month.*, 99 (1992), 734-737.
[15] A.Y.Özban, New methods for approximating square roots, *Applied mathematics and Computation*, 175 (2006), 532-540.
[16] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, 2nd Ed., Springer-Verlag, N.Y., 1993.
[17] J.F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1964.
[18] A.K. Yeyios, On two sequences of algorithms for approximating square roots, *Journal of Computational and Applied Mathematics*, 40 (1992), 63-72.