Skew group algebras of piecewise hereditary algebras are piecewise hereditary

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Abstract. The aim of this paper is twofold. First, we show that the main results of Happel-Rickard-Schofield (1988) and Happel-Reiten-Smalø (1996) on piecewise hereditary algebras are coherent with the notion of group action on an algebra. Then, we take advantage of this compatibility and show that if G is a finite group acting on a piecewise hereditary algebra A over an algebraically closed field whose characteristic does not divide the order of G, then the resulting skew group algebra $A[G]$ is also piecewise hereditary.

Let k be an algebraically closed field. For a finite dimensional k-algebra A, we denote by $\operatorname{mod} A$ the category of finite dimensional left A-modules, and by $D^b(A)$ the (triangulated) derived category of bounded complexes over $\operatorname{mod} A$ (in the sense of [**34**]). Let $\mathcal{H}$ be a connected hereditary abelian k-category which is moreover Ext-finite, that is having finite dimensional homomorphism and extension spaces. Following [**17**] (compare [**15, 19**]), we say that A is **piecewise hereditary of type** $\mathcal{H}$ if it is derived equivalent to $\mathcal{H}$, that is $D^b(A)$ is triangle-equivalent to the derived category $D^b(\mathcal{H})$ of bounded complexes over $\mathcal{H}$. Over the years, piecewise hereditary algebras have been widely investigated (see [**1, 4, 8, 24**] for instance) and proved to be related with many other topics, such as the simply connected algebras and the trivial extensions [**3, 6, 9, 21**], the self-injective algebras of polynomial growth [**2, 15, 32**] and the strong global dimension [**23**].

Hereditary categories $\mathcal{H}$ having tilting objects are of special interest in representation theory of algebras. These and the endomorphism algebras $\operatorname{End}_{\mathcal{H}} T$ of tilting objects T in $\mathcal{H}$, called **quasitilted** algebras, were introduced and studied in [**18**]. It is weel known that $\mathcal{H}$ and $\operatorname{End}_{\mathcal{H}} T$ are derived equivalent. Moreover, in the case where k is algebraically closed, it was shown by Happel [**16**] that $\mathcal{H}$ is either derived equivalent to a finite dimensional hereditary k-algebra H or derived equivalent a category of coherent sheaves $\operatorname{coh} \mathbb{X}$ on a weighted projective line $\mathbb{X}$ (in the sense of [**13**]).

On the other hand, an **action** of a group G on an algebra A is a group homomorphism from G to the group of automorphisms of A. For such an action, one defines the skew group algebra $A[G]$ (see Section 1.3). The study of the representation theory of skew group algebras was started in [**11, 29**], and pursued in [**5, 12, 33**]. We are mainly motivated by the fact that $A[G]$ retains many features from A, such as being hereditary [**29**] or quasitilted [**18**].

The aim of this paper is to marry these two concepts in order to study the skew group algebra $A[G]$, in case A is a piecewise hereditary algebra. Our main result

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(Theorem 3) shows that under some assumptions, the skew group algebra $A[G]$ is also piecewise hereditary.

An important part of our approach consists in showing that the characterizations of piecewise hereditary algebras of type $\operatorname{mod} H$, for some hereditary algebra H, and piecewise hereditary algebras of type $\operatorname{coh} \mathbb{X}$, for some weighted projective line $\mathbb{X}$, given in [**19**] and [**17**] respectively, are compatible with group actions. An appropriate use of that observation, combined with some preliminary results on the bounded derived categories of $\operatorname{mod} H$ and $\operatorname{coh}\mathbb{X}$, will prove to be sufficient to prove Theorem 3.

In order to give a clear statement of our main results, we need some additional terminology. Let G be a group and $\mathcal{A}$ be an additive category. An **action** of G on $\mathcal{A}$ is a group homomorphism $\theta : G \longrightarrow \operatorname{Aut} \mathcal{A}$ $(\sigma \longmapsto \theta_\sigma)$ from G to the group of automorphisms of $\mathcal{A}$. An object $\mathcal{M}$ in $\mathcal{A}$ is called G-**stable** with respect to θ, or briefly G-**stable** in case there is no ambiguity, if $\theta_\sigma \mathcal{M} \cong \mathcal{M}$ for all $\sigma \in G$. For such an object, the algebra $B = \operatorname{End}_{\mathcal{A}} \mathcal{M}$ inherits an action of G from θ, denoted θ_B (see (1.3.2)). Also, given another additive category $\mathcal{B}$ and an action $\vartheta : G \longrightarrow \operatorname{Aut} \mathcal{B}$ $(\sigma \longmapsto \vartheta_\sigma)$ of G on $\mathcal{B}$, a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is called G-**compatible**, with respect to the pair (θ, ϑ), if $F\theta_\sigma = \vartheta_\sigma F$ for every $\sigma \in G$.

Examples of particular interest occur when a group G acts on an artin algebra A as above. Then the action θ of G on A induces an action $\theta_{\operatorname{mod} A}$ of G on the additive category $\operatorname{mod} A$, and further an action $\theta_{D^b(A)}$ on $D^b(A)$ (see Sections 1.3 and 2.1).

This opens the way to our main results. Our first theorem deals with the case where $\mathcal{H} = \operatorname{mod} H$, for some hereditary algebra H, and stands as a generalization of the main result in [**19**].

Theorem 1. *Let A and H be k-algebras, with H hereditary, and G be a group. Let $\theta : G \longrightarrow \operatorname{Aut} A$ and $\vartheta : G \longrightarrow \operatorname{Aut} H$ be fixed actions of G on A and H.*

(a) *The following conditions are equivalent:*
  - (i) *There exists a G-compatible triangle-equivalence $E : D^b(H) \longrightarrow D^b(A)$ (with respect to the pair of induced actions $(\theta_{D^b(A)}, \vartheta_{D^b(H)})$);*
  - (ii) *There exist a sequence $H = A_0, A_1, \ldots, A_n = A$ of k-algebras and a sequence $T_0, T_1, \ldots, T_{n-1}$ of modules such that, for each i, $A_{i+1} = \operatorname{End}_{A_i} T_i$ and T_i is a G-stable tilting A_i-module (with respect to the induced action $\vartheta_{\operatorname{mod} A_i}$), and the induced action ϑ_{A_n} coincides with θ;*
  - (iii) *There exist a sequence $H = A_0, A_1, \ldots, A_n = A$ of k-algebras and a sequence $T_0, T_1, \ldots, T_{n-1}$ of modules such that, for each i, $A_{i+1} = \operatorname{End}_{A_i} T_i$ and T_i is a G-stable splitting tilting A_i-module (with respect to the induced action $\vartheta_{\operatorname{mod} A_i}$), and the induced action ϑ_{A_n} coincides with θ.*

(b) *If the above conditions are satisfied, and G is a finite group whose order is not a multiple of the characteristic of k, then $H[G]$ is hereditary and $A[G]$ is piecewise hereditary of type $\operatorname{mod} H[G]$.*

On the other hand, our second theorem concerns the case where $\mathcal{H} = \operatorname{coh}\mathbb{X}$, for some weighted projective line $\mathbb{X}$, and stands as a generalization of the main result in [**17**].

Theorem 2. *Let A be a k-algebra, $\mathcal{H} = \operatorname{coh}\mathbb{X}$ be the category of coherent sheaves on a weighted projective line $\mathbb{X}$ and G be a group. Let $\theta : G \longrightarrow \operatorname{Aut} A$ and $\vartheta : G \longrightarrow \operatorname{Aut} \mathcal{H}$ be fixed actions of G on A and $\mathcal{H}$.*

(a) *The following conditions are equivalent :*
  - (i) *There exists a G-compatible triangle-equivalence $E : D^b(\mathcal{H}) \longrightarrow D^b(A)$ (with respect to the pair of induced actions $(\theta_{D^b(A)}, \vartheta_{D^b(\mathcal{H})})$);*

(ii) *There exist a G-stable tilting object T in $\mathcal{H}$ and sequences* $\mathrm{End}_{\mathcal{H}} T = A_0, A_1, \dots, A_n = A$ *of k-algebras and $T_0, T_1, \dots, T_{n-1}$ of modules such that, for each i, $A_{i+1} = \mathrm{End}_{A_i} T_i$ and T_i is a G-stable tilting or cotilting A_i-module (with respect to the induced action $\vartheta_{\mathrm{mod}\, A_i}$), and the induced action ϑ_{A_n} coincides with θ;*

(iii) *There exist a G-stable tilting object T in $\mathcal{H}$ and sequences* $\mathrm{End}_{\mathcal{H}} T = A_0, A_1, \dots, A_n = A$ *of k-algebras and $T_0, T_1, \dots, T_{n-1}$ of modules such that, for each i, $A_{i+1} = \mathrm{End}_{A_i} T_i$ and T_i is a G-stable splitting tilting or cotilting A_i-module (with respect to the induced action $\vartheta_{\mathrm{mod}\, A_i}$), and the induced action ϑ_{A_n} coincides with θ;*

(b) *If the above conditions are satisfied and G is a finite group whose order is not a multiple of the characteristic of k, then the algebra* $(\mathrm{End}_{\mathcal{H}} T)[G]$, *where T is as in (ii) or in (iii), is quasitilted and derived equivalent to $A[G]$. In particular, $A[G]$ is piecewise hereditary.*

The equivalence of the conditions (a) of Theorem 1 and Theorem 2 were previously shown in [**19**] and [**17**] respectively in the case where, essentially, the actions θ and ϑ are the trivial actions, that is trivial homomorphisms of groups. Actually, our proofs are adaptations of the original ones.

In addition, it will become clear in Section 4 that if $\mathcal{H} = \mathrm{mod}\, H$ or $\mathcal{H} = \mathrm{coh}\mathbb{X}$, then any triangle-equivalence $E : D^b(\mathcal{H}) \longrightarrow D^b(A)$ can be converted into a G-compatible triangle-equivalence. As an application of this observation, together with Theorem 1, Theorem 2 and Happel's Theorem [**16**], we will obtain our main theorem.

Theorem 3. *Let A be a piecewise hereditary k-algebra of type $\mathcal{H}$, for some Ext-finite hereditary abelian k-category with tilting objects $\mathcal{H}$. Moreover, let G be a finite group whose order is not a multiple of the characteristic of k. Then $A[G]$ is piecewise hereditary. More precisely,*

(a) *If $\mathcal{H} = \mathrm{mod}\, H$, for some hereditary algebra H, then for any action of G on A, there exist a hereditary algebra H' and an action of G on H' such that A is piecewise hereditary of type* $\mathrm{mod}\, H'$ *and $A[G]$ is piecewise hereditary of type* $\mathrm{mod}\, H'[G]$.

(b) *If $\mathcal{H} = \mathrm{coh}\mathbb{X}$, for some category of coherent sheaves on a weighted projective line $\mathbb{X}$, then for any action of G on A there exist an action of G on $\mathcal{H}$ and a G-compatible triangle-equivalence $E : D^b(\mathcal{H}) \longrightarrow D^b(A)$. In particular, $A[G]$ is piecewise hereditary.*

As we shall see, the algebras H and H' in the statement (a) above generally differ from each other.

The paper is organized as follows. In Section 1, we fix the notations and terminologies, including brief reviews on tilting theory, bounded derived categories, piecewise hereditary algebras and skew group algebras. Most of the necessary background on weighted projective lines is however postponed to Section 4, since it is not explicitly needed until then. In Section 2, we show how an action of a group G on an additive category $\mathcal{A}$ induces an action of G on the homotopy and derived categories of $\mathcal{A}$. Then, we study the G-compatible triangle-equivalences of derived categories induced by the G-stable tilting modules. Section 3 is devoted to the proofs of Theorem 1 and Theorem 2. In Section 4, we prove Theorem 3. This first involves showing that any triangle-equivalence between $D^b(\mathcal{H})$ and $D^b(A)$ can be converted into a G-compatible equivalence when $\mathcal{H}$ is a module category over a hereditary algebra or a category of coherent sheaves on a weighted projective line. Finally, in Section 5, we give an illustrative example.

1. Preliminaries

In this paper, all considered algebras are finite dimensional algebras over an algebraically closed field k (and, unless otherwise specified, basic and connected). Moreover, all modules are finitely generated left modules. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated A-modules and by $\operatorname{proj} A$ a full subcategory of $\operatorname{mod} A$ consisting of one representative from each isomorphism class of indecomposable projective modules. Given an A-module T, we let $\operatorname{add} T$ be the full subcategory of $\operatorname{mod} A$ having as objects the direct sums of indecomposable direct summands of T. Also, the functor $D = \operatorname{Hom}_k(-, k)$ is the standard duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{op}$.

1.1. Tilting theory. Let A be an algebra. An A-module T is a **tilting module** if T has projective dimension at most one, $\operatorname{Ext}^1_A(T, T) = 0$ and there exists a short exact sequence of A-modules $0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$ in $\operatorname{mod} A$, with $T_0, T_1 \in \operatorname{add} T$. It is well known that this latter condition can be replaced by the fact that the number of pairwise nonisomorphic indecomposable direct summands in any indecomposable decomposition of T equals the number of nonisomorphic indecomposable simple A-module. We refrain from defining explicitly the dual notion of **cotilting module**.

In this case T induces torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ on $\operatorname{mod} A$ and $\operatorname{mod} B$ respectively, where $B = \operatorname{End}_A T$. In particular, the torsion-free class $\mathcal{F}$ is given by $\{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, M) = 0\}$, while $\mathcal{T} = \{M \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(T, M) = 0\}$. Then T is said to be **separating** if the torsion pair $(\mathcal{T}, \mathcal{F})$ splits, that is each indecomposable A-module lies either in $\mathcal{F}$ or in $\mathcal{T}$. Similarly, T is said to be **splitting** if the torsion pair $(\mathcal{X}, \mathcal{Y})$ splits.

For further details on tilting theory, we refer to [**2, 7**].

1.2. Bounded derived categories and piecewise hereditary algebras. Here, we recall basic features of bounded derived categories and piecewise hereditary algebras needed in the subsequent developments. For a full account, we refer to [**20, 22, 34**] and [**15, 16, 17**], respectively.

Let $\mathcal{A}$ be an additive category and $C^b(\mathcal{A})$ be the category of bounded complexes over $\mathcal{A}$. Two morphisms $f, g : M^\bullet \longrightarrow N^\bullet$ are **homotopic** if there exist morphisms $h^i : M^i \longrightarrow N^{i-1}$ such that $f^i - g^i = h^{i+1} d^i_{M^\bullet} - d^{i-1}_{N^\bullet} h^i$ for all $i \in \mathbb{Z}$.

Denote by $K^b(\mathcal{A})$ the **homotopy category**. The **suspension functor** is defined on objects by $M^\bullet[1] = (M^{i+1}, -d^{i+1}_{M^\bullet})_{i\in\mathbb{Z}}$ and on morphisms by $f[1] = (f^{i+1})_{i\in\mathbb{Z}}$. The **mapping cone** $cone(f)$ of a morphism $f : M^\bullet \longrightarrow N^\bullet$ is the complex $cone(f) = (N^i \oplus M^{i+1}, d^i_{cone(f)})$ where

$$d^i_{cone(f)} = \begin{pmatrix} d^i_{N^\bullet} & f^{i+1} \\ 0 & d^i_{M^\bullet[1]} \end{pmatrix}.$$

Then, $K^b(\mathcal{A})$, together with the suspension functor, is a triangulated category (in the sense of [**34**]), in which the **distinguished triangles** are those isomorphic to triangles of the form

$$M^\bullet \xrightarrow{\ f\ } N^\bullet \xrightarrow{(\mathrm{id},0)^\perp} cone(f) \xrightarrow{(0,\mathrm{id})} M^\bullet[1].$$

Given a complex $M^\bullet = (M^i, d^i_{M^\bullet})_{i\in\mathbb{Z}}$ in $K^b(\mathcal{A})$, we define the **cohomology objects** $\mathrm{H}^i(M^\bullet) = \operatorname{Ker} d^i_{M^\bullet} / \operatorname{Im} d^{i-1}_{M^\bullet}$. Then, a morphism of complexes $f : M^\bullet \longrightarrow N^\bullet$ is a **quasi-isomorphism** if the induced morphism $\mathrm{H}^i(f) : \mathrm{H}^i(M^\bullet) \longrightarrow \mathrm{H}^i(N^\bullet)$ is an isomorphism for each $i \in \mathbb{Z}$.

The **derived category** of bounded complexes over $\mathcal{A}$ is the triangulated category $D^b(\mathcal{A})$ obtained from $K^b(\mathcal{A})$ by localizing with respect to the set of quasi-isomorphisms. Briefly, the objects in $D^b(\mathcal{A})$ are the objects in $K^b(\mathcal{A})$ while the morphisms $u : M^\bullet \longrightarrow N^\bullet$ in $D^b(\mathcal{A})$ are represented by a pair of morphisms $M^\bullet \xleftarrow{s} L^\bullet \xrightarrow{f} N^\bullet$ in $K^b(\mathcal{A})$, where s is a quasi-isomorphism (see [**34**]). Finally, given two triangulated categories $\mathcal{C}$ and $\mathcal{C}'$, a functor $E : \mathcal{C} \longrightarrow \mathcal{C}'$ is a **triangle-equivalence** if it is an equivalence which preserves the triangles.

In most of the situations, $\mathcal{A}$ will denote the category $\operatorname{mod} A$ for some finite dimensional k-algebra A. In this case, we simply denote $D^b(\operatorname{mod} A)$ by $D^b(A)$. The bounded derived category is best understood when $\mathcal{A}$ is a hereditary abelian category, that is such that the bifunctor $\operatorname{Ext}^2(-,?)$ vanishes. Then, any indecomposable object $M^\bullet = (M^i, d^i_{M^\bullet})_{i\in\mathbb{Z}}$ in $D^b(\mathcal{A})$ is isomorphic to a stalk complex, and so we can assume that there exists $i_0 \in \mathbb{Z}$ such that $M^i \neq 0$ if and only if $i = i_0$, and where M^{i_0} is an indecomposable object in $\mathcal{A}$. In this case, we write $M^\bullet = M^{i_0}[-i_0]$. Moreover, given two objects M, N in $\mathcal{A}$ and $i \in \mathbb{Z}$, we have $\operatorname{Hom}_{D^b(\mathcal{A})}(M[i], N[i]) \cong \operatorname{Hom}_{\mathcal{A}}(M, N)$. This relation shows that derived categories of piecewise hereditary algebras behave quite well.

1.3. Skew group algebras. Let A be an algebra and G be a group with identity e. An **action** of G on A is a homomorphism of group from G to the group of automorphisms of A, or equivalently a function $G \times A \longrightarrow A$, $(\sigma, a) \longmapsto \sigma(a)$, such that:

(a) For each σ in G, the map $\sigma : A \longrightarrow A$ is an automorphism of algebra;
(b) $(\sigma_1\sigma_2)(a) = \sigma_1(\sigma_2(a))$ for all $\sigma_1, \sigma_2 \in G$ and $a \in A$;
(c) $e(a) = a$ for all $a \in A$.

For any such action, the **skew group algebra** $A[G]$ is the free left A-module endowed with the multiplication given by $(a\sigma)(b\varsigma) = a\sigma(b)\sigma\varsigma$ for all $a, b \in A$ and $\sigma, \varsigma \in G$. Clearly, $A[G]$ is an algebra (with identity element the formal product of the identity of A with the identity of G) admitting a structure of right A-module given by $(a\sigma)a' = (a\sigma(a'))\sigma$, for all $a, a' \in A$ and $\sigma \in G$. Observe that $A[G]$ is generally not connected and basic, but this will not play any major role in the sequel.

In addition, any action of G on A induces a group action on $\operatorname{mod} A$ as follows: for any $M \in \operatorname{mod} A$ and $\sigma \in G$, let ${}^\sigma M$ be the A-module with the additive structure of M and with the multiplication $a \cdot m = \sigma^{-1}(a)m$, for $a \in A$ and $m \in M$. Moreover, given a morphism of A-modules $f : M \longrightarrow N$, define ${}^\sigma f : {}^\sigma M \longrightarrow {}^\sigma N$ by ${}^\sigma f(m) = f(m)$ for each $m \in {}^\sigma M$. By [**5**], this defines an action of G on $\operatorname{mod} A$.

In order to avoid confusion, and since the A-modules M and ${}^\sigma M$ have the same elements but different external multiplication, we shall use the symbol "$\cdot$" to express the multiplication $a \cdot m$ of an element a in A by an element m in ${}^\sigma M$, while no symbol will be used to express the multiplication of a by m when m is viewed as an element in M. Hence, the expressions am and $a \cdot m$ are totally different.

Remark 1.3.1. With the above notation, we have $M = {}^\sigma M$ as abelian groups. Moreover, in this context, we have $f = {}^\sigma f$ as homomorphisms of groups.

Remark 1.3.2. In what follows, it will sometimes be convenient to consider the case where a group G acts on a category $\mathcal{A}$ (with, say, ${}^\sigma(-) : \mathcal{A} \longrightarrow \mathcal{A}$ for $\sigma \in G$) and T is an object of $\mathcal{A}$ which is G-stable under this action, that is T is isomorphic to ${}^\sigma T$ (as $\mathcal{A}$-object) for any σ in G. In this case, in order to improve the readability of the mathematics and avoid very cumbersome details, we will freely identify T with ${}^\sigma T$ instead of dealing with explicit isomorphisms between them. This approach yields a major simplification in the forthcoming technical proofs of (2.2.1) and (3.1.3), without, of course, altering their veracity. With this identification in mind, one gets a change of rings action as follows : for any $\sigma \in G$

and $f \in \mathrm{End}_{\mathcal{A}} T$, set $\sigma(f) = {}^{\sigma}f$. This defines an action of G on $\mathrm{End}_{\mathcal{A}} T$ as one can easily verify. In this case, the multiplication in $(\mathrm{End}_{\mathcal{A}} T)[G]$ is therefore given by $(f\sigma)(g\sigma') = f\sigma(g)\sigma\sigma' = f{}^{\sigma}g\sigma\sigma'$, for all $f, g \in \mathrm{End}_{\mathcal{A}} T$ and $\sigma, \sigma' \in G$.

2. Group actions and G-compatible derived equivalences

In this section, we show how an action of G on an additive category $\mathcal{A}$ induces an action of G on the homotopy and derived categories of $\mathcal{A}$. Once this is done, we show that the equivalences of derived categories induced by G-stable tilting modules are G-compatible.

2.1. Group actions on homotopy and derived categories. Let G be a group and assume that $\mathcal{A}$ is an additive category on which G acts. For each $\sigma \in G$, let ${}^{\sigma}(-) : \mathcal{A} \longrightarrow \mathcal{A}$ be the automorphism of $\mathcal{A}$ induced by σ.

Starting from this action, we define an action of G on $K^b(\mathcal{A})$ and $D^b(\mathcal{A})$ as follows. For any complex $M^\bullet = (M^i, d^i_{M^\bullet})_{i\in\mathbb{Z}}$ over $\mathcal{A}$ and $\sigma \in G$, let ${}^{\sigma}M^\bullet$ be the complex $({}^{\sigma}M^i, {}^{\sigma}d^i_{M^\bullet})_{i\in\mathbb{Z}}$. Moreover, given another complex $N^\bullet = (N^i, d^i_{N^\bullet})_{i\in\mathbb{Z}}$ and a morphism of complexes $f = (f^i : M^i \longrightarrow N^i)_{i\in\mathbb{Z}}$, let ${}^{\sigma}f = ({}^{\sigma}f^i : {}^{\sigma}M^i \longrightarrow {}^{\sigma}N^i)_{i\in\mathbb{Z}}$. Clearly, ${}^{\sigma}f$ is a morphism of complexes.

Since ${}^{\sigma}(-) : \mathcal{A} \longrightarrow \mathcal{A}$ is an automorphism, this construction is compatible with the homotopy relation (see Section 1.2). This allows to define, for each $\sigma \in G$, an endomorphism ${}^{\sigma}(-) : K^b(\mathcal{A}) \longrightarrow K^b(\mathcal{A})$. We have the following:

Lemma 2.1.1. *Let $\sigma \in G$. The mapping $M^\bullet \longmapsto {}^{\sigma}M^\bullet$ (where $M^\bullet$ is a complex over $\mathcal{A}$) induces an action of G on $K^b(\mathcal{A})$. In addition, the automorphisms ${}^{\sigma}(-) : K^b(\mathcal{A}) \longrightarrow K^b(\mathcal{A})$ induced by the elements $\sigma \in G$ are triangle-equivalences.*

Proof. The first part clearly follows from the above discussion and the fact that ${}^{\sigma}(-) \circ {}^{\sigma^{-1}}(-) = \mathrm{id}_{K^b(\mathcal{A})}$ for each $\sigma \in G$. On the other hand, let $\sigma \in G$ and Δ be a distinguished triangle in $K^b(\mathcal{A})$, that is (up to isomorphism) a triangle of the form $M^\bullet \xrightarrow{f} N^\bullet \xrightarrow{(\mathrm{id},0)^\perp} \mathit{cone}(f) \xrightarrow{(0,\mathrm{id})} M^\bullet[1]$. We want to show that ${}^{\sigma}\Delta$, that is

$$ {}^{\sigma}M^\bullet \xrightarrow{{}^{\sigma}f} {}^{\sigma}N^\bullet \xrightarrow{(\mathrm{id},0)^\perp} {}^{\sigma}(\mathit{cone}(f)) \xrightarrow{(0,\mathrm{id})} {}^{\sigma}(M^\bullet[1]), $$

is a distinguished triangle. Clearly ${}^{\sigma}(M^\bullet[1]) = ({}^{\sigma}M^\bullet)[1]$. In addition, for any $i \in \mathbb{Z}$, we have ${}^{\sigma}(\mathit{cone}(f))^i = {}^{\sigma}(N^i \oplus M^{i+1}) \cong {}^{\sigma}N^i \oplus {}^{\sigma}M^{i+1} = \mathit{cone}({}^{\sigma}f)^i$. So, ${}^{\sigma}(\mathit{cone}(f)) \cong \mathit{cone}({}^{\sigma}f)$, and therefore ${}^{\sigma}\Delta$ is given by

$$ {}^{\sigma}M^\bullet \xrightarrow{{}^{\sigma}f} {}^{\sigma}N^\bullet \xrightarrow{(\mathrm{id},0)^\perp} \mathit{cone}({}^{\sigma}f) \xrightarrow{(0,\mathrm{id})} ({}^{\sigma}M^\bullet)[1]. $$

This proves our claim. □

We now extend this relation to $D^b(\mathcal{A})$. To do so, recall from Section 1.2 that if $M^\bullet$ and $N^\bullet$ are two objects in $D^b(\mathcal{A})$ and $f \in \mathrm{Hom}_{D^b(\mathcal{A})}(M^\bullet, N^\bullet)$, then f is of the form $M^\bullet \xleftarrow{s} L^\bullet \xrightarrow{u} N^\bullet$, for some object $L^\bullet$ in $D^b(\mathcal{A})$ and homomorphisms s, u in $K^b(\mathcal{A})$ where s is a quasi-isomorphism. In this case, and for $\sigma \in G$, let ${}^{\sigma}f$ be given by ${}^{\sigma}M^\bullet \xleftarrow{{}^{\sigma}s} {}^{\sigma}L^\bullet \xrightarrow{{}^{\sigma}u} {}^{\sigma}N^\bullet$. We obtain the following result.

Proposition 2.1.2. *Let $\sigma \in G$. The mapping $M^\bullet \longmapsto {}^{\sigma}M^\bullet$ (where $M^\bullet$ is a complex over $\mathcal{A}$) induces an action of G on $D^b(\mathcal{A})$. In addition, the automorphisms ${}^{\sigma}(-) : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$ induced by the elements $\sigma \in G$ are triangle-equivalences.*

Proof. The first part follows from the above discussion and the fact that ${}^{\sigma}(-) \circ {}^{\sigma^{-1}}(-) = \mathrm{id}_{D^b(\mathcal{A})}$ for each $\sigma \in G$. On the other hand, let $\sigma \in G$ and Δ be a distinguished triangle in $D^b(\mathcal{A})$, that is (up to isomorphism) a triangle of the form

$$\begin{array}{ccccccc} & \overset{\text{id}}{\swarrow} M^\bullet \overset{\overline{f}}{\searrow} & & \overset{\text{id}}{\swarrow} N^\bullet \overset{(\text{id},0)^\perp}{\searrow} & & \overset{\text{id}}{\swarrow} cone(\overline{f}) \overset{(0,\text{id})}{\searrow} & \\ M^\bullet & \xrightarrow{f} & N^\bullet & \xrightarrow{(\text{id},0)^\perp} & cone(f) & \xrightarrow{(0,\text{id})} & M^\bullet[1] \end{array}$$

where $cone(f) = cone(\overline{f})$ and the triangle $M^\bullet \xrightarrow{\overline{f}} N^\bullet \xrightarrow{(\text{id},0)^\perp} cone(\overline{f}) \xrightarrow{(0,\text{id})} M^\bullet[1]$ is distinguished in $K^b(A)$. Then, by (2.1.1), ${}^\sigma\Delta$ is given by

$$\begin{array}{ccccccc} & \overset{\text{id}}{\swarrow} {}^\sigma M^\bullet \overset{{}^\sigma\overline{f}}{\searrow} & & \overset{\text{id}}{\swarrow} {}^\sigma N^\bullet \overset{(\text{id},0)^\perp}{\searrow} & & \overset{\text{id}}{\swarrow} cone({}^\sigma\overline{f}) \overset{(0,\text{id})}{\searrow} & \\ {}^\sigma M^\bullet & \xrightarrow{{}^\sigma f} & {}^\sigma N^\bullet & \xrightarrow{(\text{id},0)^\perp} & cone({}^\sigma f) & \xrightarrow{(0,\text{id})} & ({}^\sigma M^\bullet)[1] \end{array}$$

and is thus distinguished. □

At this point, recall that if $\mathcal{A} = \operatorname{mod} A$, for some finite dimensional k-algebra A of finite global dimension (for instance if A is piecewise hereditary [**17**, (1.2)]), then $D^b(\mathcal{A})$ has almost split triangles. We have the following result.

Proposition 2.1.3. *Let $\mathcal{A}$ be as above and $\sigma \in G$. Then, the automorphism ${}^\sigma(-) : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$ preserves the almost split triangles.*

Proof. To prove the claim, it is sufficient to show that if $u : M^\bullet \longrightarrow N^\bullet$ is left minimal almost split in $D^b(\mathcal{A})$, then so is ${}^\sigma u$. Let f be an arbitrary morphism in $D^b(\mathcal{A})$ such that $f\,{}^\sigma u = f$. Then ${}^{\sigma^{-1}}f u = {}^{\sigma^{-1}}f$. Since u is left minimal, ${}^{\sigma^{-1}}f$ is an isomorphism, and so is f. So ${}^\sigma u$ is left minimal. The fact that ${}^\sigma u$ is left almost split is shown in a similar way. □

We get the following corollary, which proof directly follows from (2.1.3).

Corollary 2.1.4. *Let $\mathcal{A}$ be as above and $\sigma \in G$. Then the Auslander-Reiten translation τ and the functor ${}^\sigma(-)$ commute, that is ${}^\sigma(\tau M^\bullet) = \tau({}^\sigma M^\bullet)$ for any indecomposable object $M^\bullet$ in $D^b(\mathcal{A})$. In particular, the functor ${}^\sigma(-)$ preserves the τ-orbits in $\Gamma(D^b(\mathcal{A}))$.* □

2.2. G-compatible derived equivalences. It is well known from [**15**] that any tilting module induced an equivalence of derived categories. In order to prove our main results, we need to show that the G-stable tilting modules induce G-compatible equivalences. We recall the following facts from [**15**, (III.2)].

Let A be a finite dimensional k-algebra of finite global dimension. Given a tilting A-module T and $B = \operatorname{End}_A T$, the functors

(i) $\operatorname{Hom}_A(T,-) : K^b(\operatorname{add} T) \longrightarrow K^b(\operatorname{proj} B)$

(ii) $\rho : K^b(\operatorname{proj} B) \hookrightarrow K^b(\operatorname{mod} B) \longrightarrow D^b(B)$

(iii) $\phi : K^b(\operatorname{add} T) \hookrightarrow K^b(\operatorname{mod} A) \longrightarrow D^b(A)$

are equivalences of triangulated categories such that the composition

(iv) $\operatorname{RHom}(T,-) = \rho \circ \operatorname{Hom}_A(T,-) \circ \phi^{-1} : D^b(A) \longrightarrow D^b(B)$ takes T to B.

Keeping (1.3.2) in mind, we can now prove the following.

Proposition 2.2.1. *Let A be an algebra and G be a group acting on A. If T is a tilting A-module which is G-stable with respect to the induced action of G on* $\operatorname{mod} A$*, then the equivalences given above are G-compatible.*

Proof. First, let $\theta : G \longrightarrow \operatorname{Aut} A$ be an action of G on A and, for each $\sigma \in G$, let ${}^\sigma(-) : \operatorname{mod} A \longrightarrow \operatorname{mod} A$ be the induced automorphism. Moreover, let T be a tilting A-module which is G-stable with respect to this action. In order to prove that the equivalences (i)-(iv) are G-compatible, we first clarify which are the induced actions of G on $K^b(\operatorname{add} T)$, $K^b(\operatorname{proj} B)$ and $D^b(B)$. The G-stability of T gives rise to a

natural action of G on $\operatorname{add} T$. Then, following Section 2.1, the additive category $K^b(\operatorname{add} T)$ inherits a (component-wise) action of G. We also denote the induced automorphism on $K^b(\operatorname{add} T)$ by ${}^{\sigma}(-)$, for $\sigma \in G$. On the other hand, since T is G-stable, it follows from (1.3.2) that B is endowed with a natural action of G, which we extend to $K^b(\operatorname{proj} B)$ and $D^b(B)$ (see Section 2.1). Again, we denote the induced automorphisms by ${}^{\sigma}(-)$, for $\sigma \in G$.
(i) $\operatorname{Hom}_A(T,-) : K^b(\operatorname{add} T) \longrightarrow K^b(\operatorname{proj} B)$. We recall that T has a structure of right B-module if we let the endomorphisms act on the right-hand side, that is for $b \in B$ and $t \in T$, $(t)b$ denotes the image of t by b. Then, any A-module M yields a left B-module $\operatorname{Hom}_A(T,M)$ (with the multiplication $(bg)(t) = g((t)b)$ for all $b \in B$, $g \in \operatorname{Hom}_A(T,M)$ and $t \in T$).

Now, let $\sigma \in G$ and

$$T^{\bullet}: \quad \cdots \longrightarrow 0 \longrightarrow T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} T_n \longrightarrow 0 \longrightarrow \cdots$$

be a complex in $K^b(\operatorname{add} T)$. We verify that ${}^{\sigma}(\operatorname{Hom}_A(T,T^{\bullet})) \cong \operatorname{Hom}_A(T, {}^{\sigma}T^{\bullet})$. To do so, for each $i \in \{0,1,\dots,n\}$, let α_i be the map

$$\begin{array}{rccc} \alpha_i : & {}^{\sigma}(\operatorname{Hom}_A(T,T_i)) & \longrightarrow & \operatorname{Hom}_A({}^{\sigma}T, {}^{\sigma}T_i) \\ & g & \longmapsto & {}^{\sigma}g \end{array}$$

Then α_i is a morphism of B-modules since we have

$$\begin{aligned} (\alpha_i(b\cdot g))(t) &= (\alpha_i(\sigma^{-1}(b)g))(t) = (\alpha_i(({}^{\sigma^{-1}}b)g))(t) = ({}^{\sigma}(({}^{\sigma^{-1}}b)g))(t) \\ &= (b({}^{\sigma}g))(t) = {}^{\sigma}g((t)b) = \alpha_i(g)((t)b) = b\alpha_i(g)(t) \end{aligned}$$

whenever $b \in \operatorname{End}_A T$ and $g \in {}^{\sigma}(\operatorname{Hom}_A(T,T_i))$. Since α_i is clearly bijective, it is an isomorphism of B-modules. Moreover, it is easily verified that $\alpha_{i+1} \circ {}^{\sigma}(\operatorname{Hom}_A(T,f_i)) = \operatorname{Hom}_A({}^{\sigma}T, {}^{\sigma}f_i) \circ \alpha_i$. So, we have an isomorphism of complexes

$$\alpha_{\bullet} : {}^{\sigma}(\operatorname{Hom}_A(T,T^{\bullet})) \longrightarrow \operatorname{Hom}_A({}^{\sigma}T, {}^{\sigma}T^{\bullet})$$

Now, since T is G-stable, we have $T \cong {}^{\sigma}T$ and an isomorphism

$$\beta_{\bullet} : {}^{\sigma}(\operatorname{Hom}_A(T,T^{\bullet})) \xrightarrow{\alpha_{\bullet}} \operatorname{Hom}_A({}^{\sigma}T, {}^{\sigma}T^{\bullet}) \xrightarrow{\cong} \operatorname{Hom}_A(T, {}^{\sigma}T^{\bullet}).$$

Similarly, if $s : T^{\bullet} \longrightarrow T'^{\bullet}$ is a morphism of complexes in $K^b(\operatorname{add} T)$, then one can check that $\operatorname{Hom}_A(T, {}^{\sigma}s) \circ \beta_{\bullet} = \beta_{\bullet} \circ {}^{\sigma}(\operatorname{Hom}_A(T,s))$. So $\operatorname{Hom}_A(T,-)$ is G-compatible.
(ii) and (iii). Since the inclusions and localization functors are clearly G-compatible, then so are ρ and ϕ. □

3. Piecewise hereditary algebras revisited

The aim of this section is to prove Theorem 1 and Theorem 2 which stand, as mentioned in the introduction, as generalizations of the well-known characterizations of piecewise hereditary algebras given in [**19**] and [**17**] respectively. Before doing so we have to recall some facts concerning skew group algebras and prove preliminary results.

3.1. Preliminary results. Let A be an algebra and $G = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ be a finite group acting on A whose order is not a multiple of the characteristic of k. Then, the natural inclusion of A in $A[G]$ induces the change of ring functors $F = A[G] \otimes_A - : \operatorname{mod} A \longrightarrow \operatorname{mod} A[G]$ and $H = \operatorname{Hom}_{A[G]}(A[G],-) : \operatorname{mod} A[G] \longrightarrow \operatorname{mod} A$, which have been extensively studied in [**5, 29, 33**], for instance.

We recall the following facts from [**29**, (1.1)(1.8)].

Remark 3.1.1.

(a) (F, H) and (H, F) are two adjoint pairs of functors. In particular, F and H are exact functors.

(b) Let $M \in \operatorname{mod} A$. For each $\sigma \in G$, the subset $\sigma \otimes_A M = \{\sigma \otimes_A m \mid m \in M\}$ of FM has a natural structure of A-module given by $a(\sigma \otimes_A m) = a\sigma \otimes_A m = \sigma\sigma^{-1}(a) \otimes_A m = \sigma \otimes_A \sigma^{-1}(a)m = \sigma \otimes_A (a \cdot m)$, so that $\sigma \otimes_A M$ and ${}^{\sigma}M$ are isomorphic as A-modules. In addition, any element $x \in FM = A[G] \otimes_A M$ can be written in the form $x = \sum_{i=1}^{n} \sigma_i^{-1} \otimes_A x_{\sigma_i}$, and in a unique way since the σ_i^{-1} form a basis for $A[G]$ as A-module. So, as A-modules, we have

$$FM \cong \bigoplus_{i=1}^{n} (\sigma_i \otimes_A M) \cong \bigoplus_{i=1}^{n} {}^{\sigma_i}M.$$

Then, since $H : \operatorname{mod} A[G] \longrightarrow \operatorname{mod} A$ is nothing else than the restriction functor, we get

$$HFM \cong \bigoplus_{i=1}^{n} (\sigma_i \otimes_A M) \cong \bigoplus_{i=1}^{n} {}^{\sigma_i}M.$$

We can now prove the following important technical results, where the simplification discussed in (1.3.2) plays a major role in the proof of the proposition.

Lemma 3.1.2. *Let A and G be as above, and T be an A-module. Then, for each f in $\operatorname{End}_{A[G]} FT$, there exists unique f_i in $\operatorname{Hom}_A(T, {}^{\sigma_i^{-1}}T)$, for $i = 1, 2, \ldots, n$, such that $f(1_{A[G]} \otimes_A t) = \sum_{i=1}^{n} \sigma_i^{-1} \otimes_A f_i(t)$ for each $t \in T$.*

Proof. Let $f \in \operatorname{End}_{A[G]} FT$. Then, for each $t \in T$, there exists by (3.1.1)(b) a unique sequence $t_1, t_2, \ldots, t_n$ of elements in T such that $f(1_{A[G]} \otimes_A t) = \sum_{i=1}^{n} \sigma_i^{-1} \otimes_A t_i$. Let $f_i : T \longrightarrow {}^{\sigma_i^{-1}}T$ be defined by $f_i(t) = t_i$ for each $t \in T$. We claim that the f_i are morphisms of A-modules for each i. Indeed, each f_i is clearly additive. Moreover, for all $a \in A$ and $t \in T$, we have

$$\begin{aligned} f(1_{A[G]} \otimes_A at) &= af(1_{A[G]} \otimes_A t) = a\textstyle\sum_{i=1}^{n} \sigma_i^{-1} \otimes_A f_i(t) = \sum_{i=1}^{n} a\sigma_i^{-1} \otimes_A f_i(t) \\ &= \textstyle\sum_{i=1}^{n} \sigma_i^{-1}\sigma_i(a) \otimes_A f_i(t) = \sum_{i=1}^{n} \sigma_i^{-1} \otimes_A \sigma_i(a) f_i(t) \end{aligned}$$

showing that $f_i(at) = \sigma_i(a) f_i(t) = a \cdot f_i(t)$ for each $t \in T$, as required (see Section 1.3). The uniqueness of the f_i is immediate from (3.1.1)(b). □

Proposition 3.1.3. *Let A and G be as above, and T be a G-stable tilting A-module. Then,*

(a) *FT is a $A[G]$-tilting module.*

(b) $\operatorname{End}_{A[G]} FT \cong (\operatorname{End}_A T)[G]$.

Proof. (a). Since T has projective dimension at most one, there exists a short exact sequence $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow T \longrightarrow 0$ with P_0, P_1 projective A-modules. Since F is exact and preserves the projectives by (3.1.1), this yields a short exact sequence $0 \longrightarrow FP_1 \longrightarrow FP_0 \longrightarrow FT \longrightarrow 0$ in $\operatorname{mod} A[G]$, and so $\operatorname{pd}_{A[G]} FT \leq 1$. In addition, we have

$$\operatorname{Ext}^1_{A[G]}(FT, FT) \cong D\operatorname{Hom}_{A[G]}(FT, \tau(FT)) \cong D\operatorname{Hom}_{A[G]}(FT, F(\tau T)),$$

by the Auslander-Reiten formula and [**29**, (4.2)]. By adjunction, this latter group is nonzero if and only if $\operatorname{Hom}_A(T, HF(\tau T)) \cong \operatorname{Hom}_A(T, \oplus_{\sigma \in G} {}^{\sigma}(\tau T))$ is nonzero, where the isomorphism follows from (3.1.1). However, if $f : T \longrightarrow {}^{\sigma}(\tau T)$ is a nonzero morphism, with $\sigma \in G$, then ${}^{\sigma^{-1}}f : T \cong {}^{\sigma^{-1}}T \longrightarrow \tau T$ is nonzero, a contradiction to $\operatorname{Hom}_A(T, \tau T) \cong D\operatorname{Ext}^1_A(T, T) = 0$. So, $\operatorname{Ext}^1_{A[G]}(FT, FT) = 0$.

Finally, since T is a tilting A-module, there exists a short exact sequence $0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$ in $\operatorname{mod} A$, with $T_0, T_1 \in \operatorname{add} T$. Applying F to this sequence yields a short exact sequence $0 \longrightarrow A[G] \longrightarrow FT_0 \longrightarrow FT_1 \longrightarrow 0$ in $\operatorname{mod} A[G]$, since $A[G] \cong FA$. So FT is a tilting $A[G]$-module. This proves (a).

(b). In order to show that $\operatorname{End}_{A[G]} FT$ and $(\operatorname{End}_A T)[G]$ are isomorphic algebras, we will construct explicit inverse isomorphisms between them. In what follows, we freely identify T with ${}^{\sigma}T$, for any $\sigma \in G$.

First, let $f \in \operatorname{End}_{A[G]} FT$. By (3.1.2), there exists, for each i, a unique homomorphism $f_i : T \longrightarrow {}^{\sigma_i^{-1}}T = T$ such that $f(1_{A[G]} \otimes_A t) = \sum_{i=1}^n \sigma_i^{-1} \otimes_A f_i(t)$ for each $t \in T$. Thus, we define

$$\begin{array}{rccl} \nu : & \operatorname{End}_{A[G]} FT & \longrightarrow & (\operatorname{End}_A T)[G], \qquad \text{by} \\ & f & \longmapsto & \sum_{i=1}^n f_i \sigma_i \end{array}$$

for each $f \in \operatorname{End}_{A[G]} FT$. Conversely, given $\overline{f} = \sum_{i=1}^n f_i \sigma_i \in (\operatorname{End}_A T)[G]$, we let $\mu(\overline{f}) \in \operatorname{End}_{A[G]} FT$ be such that

$$\mu(\overline{f})(r \otimes_A t) = \sum_{i=1}^n r\sigma_i^{-1} \otimes_A f_i(t)$$

for all $r \in A[G]$ and $t \in T$, and extend this definition by linearity. This is a well-defined application since, when viewing each f_i as a morphism of A-modules between T and ${}^{\sigma_i{}^{-1}}T$, we get

$$\begin{aligned} \mu(\overline{f})(r \otimes_A at) &= \textstyle\sum_{i=1}^n r\sigma_i^{-1} \otimes_A f_i(at) = \sum_{i=1}^n r\sigma_i^{-1} \otimes_A \sigma_i(a) f_i(t) \\ &= \textstyle\sum_{i=1}^n r\sigma_i^{-1}\sigma_i(a) \otimes_A f_i(t) = \sum_{i=1}^n ra\sigma_i^{-1} \otimes_A f_i(t) \\ &= \mu(\overline{f})(ra \otimes_A t) \end{aligned}$$

for each $a \in A$, where the second equality follows from the module structure of ${}^{\sigma_i{}^{-1}}T$ (see Section 1.3) and the fourth equality follows from that fact that $\sigma_i^{-1}\sigma_i(a) = a\sigma_i^{-1}$ in $A[G]$. Moreover, it is immediate to verify that $\mu(\overline{f})$ is a morphism of $A[G]$-modules.

We claim that ν and μ are inverse isomorphisms of algebras. Indeed, let $f \in \operatorname{End}_{A[G]} FT$, and assume as above that $f(1_{A[G]} \otimes_A t) = \sum_{i=1}^n \sigma_i^{-1} \otimes_A f_i(t)$ for each $t \in T$. Then, we get

$$(\mu\nu)(f)(r \otimes_A t) = \mu(\sum_{i=1}^n f_i\sigma_i)(r \otimes_A t) = \sum_{i=1}^n r\sigma_i^{-1} \otimes_A f_i(t) = f(r \otimes t),$$

for all $r \in A[G]$ and $t \in T$.

On the other hand, if $\overline{f} = \sum_{i=1}^n f_i\sigma_i$, then $\mu(\overline{f})(1_{A[G]} \otimes t) = \sum_{i=1}^n \sigma_i^{-1} \otimes f_i(t)$ for all $t \in T$. Consequently, $(\nu\mu)(\overline{f}) = \overline{f}$. So ν and μ are inverse bijections.

It remains to show that μ (and hence ν) is a morphism of algebras. To do so, let $\overline{f} = \sum_{i=1}^n f_i\sigma_i$ and $\overline{g} = \sum_{j=1}^n g_j\sigma_j$ be elements in $(\operatorname{End}_A T)[G]$. Keeping in mind the multiplication in $(\operatorname{End}_A T)[G]$ (see (1.3.2)), we have

$$\overline{f}\overline{g} = (\sum_{i=1}^n f_i\sigma_i)(\sum_{j=1}^n g_j\sigma_j) = \sum_{i=1}^n\sum_{j=1}^n f_i\sigma_i(g_j)\sigma_i\sigma_j = \sum_{i=1}^n\sum_{j=1}^n f_i{}^{\sigma_i}g_j\sigma_i\sigma_j$$

and consequently $\mu(\overline{f}\overline{g})(r \otimes t) = \sum_{i=1}^n\sum_{j=1}^n r(\sigma_i\sigma_j)^{-1} \otimes (f_i{}^{\sigma_i}g_j)(t)$ for all $r \in A[G]$ and $t \in T$. On the other hand, we have

$$(\mu(\overline{f})\mu(\overline{g}))(r \otimes t) = \mu(\overline{f})(\sum_{j=1}^n r\sigma_j^{-1} \otimes g_j(t)) = \sum_{i=1}^n\sum_{j=1}^n r\sigma_j^{-1}\sigma_i^{-1} \otimes (f_ig_j)(t)$$

for all $r \in A[G]$ and $t \in T$, and so $\mu(\overline{f}\overline{g}) = \mu(\overline{f})\mu(\overline{g})$ since ${}^{\sigma_i}g_j(t) = g_j(t)$ by (1.3.1).
□

3.2. Proofs of Theorem 1 and Theorem 2. In order to prove Theorem 1 and Theorem 2, we need some additional terminology. Let $\mathcal{A}$ be an arbitrary abelian category and let $M \in \mathcal{A}$ satisfying $\mathrm{Ext}^1(M,M) = 0$ and $\mathrm{Ext}^2(M,N) = 0$ for all $N \in \mathcal{A}$. Then, the **right perpendicular category** $M^\perp$ is the full subcategory of $\mathcal{A}$ containing the objects N satisfying $\mathrm{Hom}(M,N) = \mathrm{Ext}^1(M,N) = 0$. We define dually the left perpendicular category ${}^\perp M$. It was shown in [**14**] that $M^\perp$ and ${}^\perp M$ are again abelian categories.

We can now proceed with the proofs. Observe that these are adaptations of the proofs of the main results in [**19**] and [**17**] respectively, and to which we freely refer in the course of the proof for more details.

Proof of Theorem 1 :
(a). Clearly, (iii) implies (ii), while (ii) implies (i) by an easy induction using (2.2.1). Now, suppose that (i) holds and assume, without loss of generality, that A and H are basic and connected. Let $E : D^b(H) \longrightarrow D^b(A)$ be a G-compatible equivalence of triangulated categories; we shall identify the module categories $\mathrm{mod}\, A$ and $\mathrm{mod}\, H$ with their images under the natural embeddings into $D^b(A)$ and $D^b(H)$, respectively.

Let $M^\bullet$ be an object of $D^b(H)$ such that $EM^\bullet$ is isomorphic to A and assume, without loss of generality, that

$$M^\bullet = M_0 \oplus M_1[1] \oplus \cdots \oplus M_r[r]$$

where the M_i are H-modules and $M_0 \neq 0 \neq M_r$. Note that $\mathrm{Hom}_H(M_i, M_j) = 0$ if $i \neq j$, and $\mathrm{Ext}^1_H(M_i, M_j) = 0$ if $i+1 \neq j$. Moreover, since A is G-stable, then so is $M^\bullet$ and thus, since the $M_i[i]$ lie in different degrees, each of them are also G-stable.

We prove our claim by induction on r. If $r = 0$, then one can check that $M^\bullet = M_0$ is a G-stable tilting H-module, and so

$$A \cong \mathrm{End}_A(A) \cong \mathrm{End}_{D^b(A)} A \overset{E}{\cong} \mathrm{End}_{D^b(H)} M_0 \cong \mathrm{End}_H M_0$$

In addition, since the isomorphism is given by E, the actions of G on A and $\mathrm{End}_H M_0$ coincide. Also, since H is hereditary, then $M^\bullet$ is splitting and the result follows.

Now, assume inductively that the result holds true whenever r takes a smaller value, or r takes the same value and M_0 has less indecomposable direct summands.

We shall find convenient to construct a sequence of separating tilting modules instead of splitting tilting modules. We recall that T_i is a separating tilting A_i-module with endomorphism ring A_{i+i} if and only if T_i is a splitting tilting A^{op}_{i+i}-module with endomorphism ring A^{op}_i. Since A_i and A_{i+1} have equivalent derived categories if and only if so do A^{op}_i and A^{op}_{i+1}, it will be clear that this will not lead into any fallacies.

Let $L = M_1 \oplus M_2 \oplus \cdots \oplus M_r$. Then, by [**19**, (Proposition 3)], the subcategory $L^\perp$ of $\mathrm{mod}\, H$ is equivalent to $\mathrm{mod}\, \Lambda$, for some finite dimensional hereditary algebra Λ. Moreover, the inclusion functor $\mathrm{mod}\, \Lambda \hookrightarrow \mathrm{mod}\, H$ is full, faithful and exact. In what follows, we identify $L^\perp$ and $\mathrm{mod}\, \Lambda$. Observe that $M_0 \in \mathrm{mod}\, \Lambda$, and in fact is a tilting Λ-module. Also, $\mathrm{mod}\, \Lambda$ is G-stable since so is L.

Let $Q = \mathrm{Hom}_k(\Lambda, k)$, a minimal injective cogenerator for $\mathrm{mod}\, \Lambda$, and let

$$T^\bullet = Q \oplus M_1[1] \oplus M_2[2] \oplus \cdots \oplus M_r[r].$$

We observe that Q is a G-stable Λ-module. Indeed, it is easily verified that if I is an injective Λ-module, then so is ${}^\sigma I$ for each $\sigma \in G$. Because each M_i is also G-stable, then so is $T^\bullet$.

Now, according to [**19**], $ET^\bullet$ is (isomorphic to) a separating tilting A-module. Moreover, $ET^\bullet$ is G-stable since E is G-compatible and $T^\bullet$ is G-stable. Now let

$B = \mathrm{End}_A\, ET^\bullet$. By (2.2.1), we have an equivalence

$$E'' : \quad \begin{array}{ccccc} D^b(H) & \xrightarrow{E} & D^b(A) & \xrightarrow{E'} & D^b(B) \\ T^\bullet & \longmapsto & ET^\bullet & \longmapsto & B \end{array}$$

which is G-compatible since so are E and E'.

Let Q_0 be an simple indecomposable direct summand of Q : such a module exists since Λ is hereditary. Then, for each $\sigma \in G$, the Λ-module ${}^\sigma Q_0$ is also simple injective and $E''({}^\sigma Q_0) = {}^\sigma E''(Q_0)$ is a simple projective B-module. Since B is connected, $E''({}^\sigma Q_0)$ is not an injective B-module unless we are in the trivial case of a simple algebra. Observe moreover that since each ${}^\sigma Q_0$ is a simple injective Λ-module, then the set $\{{}^\sigma Q_0 \mid \sigma \in G\}$ is finite, and we denote its cardinality by n.

Now, imitating the arguments of [**19**], we find, for each $\sigma \in G$, an object R_σ in $D^b(H)$ isomorphic to $U_\sigma[1]$, for some H-module U_σ, and such that

$$E'' R_\sigma \cong \tau^{-1} E''({}^\sigma Q_0) \cong \tau^{-1}({}^\sigma(E'' Q_0)) \cong {}^\sigma(\tau^{-1} E'' Q_0).$$

So $\oplus_{\sigma \in G} E'' R_\sigma$ is a G-stable B-module. We let

$$S = (\oplus_{\sigma \in G} E'' R_\sigma) \oplus E'' N,$$

where $(\oplus_{\sigma \in G} {}^\sigma Q_0) \oplus N = T^\bullet$.

At this point, it is worthwhile to observe that $(\oplus_{\sigma \in G} R_\sigma) \oplus N = N_0 \oplus N_1[1] \oplus M_2[2] \oplus \cdots \oplus M_r[r]$ for some $N_0, N_1 \in \mathrm{mod}\, H$, and where, by definition of the R_σ, N_0 has n less indecomposable direct summands than M_0.

We shall show that S is a separating tilting B-module. If I is an injective B-module, then we have

$$\mathrm{Hom}_B(I, \tau S) = \mathrm{Hom}_B(I, \oplus_{\sigma \in G} {}^\sigma E'' Q_0) = 0$$

since ${}^\sigma E'' Q_0$ is simple projective for each $\sigma \in G$. By the Auslander-Reiten formula, S as projective dimension at most one. Moreover, $\mathrm{Ext}^1_B(S, S) \cong D\,\mathrm{Hom}_B(S, \tau S) = \mathrm{Hom}_B(S, \oplus_{\sigma \in G} {}^\sigma E'' Q_0) = 0$. Finally, the number of indecomposable direct summands in an indecomposable decomposition of S is equal to that of $T^\bullet$, which is equal to that of B. So S is a tilting B-module.

Moreover, S is a separating tilting module since the only indecomposable B-modules lying in $\mathcal{F}(S)$, the torsion-free class induced by S, are the $E''{}^\sigma Q_0$, whereas $\mathcal{T}(S)$, the corresponding torsion class, is the additive subcategory generated by all remaining indecomposables. Indeed, if V is an indecomposable B-module, then $V \notin \mathcal{T}(S)$ if and only if $0 \neq \mathrm{Ext}^1_B(S, V) \cong D\,\mathrm{Hom}_B(V, \oplus_{\sigma \in G} {}^\sigma E'' Q_0)$, and this occurs if and only if $V \cong {}^\sigma E'' Q_0$ for some $\sigma \in G$.

Let $C = \mathrm{End}_B(S)$. By (2.2.1), we have an equivalence of triangulated categories

$$E''' : \quad \begin{array}{ccccc} D^b(H) & \xrightarrow{E''} & D^b(B) & \xrightarrow{\cong} & D^b(C) \\ (\oplus_{\sigma \in G} R_\sigma) \oplus N & \longmapsto & S & \longmapsto & C \end{array}$$

which is G-compatible. Moreover, as observed earlier, we have $(\oplus_{\sigma \in G} R_\sigma) \oplus N = N_0 \oplus N_1[1] \oplus M_2[2] \oplus \cdots \oplus M_r[r]$ where $N_0 = 0$ or contains less indecomposable direct summands than M_0. By induction hypothesis, C is piecewise hereditary of type $\mathrm{mod}\, H$ and, using the separating tilting modules $ET^\bullet$ and S, and keeping in mind our preceding discussion on separating tilting modules, so is A. This shows the equivalence of conditions (i), (ii) and (iii).

(b) Now, assume that the equivalent conditions of (a) are satisfied and that G is a finite group whose order is not a multiple of the characteristic of k. Then there exists a sequence of algebras $H = A_0, A_1, \ldots, A_n = A$ on which G acts and a sequence $T_0, T_1, \ldots, T_{n-1}$ where T_i is a G-stable tilting A_i-module with endomorphism ring isomorphic to A_{i+1} for each i. Moreover, by (3.1.3), $A_i[G] \otimes_{A_i} T_i$ is a $A_i[G]$-tilting module for each i and $\mathrm{End}_{A_i[G]}(A_i[G] \otimes_{A_i} T_i) \cong A_{i+1}[G]$. Now, since the order of

G is not a multiple of the characteristic of k, $H[G]$ is hereditary by [**29**, (1.3)], and the statement thus follows from [**19**]. □

We pursue with the proof of the Theorem 2. We will give all the details until the proof of Theorem 1 carries over.

Proof of Theorem 2 :
(a). Clearly, (iii) implies (ii), while (ii) implies (i) by an easy induction using (2.2.1) and its dual.
Now, suppose that (i) holds and let $E : D^b(\mathcal{H}) \longrightarrow D^b(A)$ be a G-compatible equivalence of triangulated categories.

Let $M^\bullet$ be an object of $D^b(\mathcal{H})$ such that $EM^\bullet$ is isomorphic to A and assume, without loss of generality, that

$$M^\bullet = M_0 \oplus M_1[1] \oplus \cdots \oplus M_r[r]$$

for some $M_i \in \mathcal{H}$, with $M_0 \neq 0 \neq M_r$. Note that $\mathrm{Hom}_{\mathcal{H}}(M_i, M_j) = 0$ if $i \neq j$, and $\mathrm{Ext}^1_{\mathcal{H}}(M_i, M_j) = 0$ if $i + 1 \neq j$.

Observe that since A is G-stable, then so is $M^\bullet$ and thus, since the $M_i[i]$ lie in different degrees, each of them are also G-stable.

We prove our claim by induction on r. If $r = 0$, then one can check that $M^\bullet = M_0$ is a G-stable tilting object in $\mathcal{H}$, and

$$A \cong \mathrm{End}_A(A) \cong \mathrm{End}_{D^b(A)} A \overset{E}{\cong} \mathrm{End}_{D^b(\mathcal{H})} M_0 \cong \mathrm{End}_{\mathcal{H}} M_0$$

So A is quasitilted. In addition, since the isomorphism is given by E, the actions of G on A and $\mathrm{End}_{\mathcal{H}} M_0$ coincide.

Now, assume inductively that the result holds true in all cases where either r takes a smaller value, or r takes the same value and either M_0 or M_r has less indecomposable direct summands.

Then, M_0 is a G-stable tilting object in the abelian category $(\oplus_{i=1}^r M_i[i])^\perp$ and M_r is a G-stable cotilting object in the abelian category ${}^\perp(\oplus_{i=0}^{r-1} M_i[i])$. By [**17**], one of these two categories is a module category over a hereditary artin algebra H. The situation is then reduced to the situation of Theorem 1 and its dual, and we are done. This proves (a).
(b). Now, assume that the equivalent conditions of (a) are satisfied and that G is a finite group whose order is not a multiple of the characteristic of k. Then there exists a tilting object T in $\mathcal{H}$ and a sequence of algebras $\mathrm{End}_{\mathcal{H}} T = A_0, A_1, \ldots, A_n = A$ on which G acts and a sequence $T_0, T_1, \ldots, T_{n-1}$ where T_i is a G-stable tilting or cotilting A_i-module with endomorphism ring isomorphic to A_{i+1} for each i. Moreover, by (3.1.3) and its dual, $A_i[G] \otimes_{A_i} T_i$ is a tilting or cotilting $A_i[G]$-module for each i and $\mathrm{End}_{A_i[G]}(A_i[G] \otimes_{A_i} T_i) \cong A_{i+1}[G]$. Now, since the order of G is not a multiple of the characteristic of k, $A_0[G]$ is quasitilted by [**18**, (III.1.6)], and the statement thus follows from [**17**]. □

4. Proof of Theorem 3

The aim of this section is to prove Theorem 3. In view of Theorem 1 and Theorem 2, it would be sufficient to show that if A is piecewise hereditary of type $\mathcal{H}$, for some Ext-finite hereditary abelian category with tilting objects $\mathcal{H}$, and G is a finite group acting on A and whose order is not a multiple of the characteristic of k, then there exists a G-compatible equivalence between $D^b(A)$ and $D^b(\mathcal{H})$. If this will be shown to be true in the case where $\mathcal{H} = \mathrm{coh}\mathbb{X}$, for some weighted projective line $\mathbb{X}$, this unfortunately fails when $\mathcal{H} = \mathrm{mod}\, H$ for some hereditary algebra H. In this section, we show that the above statement holds for $\mathcal{H} = \mathrm{coh}\mathbb{X}$ and show that if $\mathcal{H} = \mathrm{mod}\, H$, then it is however possible to use the induced action of G on $D^b(A)$ in order to construct a (generally distinct) hereditary algebra H' on which

G acts and for which there exists a G-compatible equivalence between $D^b(A)$ and $D^b(H')$.

Also, new terminologies and notations are introduced in order to have a better understanding of these two cases.

4.1. The $\operatorname{mod} H$ **case.** The first situation we consider is that of a piecewise hereditary algebra A of type $\mathcal{H} = \operatorname{mod} H$, for some hereditary algebra H. Here, it will be sufficient to assume that G is a torsion group acting on A. As explained in Section 2.1, this induces an action of G on $D^b(A)$, and we let ${}^{\sigma}(-) : D^b(A) \longrightarrow D^b(A)$ be the induced automorphism for each $\sigma \in G$. In what follows, we show how one can replace H by another hereditary algebra H' for which there exists a G-compatible equivalence between $D^b(A)$ and $D^b(H')$.

First, since A has finite global dimension, then $D^b(A)$ has Auslander-Reiten sequences and we denote by $\Gamma(D^b(A))$ its Auslander-Reiten quiver and by τ its Auslander-Reiten translation. By a **component** of $\Gamma(D^b(A))$, we mean a connected component. Moreover, given a component Γ of $\Gamma(D^b(A))$ a **walk** between two objects M and N in Γ is a sequence :

$$\omega : \quad M = M_0 \overset{f_1}{\text{——}} M_1 \overset{f_2}{\text{——}} \cdots \overset{f_t}{\text{——}} M_t = N, \quad (t \geq 0)$$

where, for each i, M_i is an object in Γ and f_i is a nonzero irreducible morphism either from M_{i-1} to M_i or from M_i to M_{i-1}. Such a walk is called a **path** from M to N provided each f_i is a morphism from M_{i-1} to M_i. In this case, we write $M \rightsquigarrow N$, and then ω is called **sectional** if $\tau M_{i+1} \neq M_{i-1}$ for each i, with $1 \leq i < t$. Also, a path ω is a **oriented cycle** if $M = N$ and a component Γ of $\Gamma(D^b(A))$ containing no oriented cycles is called **directed**. Finally, a subquiver Γ' of a component Γ in $\Gamma(D^b(A))$ is **full** in Γ if any arrow in Γ having its extremities in Γ' lies in Γ', and **convex** in Γ if any path ω in Γ having its extremities M and N in Γ' consists only of morphisms in Γ'.

Now, let A be as above and $Q = (Q_0, Q_1)$ be a finite and acyclic quiver such that $H \cong kQ$, where Q_0 and Q_1 respectively denotes the set of vertices and arrows of Q. It is then well known that the family $\{\Gamma_i\}_{i \in I}$ of directed components of $\Gamma(D^b(A))$ contains only one element Γ if H is representation-finite, and then $\Gamma(D^b(A)) = \Gamma$, or is parametrized by $I = \mathbb{Z}$ otherwise, and $\operatorname{Hom}_{D^b(A)}(\Gamma_i, \Gamma_j) \neq 0$ if and only if $j \in \{i, i+1\}$. Moreover, in both cases, all these directed components are isomorphic to $\mathbb{Z}Q$ as translation quivers, and so have only finitely many τ-orbits.

Our first aim is to show that any such directed component Γ admits a section which is stable under the induced action of G on $D^b(A)$. As we shall see, the endomorphism ring of such sections will turn out to be suitable hereditary algebras. Recall that a full and connected subquiver Ω of Γ is a **section** if :

(a) Ω contains no oriented cycles;
(b) Ω intersects each τ-orbit of Γ exactly once;
(c) Ω is convex.

Let Γ be a directed component in $\Gamma(D^b(A))$. We have the following lemma, in which we set ${}^{\sigma}\Gamma := \{{}^{\sigma}M^{\bullet} \mid M^{\bullet} \in \Gamma\}$ (compare with [**1**, (5.3)]).

Lemma 4.1.1. *Under the above assumptions, we have*

(a) ${}^{\sigma}\Gamma = \Gamma$ *for every* $\sigma \in G$*;*
(b) *For each* $M^{\bullet}$ *in* Γ*, we have* $|\{{}^{\sigma}M^{\bullet} \mid \sigma \in G\}| \leq |Q_0|$.

Proof. (a). If H is representation-finite, this is obvious. Otherwise, let $\{\Gamma_i\}_{i \in \mathbb{Z}}$ be the family of directed components of $\Gamma(D^b(A))$, and let $\sigma \in G$. Since ${}^{\sigma}(-)$ is an automorphism, there exists $m_\sigma \in \mathbb{Z}$ such that ${}^{\sigma}(\Gamma_i) = \Gamma_{i+m_\sigma}$ for each directed component Γ_i of $\Gamma(D^b(A))$. Let k be the order of σ in G. Then, applying repeatedly

the automorphism ${}^{\sigma}(-)$ yields to $\Gamma_i = \Gamma_{i+km_\sigma}$ for each $i \in \mathbb{Z}$. So, $km_\sigma = 0$, and thus $m_\sigma = 0$. This proves (a).
(b). Let $M^\bullet \in \Gamma$ and assume to the contrary that $|\{{}^{\sigma}M^\bullet \mid \sigma \in G\}| > |Q_0|$. Since $\Gamma \cong \mathbb{Z}Q$ as translation quiver, it follows from (a) that there exists a τ-orbit $\mathcal{O}$ of Γ and $\sigma_1, \sigma_2 \in G$ such that ${}^{\sigma_1}M, {}^{\sigma_2}M \in \mathcal{O}$. Assume without loss of generality that there exists a path $\delta : {}^{\sigma_1}M^\bullet \rightsquigarrow {}^{\sigma_2}M^\bullet$ in Γ. Let $\lambda = \sigma_2\sigma_1^{-1}$ and l be the order of λ in G. Then, by (a), this yields an oriented cycle

$$ {}^{\sigma_1}M^\bullet \overset{\delta}{\rightsquigarrow} {}^{\lambda\sigma_1}M^\bullet \overset{{}^{\lambda}\delta}{\rightsquigarrow} {}^{\lambda^2\sigma_1}M^\bullet \overset{{}^{\lambda^2}\delta}{\rightsquigarrow} \cdots \overset{{}^{\lambda^{l-1}}\delta}{\rightsquigarrow} {}^{\lambda^l\sigma_1}M^\bullet = {}^{\sigma_1}M^\bullet $$

in Γ, a contradiction since Γ is directed. □

We now construct a G-stable section in Γ as follows :

Definition 4.1.2. Let G, A and Γ be as above, and let $X^\bullet$ be a fixed object in Γ. We define Σ $(= \Sigma_{X^\bullet})$ to be the full subquiver of Γ formed by the objects $M^\bullet$ in Γ such that there exists a path ${}^{\sigma}X^\bullet \rightsquigarrow M^\bullet$ for some $\sigma \in G$ and any such path is sectional.

Observe that $\Sigma \subseteq \Gamma$ by (4.1.1)(a). We prove that Σ is a section in Γ.

Lemma 4.1.3. *Let Σ and Γ be as above. Then Σ intersects each τ-orbit of Γ exactly once.*

Proof. Let $M^\bullet \in \Gamma$. For each $\sigma \in G$, there exists by (the proof of) [**27**, (3.3)] an integer r_σ such that there exists a path from ${}^{\sigma}X^\bullet$ to $\tau^r M^\bullet$ in Γ if and only if $r \leq r_\sigma$. Clearly, any path from ${}^{\sigma}X^\bullet$ to $\tau^{r_\sigma}M^\bullet$ is sectional. By (4.1.1)(b), there exists an integer s which is maximal for the property that there exists a path ${}^{\sigma}X^\bullet \rightsquigarrow \tau^s M^\bullet$ in Γ, for some $\sigma \in G$. The maximality of s then gives $\tau^s M^\bullet \in \Sigma$. The uniqueness of $\tau^s M^\bullet$ follows from the fact that any two distinct objects on a common τ-orbit are related by a nonsectional path. □

Lemma 4.1.4. *Let Σ and Γ be as above.*
(a) *If $f : M^\bullet \longrightarrow N^\bullet$ is an irreducible morphism in Γ, with $M^\bullet \in \Sigma$, then $N^\bullet \in \Sigma$ or $\tau N^\bullet \in \Sigma$.*
(b) *If $f : M^\bullet \longrightarrow N^\bullet$ is an irreducible morphism in Γ, with $N^\bullet \in \Sigma$, then $M^\bullet \in \Sigma$ or $\tau^{-1}M^\bullet \in \Sigma$.*
(c) *If $n \geq 1$ and $\omega : M^\bullet — M_1^\bullet — \cdots — M_n^\bullet$ is a walk in Γ, with $M^\bullet \in \Sigma$, then $\tau^k M_n^\bullet \in \Sigma$ for some integer k. Moreover, $M^\bullet$ and $\tau^k M_n^\bullet$ belong to the same connected component of Σ.*

Proof. (a). Since Σ intersects the τ-orbit of $N^\bullet$ by (4.1.3), there exists $n \in \mathbb{Z}$ such that $\tau^n N^\bullet \in \Sigma$. Moreover, since there is an irreducible morphism from $M^\bullet$ to $N^\bullet$, then there exists such a morphism from $\tau N^\bullet$ to $M^\bullet$. Consequently, if $n > 1$, then there exists a nonsectional path from $\tau^n N^\bullet$ to $M^\bullet$ in Γ. But $\tau^n N^\bullet \in \Sigma$, and so there exists a nonsectional path from an element ${}^{\sigma}X^\bullet$ to $M^\bullet$ in Γ, a contradiction to $M^\bullet \in \Sigma$. On the other hand, if $n < 0$, then there exists a nonsectional path from $M^\bullet$ to $\tau^n N^\bullet$ and this contradicts the fact that both M and $\tau^n N^\bullet$ belong to Σ. Therefore $0 \leq n \leq 1$ and this proves (a).
(b). The proof of (b) is dual to that of (a).
(c). Let ω be as in the statement. We prove it by induction on n. If $n = 1$, then the claim follows from fullness of Σ and (a) or (b). Now, assume that the statement holds for $n-1$. So, there exists $k \in \mathbb{Z}$ such that $\tau^k M_{n-1}^\bullet$ belongs to the same connected component of Σ than $M^\bullet$. By translation, there exists an irreducible morphism between $\tau^k M_{n-1}^\bullet$ and $\tau^k M_n^\bullet$. Another application of (a) or (b) gives the result. □

Proposition 4.1.5. *Let Σ and Γ be as above. Then Σ is a G-stable section in Γ.*

Proof. First, Σ is a full subquiver of Γ by definition. Moreover, Σ contains no oriented cycles (since Γ is directed) and intersects each τ-orbits of Γ exactly once by (4.1.3). It remains to show that Σ is convex, connected and G-stable. In order to show the convexity, assume that $M_0^\bullet \longrightarrow M_1^\bullet \longrightarrow \cdots \longrightarrow M_n^\bullet$ is a path of indecomposable objects in $D^b(A)$ with $M_0^\bullet, M_n^\bullet \in \Sigma$. Let $i \in \{0, 1, \dots, n\}$. Since, $M_0^\bullet \in \Sigma$, there exists a path ${}^\sigma X^\bullet \rightsquigarrow M_0^\bullet \rightsquigarrow M_i^\bullet$ for some $\sigma \in G$. Moreover, since any such path can be extended to a path from ${}^\sigma X^\bullet$ to $M_n^\bullet$, then any such path is sectional because $M_n^\bullet \in \Sigma$. So $M_i^\bullet \in \Sigma$ and Σ is convex. About the connectedness, assume that $M^\bullet, N^\bullet$ are two objects in Σ. Since Γ is connected, there exists a walk from $M^\bullet$ to $N^\bullet$ in Γ. By (4.1.4), there exists $r \in \mathbb{Z}$ such that $\tau^r N^\bullet$ belongs to the same connected component of Σ as $M^\bullet$. Since Σ intersects each τ-orbit exactly once, we get $r = 0$, and so Σ is connected. Finally, Σ is G-stable since, for each $\sigma \in G$, the functor ${}^\sigma(-) : D^b(A) \longrightarrow D^b(A)$ commutes with the Auslander-Reiten translation τ by (2.1.4), and thus preserves the sectional paths. □

The following proposition shows how one can convert any triangle-equivalence between $D^b(A)$ and $D^b(H)$ into a G-compatible triangle-equivalence between $D^b(A)$ and $D^b(H')$, for a suitably chosen hereditary algebra H' arising from a G-stable section.

Proposition 4.1.6. *Let A be a piecewise hereditary algebra of type $\operatorname{mod} H$, for some hereditary algebra H, and G be a torsion group. Then, for any action of G on A, there exists a hereditary algebra H' and an action of G on H' inducing a G-compatible equivalence of triangulated categories between $D^b(A)$ and $D^b(H')$.*

Proof. Let A be as in the statement. Since G is a torsion group, it follows from (4.1.5) that $D^b(A)$ admits a section Σ which is G-stable. Let $H' = \operatorname{End}_{D^b(A)} \Sigma$. By Rickard's Theorem [**30**], there exists an equivalence of triangulated categories $E : D^b(A) \longrightarrow D^b(H')$ which takes Σ to the full subquiver Ω of $D^b(H')$ generated by the complexes concentrated in degree zero and having as unique nonzero cell an indecomposable projective H'-module. Now, under the identification $D^b(A) \cong D^b(H')$, $D^b(H')$ is endowed with an action of G and, for any $\sigma \in G$, we let ${}_\sigma(-) : D^b(H') \longrightarrow D^b(H')$ denotes the induced automorphism. Also, these automorphisms restrict to automorphisms of $(\operatorname{mod} H')[i]$, for any $i \in \mathbb{Z}$, by [**15**, (IV.5.1)]. In order to prove our claim, it then remains to show that there exists an action of G on H' such that the induced action on $D^b(H')$ (in the sense of Section 2.1) coincides with the action carried from $D^b(A)$.

For this sake, observe that since Σ is G-stable, then so is Ω. Moreover, Ω is the ordinary quiver associated to H', and so $H' \cong k\Omega$. We then define an action of G on H' as follows: let $\{e_1, e_2, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of H' and let $\{P_1, P_2, \dots, P_n\}$ be the associated indecomposable projective H'-modules, each of them being a vertex of Ω. Then, for $\sigma \in G$, set $\sigma(e_i) = e_j$ if ${}_\sigma P_i = P_j$. Moreover, if α is an arrow of Ω, then set $\sigma(\alpha) = {}_\sigma\alpha$. This defines an action of G on H', and further on $D^b(H')$. For each $\sigma \in G$, we let ${}^\sigma(-) : D^b(H') \longrightarrow D^b(H')$ denotes the induced automorphism. Our aim is to show that the equivalences ${}_\sigma(-)$ and ${}^\sigma(-)$ coincide, up to a functorial isomorphism, that we shall denote by ${}^\sigma\phi$.

Let $\sigma \in G$ and $P_i = H'e_i$ be an indecomposable projective H'-module (as described above). Then, taking into account the multiplication in ${}^\sigma P_i$ by the elements of H', we have an isomorphism of H'-modules ${}^\sigma\phi_{P_i} : {}^\sigma P_i \longrightarrow {}_\sigma P_i$ taking an element he_i in $H'e_i$ to the element $\sigma(h)\sigma(e_i)$. Moreover, one can verify that if $f : P \longrightarrow P'$ is a morphism between projective H'-modules, then the induced isomorphisms ${}^\sigma\phi_P$

and ${}^{\sigma}\phi_{P'}$ are such that ${}^{\sigma}f \circ {}^{\sigma}\phi_P = {}^{\sigma}\phi_{P'} \circ {}_{\sigma}f$. Now, since any H' is hereditary, this relation extends to $\operatorname{mod} H'$, that is if $f : M_1 \longrightarrow M_2$ is a morphism between H'-modules, then there exist isomorphisms ${}^{\sigma}\phi_{M_j} : {}^{\sigma}M_j \longrightarrow {}_{\sigma}M_j$, for $j = 1, 2$, such that ${}^{\sigma}f \circ {}^{\sigma}\phi_{M_1} = {}^{\sigma}\phi_{M_2} \circ {}_{\sigma}f$. Since the equivalences ${}^{\sigma}(-)$ and ${}_{\sigma}(-)$ both restrict to $(\operatorname{mod} H')[i]$ for any $i \in \mathbb{Z}$, this shows that we have a functorial isomorphism ${}^{\sigma}\phi : {}^{\sigma}(-) \longrightarrow {}_{\sigma}(-)$ when we restrict to $\cup_{i\in\mathbb{Z}}(\operatorname{mod} H')[i]$.

It then remains to show that the same holds for the morphisms linking the components $(\operatorname{mod} H')[i]$ of $D^b(H')$. To do so, it is sufficient to show that for any morphism of the form $f : I[0] \longrightarrow P[1]$, for some indecomposable injective H'-module I and indecomposable projective H'-module P, we have ${}^{\sigma}f \circ {}^{\sigma}\phi_{I[0]} = {}^{\sigma}\phi_{P[1]} \circ {}_{\sigma}f$. But this easily follows from $\operatorname{Hom}_{D^b(H')}(I[0], P[1]) \cong \operatorname{Ext}^1_{H'}(I, P)$ and the fact that any short exact sequence $0 \longrightarrow P \xrightarrow{f} E \xrightarrow{g} I \longrightarrow 0$ yields to a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^{\sigma}P & \xrightarrow{{}^{\sigma}f} & {}^{\sigma}E & \xrightarrow{{}^{\sigma}g} & {}^{\sigma}I & \longrightarrow & 0 \\ & & \cong\downarrow {}^{\sigma}\phi_P & & \cong\downarrow {}^{\sigma}\phi_E & & \cong\downarrow {}^{\sigma}\phi_I & & \\ 0 & \longrightarrow & {}_{\sigma}P & \xrightarrow{{}_{\sigma}f} & {}_{\sigma}E & \xrightarrow{{}_{\sigma}g} & {}_{\sigma}I & \longrightarrow & 0 \end{array}$$

So ${}^{\sigma}\phi : {}^{\sigma}(-) \longrightarrow {}_{\sigma}(-)$ is a functorial isomorphism, and we are done. □

As we will see, the above proposition will play a major role in the proof of Theorem 3.

4.2. The coh$\mathbb{X}$ case. We now consider the case where A is a piecewise hereditary algebra of type $\mathcal{H} = \operatorname{coh}\mathbb{X}$, for some weighted projective line $\mathbb{X}$, and G is a group acting on A. In what follows, we collect the necessary background on the categories of coherent sheaves on weighted projective lines to show that if G is a group acting on a piecewise hereditary algebra A of type $\mathcal{H}$, then the induced action of G on $D^b(\mathcal{H})$ restricts to an action of G on $\mathcal{H}$, showing that this equivalence is G-compatible. This is actually the last needed ingredient for the proof of Theorem 3.

For more details concerning the categories of coherent sheaves on a weighted projective lines, we refer to [**13, 26, 25, 28**].

Let $p_1, p_2, \ldots, p_r$ be a set of natural numbers and $\mathbb{X} = \mathbb{X}(p_1, p_2, \ldots, p_r)$ be a weighted projective line over k of type $p_1, p_2, \ldots, p_r$ (in the sense of [**13**]). Let $\mathcal{H} = \operatorname{coh}\mathbb{X}$ be the category of coherent sheaves on $\mathbb{X}$. Then $\mathcal{H}$ is a hereditary abelian category with tilting objects. In fact, it is known that there exists a tilting object $T \in \mathcal{H}$ such that $\operatorname{End}_{\mathcal{H}} T = C(p_1, p_2, \ldots, p_r)$, where $C(p_1, p_2, \ldots, p_r)$ is the canonical algebra of type $p_1, p_2, \ldots, p_r$ (in the sense of [**31**]).

Two of the main classification tools are the **slope function** $\mu : \mathcal{H} \longrightarrow \mathbb{Q} \cup \infty$ and the **Euler characteristic** $\chi_{\mathcal{H}} = 2 - \sum_{i=1}^{t}(1 - 1/p_i)$. The role of the Euler characteristic is to reflect the representation type of $\mathcal{H}$. If $\chi_{\mathcal{H}} = 0$, then all Auslander-Reiten components of $\mathcal{H}$ turn out to be tubes, we say then that $\mathcal{H}$ as **tubular type**. It is shown in [**26**, (Section 4)] that each automorphism F of $D^b(\mathcal{H})$ acts on slopes as follows:

(a) If $\mathcal{H}$ is not tubular, then there exists a rational number δ such that $\mu(FX) = \mu(X) + \delta$ for each indecomposable object X in $D^b(\mathcal{H})$.
(b) If $\mathcal{H}$ is tubular, then there exists a linear fractional transformation $g(x) = \frac{ax+b}{cx+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{SL}(2, \mathbb{Z})$ such that $\mu(FX) = g(\mu(X))$.

Then, we get the following proposition, whose proof easily follows from [**26**, (4.4)]. We include a proof for the convenience of the reader.

Proposition 4.2.1. *Let A be a piecewise hereditary algebra of type* $\text{coh}\mathbb{X}$*, for some weighted projective line* $\mathbb{X}$*, and G be a group. For any action of G on A, there exists an action of G on* $\text{coh}\mathbb{X}$ *and a G-compatible equivalence of triangulated categories between $D^b(A)$ and* $D^b(\text{coh}\mathbb{X})$.

Proof. Assume that G acts on A, and let ${}^{\sigma}(-) : D^b(A) \longrightarrow D^b(A)$ be the induced isomorphism for each $\sigma \in G$. Also, let $P_1, P_2, \ldots, P_n$ be a complete set of indecomposable projective A-modules (up to isomorphism). Since A and $\text{coh}\mathbb{X}$ are derived equivalent, it follows from Rickard's Theorem [**30**] that there exists a tilting complex $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ such that $A = \text{End}_{D^b(\text{coh}\mathbb{X})} T$. Moreover, we may assume that the equivalence sends each indecomposable direct summand T_i of T to P_i, for $i = 1, 2, \ldots, n$. With this equivalence, G acts on $D^b(\text{coh}\mathbb{X})$ and, for each $\sigma \in G$, the induced automorphism ${}^{\sigma}(-) : D^b(\text{coh}\mathbb{X}) \longrightarrow D^b(\text{coh}\mathbb{X})$ yields a permutation of $T_1, T_2, \ldots, T_n$ hence of their slopes. The above form of the action of ${}^{\sigma}(-)$ on slopes now implies that ${}^{\sigma}(-)$ is slope-preserving and then, by [**26**, (4.1)], ${}^{\sigma}(-) = T^m \circ f_{\sigma}$, where T is the translation functor of $D^b(\text{coh}\mathbb{X})$ and f_{σ} is an automorphism of $\text{coh}\mathbb{X}$. Now, since ${}^{\sigma}(-)$ permutes $T_1, T_2, \ldots, T_n$, we further deduce that $m = 0$, and thus ${}^{\sigma}(-)$ restricts to $\text{coh}\mathbb{X}$. This shows that G acts on $\text{coh}\mathbb{X}$, hence the above equivalence between $D^b(A)$ and $D^b(\text{coh}\mathbb{X})$ is G-compatible. □

We can now prove Theorem 3.

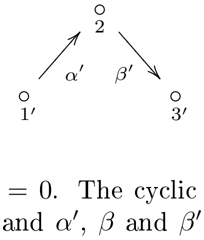
Proof of Theorem 3 : (a). Let A be as in the statement and fix an action of G on A. By (4.1.6), there exist a hereditary algebra H' and an action of G on H' such that there exists a G-compatible equivalence $D^b(H') \longrightarrow D^b(A)$. The statement now follows from Theorem 1.

(b). Let A be as in the statement and fix an action of G on A. By (4.2.1), there exist an action of G on $\text{coh}\mathbb{X}$ and a G-compatible equivalence of triangulated categories between $D^b(A)$ and $D^b(\text{coh}\mathbb{X})$. The statement now follows from Theorem 2. □

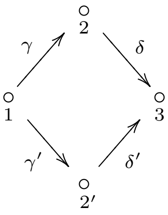
5. An example

In this section, we illustrate Theorem 3 and the mechanics of (3.1.3) by a small example.

Let A be the path algebra of the quiver

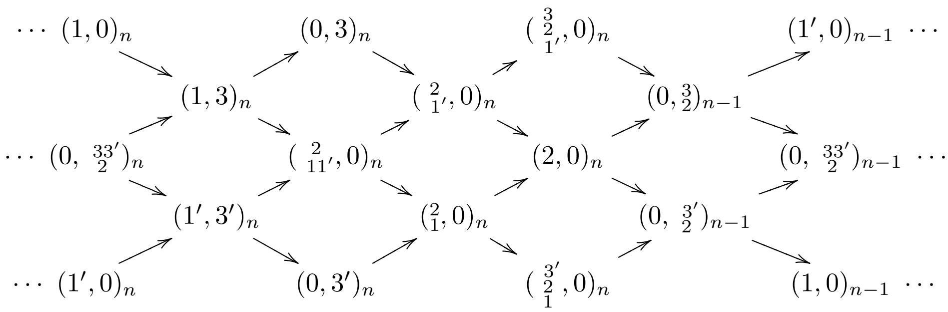


with relations $\alpha\beta = 0$ and $\alpha'\beta' = 0$. The cyclic group $G = \mathbb{Z}/2\mathbb{Z}$ acts on A by switching 1 and $1'$, 3 and $3'$, α and α', β and β', and fixing the vertex 2. Then applying the method explained in [**29**, (Section 2.3)], we get that the skew group algebra $A[G]$ is (Morita equivalent to) the path algebra of the quiver



with relations $\gamma\delta = \gamma'\delta'$.

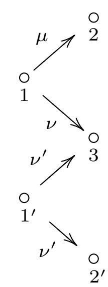
On the other hand, the Auslander-Reiten quiver of $D^b(A)$ consists of a unique directed component Γ given as follows, where the pair $(M, N)_n$ indicates that the homology in degree n is M, and the homology in degree $n+1$ is N, for some $n \in \mathbb{Z}$. The A-modules M and N are represented by their Loewy series.



Clearly, A is a piecewise hereditary algebra of type $\operatorname{mod} H$, where H is the path algebra of the quiver $Q_H = \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ$. However, each action of G on H fixes all vertices of Q_H and sends each arrow α to $\pm\alpha$. Thus, for any action of G on H, the resulting skew group algebra $H[G]$ is (Morita equivalent to) the hereditary algebra given by the disjoint union of two copies of Q_H. Therefore, $A[G]$ is not piecewise hereditary of type $\operatorname{mod} H[G]$ since, for instance, their quivers do not have the same number of vertices.

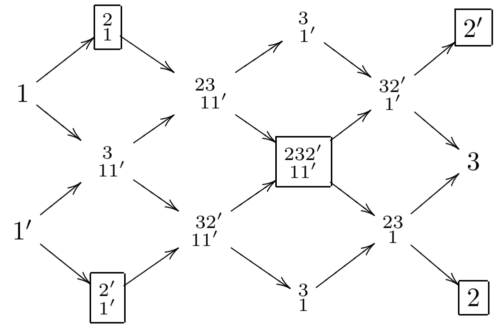
Now, let $X^\bullet$ be a fixed object in the above directed component Γ, say $X^\bullet = (1,3)_n$, and construct, as in (4.1.2), the unique section $\Sigma = \Sigma_{X^\bullet}$ of Γ having the objects of the form ${}^\sigma X^\bullet$ as sources, for $\sigma \in G$. Clearly, the induced action of $G = \mathbb{Z}/2\mathbb{Z}$ of $D^b(A)$ switches the objects $(1,3)_n$ and $(1',3')_n$, and so Σ is the full subquiver of Γ generated by the objects $(1,3)_n, (1',3')_n, (0,3)_n, (0,3')_n$ and $(\,{}^{\;2}_{11'}, 0)_n$. Now, let $H' = \operatorname{End}\Sigma$. Thus, H' is the path algebra of the quiver

$Q_{H'} =$



and, by (the proof of) (4.1.6), the action of G on H' switching 1 and 1′, 2 and 2′, μ and μ', ν and ν' and fixing the vertex 3 induces a G-compatible equivalence between $D^b(A)$ and $D^b(H')$. Then, by Theorem 1, there exist a sequence $H' = A_0, A_1, \ldots, A_n = A$ of algebras and a sequence $T_0, T_1, \ldots, T_{n-1}$ of modules such that, for each i, $A_{i+1} = \operatorname{End}_{A_i} T_i$ and T_i is a G-stable tilting A_i-module. Here,

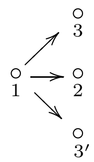
$n = 1$, and so A is a tilted algebra of type H'. Indeed, the Auslander-Reiten quiver of H' is given by



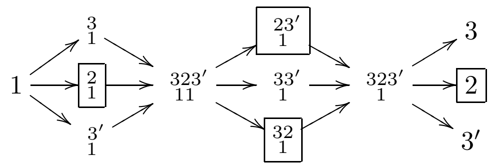
and it is easily seen that if T is the direct sum of the squared indecomposable modules in the above diagram, then T is a G-invariant tilting H'-module such that $\mathrm{End}_{H'} T \cong A$.

Now, since G is cyclic, the method explained in [**29**, (Section 2.3)] gives that $H'[G]$ is (Morita equivalent to) the path algebra of the quiver

$Q_{H'[G]} =$



Moreover, the Auslander-Reiten quiver of $H'[G]$ is given by



where the squared indecomposable modules correspond with the indecomposable direct summand of the tilting $H'[G]$-module FT of (3.1.3). It is then easily verified, as predicted by (3.1.3), that $\mathrm{End}_{H'[G]} FT \cong (\mathrm{End}_{H'} T)[G] \cong A[G]$.

In this particular example, the reader will observe that G also acts on $A[G]$. Moreover, the resulting skew group algebra $(A[G])[G]$ is Morita equivalent to A, showing that we can recover A from $A[G]$. This is not an accident. Indeed, is was shown in [**29**, (Corollary 5.2)] that if G is abelian, then G acts on $A[G]$ in such a way that $(A[G])[G]$ is Morita equivalent to A.

In fact, the following proposition, borrowed from [**29**], gives a stronger version of that observation.

Proposition 5.0.2. [**29**, (Proposition 5.4)] *If G is a finite solvable group, whose order is not a multiple of the characteristic of k, then $A[G]$ can be obtained from A using a finite number of skew group algebra constructions with cyclic groups, combined with Morita equivalences, and conversely.*

As a direct consequence of the above proposition and Theorem 3, we then get the following corollary.

Corollary 5.0.3. *Let A be an algebra and G be a finite solvable group whose order is not a multiple of the characteristic of k. Then A is a piecewise hereditary algebra of type a certain Ext-finite hereditary abelian k-category with tilting objects if and only if so is $A[G]$, where the types of A and $A[G]$ may differ.* □

Hence, the above statement shows that if G is a finite solvable group whose order is not a multiple of the order of k, then A is piecewise hereditary when so is $A[G]$.

We mention another situation where such a relation holds. Let A be an algebra, viewed as a k-category having a complete set $\{e_1, \ldots, e_n\}$ of pairwise orthogonal primitive idempotents as set of objects, whose space of morphisms from e_i to e_j is e_jAe_i and whose composition of morphisms is induced by the product of A. Assume that G is a (not necessarily finite) group acting freely on A, that is such that the action on the objects of A is free, and consider the Galois covering $A \longrightarrow A/G$, where A/G is the quotient category of A by G (see [**10**] for instance). By [**10**, (2.4)(2.8)], $A[G]$ and A/G (now viewed as an algebra) are Morita equivalent. Then, it has recently been shown by Le Meur in [**24**, (Theorem 3)] that if A/G is piecewise hereditary of type $\operatorname{mod} H$, for some hereditary algebra H (or equivalently if so is $A[G]$), then A is piecewise hereditary of type $\operatorname{mod} H'$, for a certain hereditary algebra H'.

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