Almost laura algebras

David Smith

Abstract. In this paper, we propose a generalization for the class of laura algebras of [**AC2**] and [**RS**], which we call almost laura. We show that this new class of algebras retains most of the essential features of laura algebras, especially concerning the important role played by the non-semiregular components in their Auslander-Reiten quivers. Also, we study more intensively the left supported almost laura algebras, showing that these are characterized by the presence of a generalized standard, convex and faithful component. Finally, we prove that almost laura algebras behave well with respect to full subcategories, split-by-nilpotent extensions and skew group algebras.

In the representation theory of algebras, a prevalent technique consists in modifying certain features of a well-known family of algebras in order to obtain one whose representation theory is, to a large extent, predictable. For instance, in [**HRS**], Happel, Reiten and Smalø [**HRS**] defined the quasitilted algebras (that is the endomorphism algebras of tilting objects over a hereditary abelian category), thus obtaining a common treatment of both the classes of tilted and canonical algebras. To overcome some difficulties caused by the categorical language, they introduced the left and the right parts of the module category of an algebra A, respectively denoted $\mathcal{L}_A$ and $\mathcal{R}_A$. They showed that an algebra A is quasitilted if and only if its global dimension is at most two and any indecomposable A-module lies in $\mathcal{L}_A \cup \mathcal{R}_A$.

Since then, many generalizations of quasitilted algebras, based on the behavior of $\mathcal{L}_A$ and $\mathcal{R}_A$ have appeared, such as the shod, the weakly shod, the laura and the supported algebras (see the survey [**ACLST**]). Among them, laura algebras have been introduced independently by Assem and Coelho [**AC2**] and Reiten and Skowroński [**RS**] as a generalization of representation-finite algebras and weakly shod algebras. Their nice properties have made them rather interesting and hugely investigated (see [**Sko5, AC3, ALR, LS, DS, Sm**], for instance). The aim of this paper is to introduce a new class of algebras, called almost laura, determined by the behavior of the infinite radical of $\operatorname{mod} A$ and generalizing laura algebras.

This paper is organized as follows. In Section 1, we fix the terminology and prove some preliminary results. In Section 2, we give the definition of almost laura algebras and discuss examples. In Section 3, we study the Auslander-Reiten quiver of an almost laura algebra and we classify those almost laura algebras which are laura. Section 4 is devoted to the left or right supported almost laura algebras (in the sense of [**ACT**]). Our main result (see (4.9)) is an analogue of the result of [**RS**, (3.1)] for laura algebras (see also [**LS**, (4.2.5)]), and states that if A is left (or right) supported, then A is almost laura if and only if its Auslander-Reiten quiver admits

---

2000 Mathematics Subject Classification: 16G70, 16G30, 18G05, 16E10.

Key words: almost laura algebras, infinite radical, laura algebras, left and right supported algebras, skew group algebras

a generalized standard, convex and faithful component. Finally, in Section 5, we show that almost laura algebras behave well with respect to some constructions preserving homological properties, such as dealing with full subcategories, split-by-nilpotent extensions and skew group algebras. The main result of this section states that if G is a finite group acts on an algebra A and whose order is invertible in A, then A is almost laura if and only if so is the skew group algebra $A[G]$ (see (5.11)). As a consequence, we get that the infinite radical of A is nilpotent if and only if so is the infinite radical of $A[G]$, and in this case, they have the same index of nilpotency. We also deduce that A is cycle-finite (in the sense of [**AS1**]) if and only if so is $A[G]$.

1. Preliminaries

Throughout this paper, all algebras are basic connected artin algebras over an artinian ring k. For an algebra A, we denote by $\mathrm{mod}A$ its category of finitely generated left modules and by $\mathrm{ind}A$ a full subcategory of $\mathrm{mod}\,A$ consisting of one representative from each isomorphism class of indecomposable modules. For a subcategory $\mathcal{C}$ of $\mathrm{mod}\,A$, we write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$, and denote by $\mathrm{add}\,\mathcal{C}$ the full subcategory of $\mathrm{mod}\,A$ having as objects the direct sums of indecomposable summands of objects in $\mathcal{C}$. For an A-module M, we denote by $\mathrm{pd}\,M$ its projective dimension and by $\mathrm{id}\,M$ its injective dimension. We denote by $D := \mathrm{Hom}_k(-, J) : \mathrm{mod}\,A \longrightarrow \mathrm{mod}\,A^{op}$ the usual duality, where J is the injective envelope of $k/\,\mathrm{rad}\,k$.

We denote by $\Gamma(\mathrm{mod}\,A)$ the Auslander-Reiten quiver (AR-quiver for short) of A and by $\tau_A$the usual AR-translation. By an **AR-component** Γ of $\Gamma(\mathrm{mod}A)$, we mean a connected component of $\Gamma(\mathrm{mod}A)$. Then Γ is **non-semiregular** if it contains a projective module and an injective module, and **semiregular** otherwise. Also, Γ is **faithful** if it contains a **faithful** module, that is a module M which cogenerates A. Finally, an indecomposable module $M \in \Gamma$ is **left stable** if $\tau^n M \neq 0$ for each $n \geq 0$ and we define the **left stable part** of Γ to be the full subquiver of Γ consisting of the left stable modules in Γ. We define dually the **right stable** modules and the **right stable part** of Γ.

We call **radical** of $\mathrm{mod}A$ and we denote by $\mathrm{rad}(\mathrm{mod}A)$ the ideal in $\mathrm{mod}A$ generated by all non-isomorphisms between indecomposable modules. The **infinite radical** $\mathrm{rad}^\infty(\mathrm{mod}A)$ of $\mathrm{mod}A$ is the intersection of all powers $\mathrm{rad}^n(\mathrm{mod}A)$, with $n \geq 1$, of $\mathrm{rad}(\mathrm{mod}A)$. A component Γ of $\Gamma(\mathrm{mod}\,A)$ is **generalized standard** [**Sko1**] if $\mathrm{rad}^\infty(M, N) = 0$ for each $M, N \in \Gamma$.

A **path of length** t is a sequence

$$\delta : \qquad M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = N \ , \quad (t \geq 0)$$

where $M_i \in \mathrm{ind}A$ and f_i is a non-zero morphism for each i. In this case, we write $M \overset{\delta}{\rightsquigarrow} N$ and we say that M is a **predecessor** of N and N is a **successor** of M. Following [**Sko1**], a path δ is **infinite** if $f_i \in \mathrm{rad}^\infty(\mathrm{mod}\,A)$ for some i, and **finite** otherwise. If each f_i is irreducible, δ is a **path of irreducible morphisms** and, in this case, δ is **sectional** if it contains no triple (M_{i-1}, M_i, M_{i+1}) such that $\tau_A M_{i+1} = M_{i-1}$. A **refinement** of δ is a path $M = M'_0 \xrightarrow{f'_1} M'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_s} M'_s = N$, with $s \geq t$, with an injective order-preserving function $\sigma : \{1, \ldots, t-1\} \longrightarrow \{1, \ldots, s-1\}$ such that $M_i = M'_{\sigma(i)}$ when $1 \leq i \leq t-1$. Finally, a path δ is a **cycle** if $M = N$ and at least one f_i is not an isomorphism. An A-module M is **directing** if it does not lie on any cycle and a component Γ of $\Gamma(\mathrm{mod}\,A)$ is **directed** if it contains only directing modules. Also, Γ is **almost directed** if it contains only finitely many non-directing modules, and **quasi-directed** if it is also generalized standard.

Moreover, Γ is **convex** if any path from M to N, with M, N in Γ, contains only modules from Γ.

Let A be an artin algebra. Following [**HRS**], we define the left part $\mathcal{L}_A$ and the right part $\mathcal{R}_A$ of $\operatorname{mod} A$ as follows:

$$\mathcal{L}_A = \{\ M \in \operatorname{ind} A \mid \operatorname{pd}_A N \leq 1 \text{ for each predecessor } N \text{ of } M\ \},$$
$$\mathcal{R}_A = \{\ M \in \operatorname{ind} A \mid \operatorname{id}_A N \leq 1 \text{ for each successor } N \text{ of } M\ \}.$$

The next result is helpful to detect the modules which lie in $\mathcal{L}_A$ or in $\mathcal{R}_A$.

LEMMA 1.1. [**AC2**, (1.6)] *Let A be an algebra.*

(a) *$\mathcal{R}_A$ consists of the modules $N \in \operatorname{ind} A$ such that, if there exists a path from N to an indecomposable projective module, then this path can be refined to a path of irreducible morphisms, and any such refinement is sectional.*
(b) *$\mathcal{L}_A$ consists of the modules $M \in \operatorname{ind} A$ such that, if there exists a path from an indecomposable injective module to M, then this path can be refined to a path of irreducible morphisms, and any such refinement is sectional.* □

We conclude this section with some preliminary results, needed later on.

LEMMA 1.2. *Let A be an algebra and Γ be a component of $\Gamma(\operatorname{mod} A)$. Assume that $\operatorname{rad}^{\infty}(M, N) \neq 0$ for some indecomposable modules M, N with $N \in \Gamma$. Then, for each $L \in \Gamma$, there exists $N' \in \Gamma$ such that :*

(a) *There exists a path of irreducible morphisms from N' to N;*
(b) *N' is a predecessor of L or is a predecessor of a projective module in Γ;*
(c) $\operatorname{rad}^{\infty}(M, N') \neq 0$.

Proof. Let M and N be as in the statement. By [**Sko2**, (2.1)], there exists a path of infinite length of irreducible morphisms

$$\cdots \longrightarrow N_r \xrightarrow{h_r} N_{r-1} \longrightarrow \cdots \xrightarrow{h_2} N_1 \xrightarrow{h_1} N_0 = N$$

in $\operatorname{ind} A$ such that there exists $u_r \in \operatorname{rad}^{\infty}(M, N_r)$ with $h_1 h_2 \cdots h_r u_r \neq 0$ for each $r \geq 1$. Let $L \in \Gamma$. We claim that there exists $s \geq 1$ such that N_s is a predecessor of L or is a predecessor of a projective module in Γ. Indeed, if this is not the case, then N_i is not projective for all i and it follows from [**CL**, (1.1)] that there exists an integer $r \geq 1$ which is minimal for the property that N_i is not a predecessor of $\tau^r N$ for all i. By the choice of r, there exists N_j such that N_j is a predecessor of $\tau^{r-1} N$. We claim that the path $N_m \xrightarrow{h_m} N_{m-1} \xrightarrow{h_{m-1}} \cdots \xrightarrow{h_{j+1}} N_j$ is sectional for each $m > j$. Indeed, if this is not the case, then there exists n with $j \leq n \leq m-2$ such that $N_{n+2} = \tau N_n$. This yields a path $N_{n+2} = \tau N_n \rightsquigarrow \tau N_j \rightsquigarrow \tau^r N$, a contradiction to the choice of r. In particular, $N_m \neq N_n$ whenever $m \neq n$ and $m, n \geq j$. Therefore, $\operatorname{Hom}(N_m, \tau N_n) \neq 0$ for some $m, n \geq j$ by [**Sko3**, (Lemma 2)]. Again, this yields a path from N_m to $\tau^r N$, a contradiction. Whence there exists $s \geq 1$ such that N_s is a predecessor of L or is a predecessor of a projective in Γ. □

As immediate consequences, we obtain the following corollary which generalizes results obtained in [**AC2**, (1.4)] and [**Sm**, (1.4)].

COROLLARY 1.3. *Let A be an algebra, Γ be a component of $\Gamma(\operatorname{mod} A)$ and assume that M is a non-directing module in Γ.*

(a) *If Γ contains projective modules, then there exists a path from M to a projective module in Γ.*
(b) *If Γ contains injective modules, then there exists a path from an injective module in Γ to M.*

Proof. We only prove (a) since the proof of (b) is dual.

(a). Let $M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = M$ be a cycle in $\operatorname{ind} A$. If no f_i belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$, then this cycle can be refined to a cycle of irreducible morphisms in Γ, and the result follows from [**AC2**, (1.4)]. Otherwise, we have $f_i \in \operatorname{rad}^\infty(M_{i-1}, M_i)$ for some $M_i \in \Gamma$, and it follows from (1.2) that there exists a projective module P in Γ and a path from M_{i-1} to P. This gives a path from M to P as required. □

We also deduce from (1.2) the following generalization of [**ACT**, (1.5)].

COROLLARY 1.4. *Let A be an algebra and Γ be a component of $\Gamma(\operatorname{mod} A)$.*

(a) *If Γ contains projectives, then $\mathcal{R}_A \cap \Gamma$ contains only directing modules.*
(b) *If Γ contains injectives, then $\mathcal{L}_A \cap \Gamma$ contains only directing modules.*

Proof. We only prove (a) since the proof of (b) is dual.
(a). Assume that $M \in \mathcal{R}_A \cap \Gamma$ and $\omega : M \rightsquigarrow M$ is a cycle in $\operatorname{ind} A$. By (1.3), there exists a path $M \overset{\omega}{\rightsquigarrow} M \rightsquigarrow P$ where P is projective. By (1.1), this path can be refined to a sectional path of irreducible morphisms. But this contradicts the non-sectionality of cycles [**BS, B, IT**]. □

2. Almost laura algebras : definition and examples

We recall from [**AC2**] that an artin algebra A is called **laura** if the set $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ is finite. Since the left and the right part generally behave well, the spirit of laura algebras is to deal with algebras having potentially only finitely many "unpredictable" modules. This idea behind almost laura algebras is to accept infinitely such modules but restrict their scope by adding a condition on the morphisms between them.

DEFINITION 2.1. An artin algebra is called **almost laura** if $\operatorname{rad}^\infty(M, N)$ vanishes for all $M, N \in \operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$.

In the vein of [**ACLST**], we also say that an almost laura algebra is **strict** if it is not quasitilted. The following proposition provides many equivalent useful conditions for an algebra to be almost laura.

PROPOSITION 2.2. *Let A be an algebra. The following are equivalent:*

(a) *A is almost laura.*
(b) *For all $M \in \operatorname{ind} A \setminus \mathcal{L}_A$ and $N \in \operatorname{ind} A \setminus \mathcal{R}_A$, we have $\operatorname{rad}^\infty(M, N) = 0$.*
(c) *There is no infinite path between modules in $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$.*
(d) *There is no infinite path from a module not in $\mathcal{L}_A$ to a module not in $\mathcal{R}_A$.*
(e) *There is no infinite path from an injective module to a projective module.*
(f) *There is no infinite path from a module M, with $\operatorname{pd} M \geq 2$, to a module N, with $\operatorname{id} N \geq 2$.*

Proof. The equivalence of (a), (b), (c) and (d) follows from the fact that $\mathcal{L}_A$ is closed under predecessors and $\mathcal{R}_A$ is closed under successors.
(e) implies (f). Let $M \overset{\omega}{\rightsquigarrow} N$ be a path in $\operatorname{ind} A$, with $\operatorname{pd} M \geq 2$ and $\operatorname{id} N \geq 2$. Since $\operatorname{pd} M \geq 2$, we have $\operatorname{Hom}_A(DA, \tau M) \neq 0$ and so there exists a path $\omega' : I \rightsquigarrow M$ in $\operatorname{ind} A$, for some indecomposable injective module I. Dually, there exists a path $\omega'' : N \rightsquigarrow P$ for some indecomposable projective module P. This yields a path $I \overset{\omega'}{\rightsquigarrow} M \overset{\omega}{\rightsquigarrow} N \overset{\omega''}{\rightsquigarrow} P$. By assumption, this latter path is finite, and so is ω.
(f) implies (d). This clearly follows from the definitions of $\mathcal{L}_A$ and $\mathcal{R}_A$, since any path from a module not in $\mathcal{L}_A$ to a module not in $\mathcal{R}_A$ can be extended to a path from a module having projective dimension at least two to a module having injective dimension at least two.

(d) implies (e). Let $\delta : I = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = P$ be a path in $\operatorname{ind} A$ from an injective I to a projective P. Assume that $f_i \in \operatorname{rad}^\infty(\operatorname{mod} A)$, for some $1 \leq i \leq t$. For any $n \geq 0$, it follows from [**Sm**, (1.1)] that δ may be refined to a path

$$\delta' : I = M_0 \rightsquigarrow M_{i-1} \xrightarrow{h_0} N_0 \xrightarrow{h_1} N_1 \xrightarrow{h_2} \cdots \xrightarrow{h_n} N_n \xrightarrow{g_n} M_i \rightsquigarrow P$$

where $g_n \in \operatorname{rad}^\infty(\operatorname{mod} A)$ and $N_k \neq N_l$ whenever $k \neq l$. Since there are only finitely many modules in $\mathcal{L}_A$ which are successors of an injective by [**AC2**, (1.5)] (see also [**LS**, (3.2.6)]), there exists $n \geq 0$ such that $N_n \notin \mathcal{L}_A$. Applying the dual argument to g_n yields an infinite path $\delta'' : N_n \rightsquigarrow M$, with $M \notin \mathcal{R}_A$, a contradiction to the hypothesis. □

We get the following corollary ss an immediate consequence of (2.2)(e).

COROLLARY 2.3. *If A is an almost laura algebra, then* $\operatorname{rad}^\infty(DA, A) = 0$.

REMARK 2.4. We stress that the converse of the above corollary is false, as it can be easily verified with the radical square zero algebra A given by the quiver $1 \rightrightarrows 2 \rightrightarrows 3 \rightrightarrows 4$.

We now gives few examples of almost laura algebras.

EXAMPLES 2.5.

(a) By [**AC2**, (3.3)], any laura algebra is almost laura. In particular, so is any representation-finite or quasitilted algebra.

(b) Let A be the algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "\beta_2"'] & 2 \arrow[dr, "\alpha"] & \\ & & 5 \\ 3 \arrow[r, shift left, "\delta_1"] \arrow[r, shift right, "\delta_2"'] & 4 \arrow[ur, "\gamma"'] & \end{tikzcd}$$

bound by $\alpha\beta_2 = \gamma\delta_1 = \gamma\delta_2 = 0$. Then $\Gamma(\operatorname{mod} A)$ has the shape presented in Fig. 1 below (where indecomposable modules are represented by their Loewy series), where we identify both copies of the module $\begin{smallmatrix}2\\1\end{smallmatrix}$ along the vertical dashed line, and both copies of the module 2 along the horizontal dashed line. The horizontal dotted lines represent the AR-translations. One can verify that A is an almost laura algebra, but not a laura algebra.

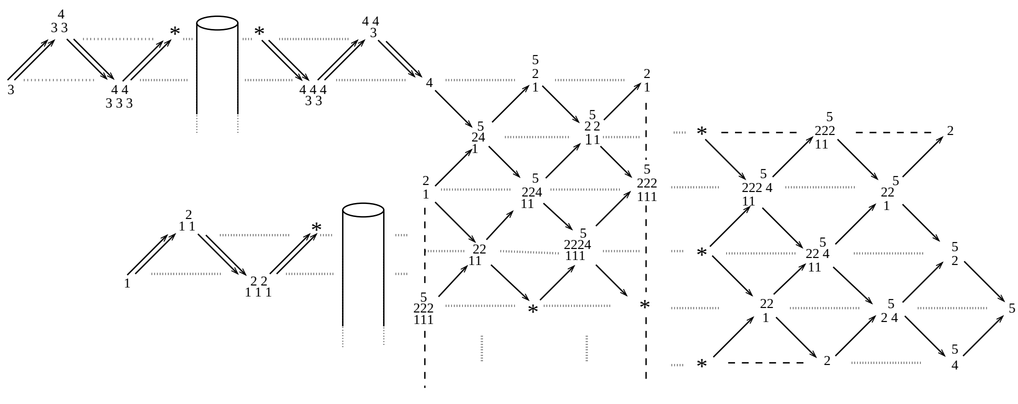

FIGURE 1. $\Gamma(\operatorname{mod} A)$

In this latter example, the algebra has been obtained by performing a one-point extension in a chosen homogeneous tube of the Kronecher algebra formed by the vertices 1 and 2, and by "gluing" another Kronecker algebra to the resulting ray tube. Repeating the same procedure in another homogeneous tube would result in an almost laura algebra having two non-semiregular components. Since there are infinitely many such tubes, this shows that one can construct almost laura algebras having arbitrarily many non-semiregular components.

We would like to propose the following problem, which is is an analogue to the Skowroński's conjecture for laura algebras [**Sko5**].

PROBLEM 1. *Let A be an algebra. Are the following conditions equivalent?*

(a) *A is almost laura.*
(b) $\operatorname{rad}^{\infty}(M,N)=0$ *for all* $M,N\in\operatorname{ind}A$, *with* $\operatorname{pd}M\geq 2$ *and* $\operatorname{id}N\geq 2$.
(c) *There is no infinite path from a module* $M\in\operatorname{ind}A$ *with* $\operatorname{pd}M\geq 2$ *and* $\operatorname{id}M\geq 2$ *to a module* $N\in\operatorname{ind}A$ *with* $\operatorname{pd}N\geq 2$ *and* $\operatorname{id}N\geq 2$.

3. Those almost laura algebras which are laura

The definition of almost laura algebras is closely related to that of laura algebras. In this section, we are interested in determining when an almost laura algebra is laura. We recall that strict laura algebras are characterized by the existence of a unique non-semiregular component in their AR-quiver, which is moreover quasi-directed and faithful (see [**AC2, RS**]). Our approach consists in studying the behavior of the non-semiregular components in the AR-quiver of almost laura algebras. As we shall see, those components behave similarly as for laura algebras. We infer some characterizations of almost laura algebras which are laura. Our results on the non-semiregular components will also play a major role in Section 4.

3.1. Non-semiregular components and almost laura algebras. We begin our investigation of non-semiregular components over almost laura algebras with the following key lemma, whose proof is a routine application of (2.2) and (1.2). We leave the verification to the reader.

LEMMA 3.1. *An algebra is almost laura if and only if there is no infinite path from a module M lying in a component containing injectives to a module N lying in a component containing projectives.* □

As a first application, we get the following corollary.

COROLLARY 3.2. *Let A be an almost laura algebra. If $M\in\operatorname{ind}A\setminus(\mathcal{L}_A\cup\mathcal{R}_A)$, then M belongs to a non-semiregular component of $\Gamma(\operatorname{mod}A)$.*

Proof. Let $M\notin\mathcal{L}_A\cup\mathcal{R}_A$. Then there exists a path $I\rightsquigarrow M\rightsquigarrow P$ for some injective module I and some projective module P. Since A is almost laura, this path is finite by (2.2) and so I and P belong to the same component as M. □

We also deduce the following result whose proof is immediate from the lemma.

PROPOSITION 3.3. *Let A be an almost laura algebra. Each non-semiregular component of $\Gamma(\operatorname{mod}A)$ is generalized standard and convex.* □

The above result has a direct nice consequence. In fact, a well-known conjecture in representation theory of algebras states that if A is an algebra having a connected AR-quiver, then A is representation-finite. The following provides a positive solution for almost laura algebras (and thus also for laura algebras).

COROLLARY 3.4. *Let A be an almost laura algebra such that $\Gamma(\operatorname{mod}A)$ is connected. Then A is representation-finite.*

Proof. Let A be as in the statement, and set $\Gamma=\Gamma(\operatorname{mod}A)$. If A is representation-infinite, then $\operatorname{rad}^{\infty}(\operatorname{mod}A)\neq 0$ by Auslander's theorem (see [**ARS**, (V.7.7)]), and so there exist $X, Y \in \Gamma$ such that $\mathrm{rad}^\infty(X,Y) \neq 0$. But Γ is non-semiregular, a contradiction to (3.3). Hence A is representation-finite. $\square$

Coming back to our original aim, we recall from [**CS**] that the AR-quiver of a quasitilted algebra generally does not contain non-semiregular components. It however turns out that, as for laura algebras (see [**AC2**]), any strict almost laura algebra admits such components.

PROPOSITION 3.5. *Let A be a strict almost laura algebra. Then $\Gamma(\mathrm{mod}A)$ has non-semiregular components, and these are generalized standard and convex.*

Proof. Since A is not quasitilted, it follows from [**HRS**, (II.1.14)] that there exists an indecomposable projective module P not lying in $\mathcal{L}_A$. So, there is a path from an injective module I to P in $\mathrm{ind}\, A$. Since A is almost laura, the modules P and I belong to the same component of $\Gamma(\mathrm{mod}\, A)$, which is thus non-semiregular, and it is generalized standard and convex by (3.3). $\square$

For the remaining part of this subsection, we let A be an almost laura algebra and Γ be a non-semiregular component of $\Gamma(\mathrm{mod}\, A)$. Here and in the sequel, we also use the following notation: if $\mathcal{A}$ and $\mathcal{B}$ are two classes of A-modules, then we write $\mathrm{Hom}_A(\mathcal{A}, \mathcal{B}) \neq 0$ to express that there exists a non-zero morphism from a module in $\mathcal{A}$ to a module in $\mathcal{B}$.

The following are generalizations of [**AC2**, (4.1)] and [**AC2**, (4.2)]. The proof of the lemma follows directly from (3.1) and it is omitted.

LEMMA 3.6. *Let A and Γ be as above.*

(a) *Assume that I is an indecomposable injective module such that there exists a path $I \rightsquigarrow M$ with $M \in \Gamma$, then I belongs to Γ.*
(b) *Assume that P is an indecomposable projective module such that there exists a path $M \rightsquigarrow P$ with $M \in \Gamma$, then P belongs to Γ.* $\square$

PROPOSITION 3.7. *Let A and Γ be as above, and let Γ' be a component of $\Gamma(\mathrm{mod}\, A)$ distinct from Γ.*

(a) *If* $\mathrm{Hom}_A(\Gamma', \Gamma) \neq 0$, *then* $\Gamma' \subseteq \mathcal{L}_A \setminus \mathcal{R}_A$.
(b) *If* $\mathrm{Hom}_A(\Gamma, \Gamma') \neq 0$, *then* $\Gamma' \subseteq \mathcal{R}_A \setminus \mathcal{L}_A$.
(c) *Either* $\mathrm{Hom}_A(\Gamma', \Gamma) = 0$, *or* $\mathrm{Hom}_A(\Gamma, \Gamma') = 0$.

Proof. (a). Let $M, M' \in \Gamma'$, $N \in \Gamma$ and assume that $0 \neq f \in \mathrm{Hom}_A(M, N)$. We need to show that $M' \in \mathcal{L}_A \setminus \mathcal{R}_A$. Clearly $f \in \mathrm{rad}^\infty(\mathrm{mod}\, A)$. By (1.2), there exists $N' \in \Gamma$ such that N' is a predecessor of a projective P in Γ and $\mathrm{rad}^\infty(M, N') \neq 0$. Dually, there exists $M'' \in \Gamma'$ such that M'' is a successor of M' or a successor of an injective module in Γ' and $\mathrm{rad}^\infty(M'', N') \neq 0$. By (3.1), M'' is not a successor of an injective. So we get a path $M' \rightsquigarrow M'' \xrightarrow{g} N' \rightsquigarrow P$ where g is a non-zero morphism in $\mathrm{rad}^\infty(M'', N')$. Then $M' \in \mathcal{L}_A \setminus \mathcal{R}_A$ by (1.1). So $\Gamma' \subseteq \mathcal{L}_A \setminus \mathcal{R}_A$.
(b). The proof is dual to that of (a).
(c). This follows directly from (a) and (b). $\square$

We prove in (4.7) below a stronger version of this result when A is left or right supported. We conclude with an observation on semiregular components.

PROPOSITION 3.8. *Let A be an almost laura algebra and Γ' be a semiregular component of $\Gamma(\mathrm{mod}\, A)$.*

(a) $\Gamma' \subseteq \mathcal{L}_A \cup \mathcal{R}_A$.
(b) *If Γ' contains injectives but no projectives, then $\Gamma' \subseteq \mathcal{R}_A$.*
(c) *If Γ' contains projectives but no injectives, then $\Gamma' \subseteq \mathcal{L}_A$.*
(d) *If Γ' is regular, that is it contains neither injectives nor projectives, then Γ' lies in $\mathcal{L}_A \setminus \mathcal{R}_A$, $\mathcal{R}_A \setminus \mathcal{L}_A$ or $\mathcal{L}_A \cap \mathcal{R}_A$.*

Proof. (a). This directly follows from (3.2).
(b). Assume that M is a module in Γ' which does belong to $\mathcal{R}_A$. By (1.1) there exists a path δ from M to a projective module P. Since $P \notin \Gamma'$ by assumption, this path is infinite. By the dual of (1.2), there exists an infinite path from an injective module in Γ' to P, which contradicts the fact that A is almost laura by (2.2).
(c). The proof is dual to that of (b).
(d). In view of (a), it suffices to show that if $\Gamma' \cap \mathcal{L}_A \neq \emptyset$ (or $\Gamma' \cap \mathcal{R}_A \neq \emptyset$), then $\Gamma' \subseteq \mathcal{L}_A$ (or $\Gamma' \subseteq \mathcal{R}_A$ respectively). Assume that $\Gamma' \cap \mathcal{L}_A \neq \emptyset$ and let $M, N \in \Gamma'$ be such that $M \in \mathcal{L}_A$. If $N \notin \mathcal{L}_A$, then there exists by (1.1) a path δ from N to an injective module I. But then, since $I \notin \Gamma'$, this path is infinite and it follows from (1.2) that there exists an infinite path from M to I, contradicting the fact that $M \in \mathcal{L}_A$. So $\Gamma' \subseteq \mathcal{L}_A$. Similarly $\Gamma' \cap \mathcal{R}_A \neq \emptyset$ implies $\Gamma' \subseteq \mathcal{R}_A$. □

3.2. On almost laura algebras which are laura. In this section, we provide necessary and sufficient conditions for an almost laura algebra to be laura. We also deduce from these new characterizations of laura and weakly shod algebras. We begin with the following key lemma.

LEMMA 3.9. *Let A be an algebra and Γ be a generalized standard and convex component of $\Gamma(\operatorname{mod} A)$. For all $L, N \in \Gamma$, there are only finitely many directing modules M lying on a path $L \rightsquigarrow M \rightsquigarrow N$.*

Proof. Let $L, N \in \Gamma$ and assume to the contrary that there exists an infinite set of indecomposable directing modules $\mathcal{M} = \{M_\lambda\}_{\lambda \in \Lambda}$ such that, for each $\lambda \in \Lambda$, there is a path $L \rightsquigarrow M_\lambda \rightsquigarrow N$ in $\operatorname{ind} A$. Since $\mathcal{M}$ is infinite and Γ has only finitely many non-periodic τ-orbits by [**Sko1**, (2.3)], there exists an orbit $\mathcal{O}$ of Γ with $|\mathcal{O} \cap \mathcal{M}| = \infty$. Let $M \in \mathcal{O}$ and assume without loss of generality that $\tau^m M \in \mathcal{M}$ for infinitely many $m \geq 0$. Then, M is left stable. Let ${}_l\Gamma$ be the connected component of the left stable part of Γ containing M. It then easily follows from [**CS**, (1.4)] that ${}_l\Gamma$ contains no cycle and ${}_l\Gamma$ has only finitely many τ-orbits. Then, ${}_l\Gamma$ admits a section Δ such that ${}_l\Gamma$ is isomorphic to a full subquiver of $\mathbb{Z}\Delta$, and is closed under predecessors by paths of irreducible morphisms (see [**L1**, (3.4)]). Moreover, for any predecessors Q, Q' of Δ, there exist at most finitely many integers $n \geq 0$ such that Q is a predecessor of $\tau^n Q'$. However, since there exists $s \geq 0$ such that $\tau^m M$ is a predecessor of Δ for all $m \geq s$, and since Γ is generalized standard and convex, L and $\tau^s M$ are two predecessors of Δ such that L is a predecessor of infinitely many $\tau^m M$, with $m \geq s$, a contradiction. □

PROPOSITION 3.10. *Let A be an almost laura algebra. Then A satisfies the following equivalent conditions:*

(a) $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ *contains only finitely many directing A-modules.*
(b) *There are only finitely many indecomposable directing A-modules M with a path $I \rightsquigarrow M \rightsquigarrow P$ in* $\operatorname{ind} A$ *where I is an injective module and P a projective module.*
(c) *There are only finitely many indecomposable directing A-modules M with a path $L \rightsquigarrow M \rightsquigarrow N$ in* $\operatorname{ind} A$ *where $L \notin \mathcal{L}_A$ and $N \notin \mathcal{R}_A$.*

Proof. We first show the equivalence of statement (a), (b) and (c).
(a) implies (b). This follows from the fact that any injective module (or projective module) has only finitely many successors (or predecessors) lying in $\mathcal{L}_A$ (or in $\mathcal{R}_A$, respectively) by [**AC2**, (1.5)] (see also [**LS**, (3.2.6)]).
(b) implies (c). This follows from (1.1).
(c) implies (a). Assume $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ contains an infinite class $(M_\lambda)_{\lambda \in \Lambda}$ of directing modules. The set of trivial paths $M_\lambda \xrightarrow{\mathrm{id}} M_\lambda \xrightarrow{\mathrm{id}} M_\lambda$ then contradicts (c).

Now, assume that A is an almost laura algebra not satisfying the condition (b). Then, there exist an injective I, a projective P and infinitely many directing modules M lying on a path $I \rightsquigarrow M \rightsquigarrow P$. By (2.2) and (3.1), all these modules, including I and P, belong to a unique component Γ of $\Gamma(\operatorname{mod} A)$. By (3.3), Γ is generalized standard and convex. This contradicts (3.9). $\square$

As a consequence, we get the following theorem:

THEOREM 3.11. *The following are equivalent for an almost laura algebra A.*

(a) *A is laura.*
(b) $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ *contains only finitely many non-directing modules.*
(c) *Any non-semiregular component of* $\Gamma(\operatorname{mod} A)$ *is almost directed.*
(d) *Any non-semiregular component of* $\Gamma(\operatorname{mod} A)$ *is quasi-directed.*

Proof. (a) implies (b). This is obvious.
(b) implies (d). Assume that Γ is a non-semiregular component of $\Gamma(\operatorname{mod} A)$ and M is a non-directing module in Γ. By (1.4), $M \in \operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ and the claim follows from the assumption and (3.3).
(d) implies (c). This is obvious.
(c) implies (a). Assume that A is not laura. So $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ is infinite and, by (1.1), there exist an injective module I, a projective module P and infinitely many modules M lying on a path $I \rightsquigarrow M \rightsquigarrow P$. By assumption, we may assume that these modules are directing. Since A is almost laura, it follows from (2.2) and (3.1) that all these modules, including I and P, belong to the same component Γ of $\Gamma(\operatorname{mod} A)$. By (3.3), Γ is generalized standard and convex, which contradicts (3.9). $\square$

We get a similar characterization of almost laura algebras which are weakly shod. Recall from [**CL**] that an artin algebra A is **weakly shod** if and only if it is laura and none of the non-semiregular components of $\Gamma(\operatorname{mod} A)$ contains cycles. Recall also from [**CL**] that a non-semiregular component Γ is **pip-bounded** if there exists an n_0 such that any path of non-isomorphisms in $\operatorname{ind} A$ from an injective module in Γ to a projective module in Γ has length at most n_0.

PROPOSITION 3.12. *The following are equivalent for an almost laura algebra A.*

(a) *A is weakly shod.*
(b) $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ *contains only directing modules.*
(c) *Any non-semiregular component of* $\Gamma(\operatorname{mod} A)$ *is directed.*
(d) *Any non-semiregular component of* $\Gamma(\operatorname{mod} A)$ *is pip-bounded.*

Proof. (a) implies (c). This follows from the above discussion.
(c) implies (d). This follows from (3.3) and [**LS**, (4.2.6)] (see also [**Sm**, (3.12)]).
(d) implies (b). Assume that M is a non-directing module in $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$. By (1.1), there exists a path $I \rightsquigarrow M \rightsquigarrow P$ in $\operatorname{ind} A$ for some injective module I and projective module P. Since A is almost laura, the modules I, M and P belong to the same component Γ of $\Gamma(\operatorname{mod} A)$, which is therefore non-semiregular. Obviously, Γ is not pip-bounded, a contradiction.
(b) implies (a). By (3.11), A is laura. Now, assume that Γ is a non-semiregular component of $\Gamma(\operatorname{mod} A)$ containing a non-directing module M. By (1.3), there exist an indecomposable injective I, a projective module P and a path $I \rightsquigarrow M \rightsquigarrow P$. By non-sectionality of cycles [**BS, B, IT**] and (1.1), we get $M \in \operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$, a contradiction. So A is weakly shod. $\square$

The preceding results provide new characterizations for laura and weakly shod algebras. We need one further lemma.

LEMMA 3.13. *Let A be an algebra such that* $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ *contains only finitely many non-directing modules. Then A is almost laura.*

Proof. Assume that A is not almost laura. Then, there exist $L, N \notin \mathcal{L}_A \cup \mathcal{R}_A$ such that $\operatorname{rad}^{\infty}(L, N) \neq 0$. Invoking [**LS**, (4.2.2)], there exist infinitely many non-directing modules M_λ lying on a path from L to N. Since $\mathcal{L}_A$ is closed under predecessors and $\mathcal{R}_A$ is closed under successors, we have $M_\lambda \notin \mathcal{L}_A \cup \mathcal{R}_A$ for any λ. This contradicts our assumption, and so A is almost laura. □

We get the following result whose proof follows from (3.11), (3.12) and (3.13).

COROLLARY 3.14. *Let A be an algebra.*

(a) *A is laura if and only if* $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ *contains only finitely many non-directing modules.*
(b) *A is weakly shod if and only if* $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ *contains only directing modules.* □

3.3. Left glued algebras revisited. A particular class of laura algebras is given by the so-called left (or right) glued algebras. Recall from [**AC1, AC2**] that an algebra A is called **left glued** if the set $\operatorname{ind} A \setminus \mathcal{R}_A$ is finite. The **right glued** algebras are defined dually. The origin of their names comes from the fact that, roughly speaking, the AR-quiver of any left glued algebra is obtained by "gluing", on the left-hand-side of the AR-quiver of a representation-finite algebra, some AR-components (without injectives) arising from tilted algebras (see [**AC1**] for details).

It is well-known that left glued (or right glued) algebras are characterized by the existence, in their AR-quiver, of a faithful π-component (or ι-component respectively). Recall from [**C**] that an AR-component Γ is called a **π-component** (or a **ι-component**) provided all but finitely many modules in Γ are directing and lie in the τ-orbit of a projective (or an injective, respectively). We refer to [**ACLST, L2**] for more details concerning left (or right) glued algebras.

The aim of this section is to show that, although laura and almost laura algebras differ from many points of view, the "left glued" and "right glued" versions for almost laura algebras coincide with the usual left and right glued algebras arising from laura algebras.

THEOREM 3.15. *Let A be an algebra.*

(a) *A is left glued if and only if* $\operatorname{rad}^{\infty}(M, N) = 0$ *provided* $M, N \in \operatorname{ind} A \setminus \mathcal{R}_A$.
(b) *A is right glued if and only if* $\operatorname{rad}^{\infty}(M, N) = 0$ *provided* $M, N \in \operatorname{ind} A \setminus \mathcal{L}_A$.

Proof. We only prove (a) since the proof of (b) is dual.
(a). The necessity clearly follows from the definition of left glued algebras and [**Sm**, (1.1)], for instance. Conversely, assume that $\operatorname{rad}^{\infty}(M, N) = 0$ provided $M, N \in \operatorname{ind} A \setminus \mathcal{R}_A$. If $\operatorname{ind} A = \mathcal{R}_A$, then there is nothing to prove. Otherwise, let $M \in \operatorname{ind} A \setminus \mathcal{R}_A$ and Γ be the AR-component containing M. We show that Γ is a faithful π-component. Let P be an indecomposable projective module such that $\operatorname{Hom}(P, M) \neq 0$. Since $M \notin \mathcal{R}_A$ and $\mathcal{R}_A$ is closed under successors, we have $P \notin \mathcal{R}_A$. It then follows from our hypothesis that $\operatorname{rad}^{\infty}(P, M) = 0$, and so P lies in Γ. So Γ contains projective modules. We now claim that Γ contains every projective module. Indeed, if this is not that case, then there exist a projective module P in Γ and a projective module P' not in Γ such that $\operatorname{rad}^{\infty}(P, P') \neq 0$ or $\operatorname{rad}^{\infty}(P', P) \neq 0$. Assume that $\operatorname{rad}^{\infty}(P, P') \neq 0$. Then, since there are only finitely many predecessors of P' lying in $\mathcal{R}_A$ by [**AC2**, (1.5)] and [**LS**, (3.2.6)], it follows from [**Sm**, (1.1)], for instance, that there exists a predecessor N of P' such that $N \notin \mathcal{R}_A$ but $\operatorname{rad}^{\infty}(P, N) \neq 0$, which contradicts our hypothesis. The same argument shows that $\operatorname{rad}^{\infty}(P', P) \neq 0$. So Γ contains every indecomposable projective module. In particular, Γ is faithful.

Moreover, we have $\mathrm{rad}^{\infty}(-,\Gamma)=0$. Indeed, assume that $\mathrm{rad}^{\infty}(M',N')\neq 0$ for some indecomposables M',N' with $N'\in\Gamma$. Then, invoking (1.2), and recalling that there exist only finitely many predecessors of a projective module lying in $\mathcal{R}_A$, there exists a projective module P'' in Γ and an indecomposable module $M''\notin\mathcal{R}_A$ such that $\mathrm{rad}^{\infty}(M'',P'')\neq 0$. This contradicts our assumption. Hence $\mathrm{rad}^{\infty}(-,\Gamma)=0$, and it then follows from [**L2**, (2.1)-(2.3)] that Γ is a π-component. Since Γ is also faithful, then A is left glued. □

4. Supported almost laura algebras

As pointed out in the discussion following (2.5), the AR-quivers of almost laura algebras usually have many non-semiregular components. It is also easy to construct examples of almost laura algebras having multicoils (in the sense of [**AS2**]). With this in mind, it seems that the general shape of the AR-quiver of an almost laura algebra is not easy to describe. In this section, we propose to study the AR-quiver of left (or right) supported almost laura algebras (in the sense of [**ACT**]).

Informally, left (or right) supported algebras A are those whose left part $\mathcal{L}_A$ (or right part $\mathcal{R}_A$) "behaves well". For instance, any strict laura algebra is left and right supported (see [**ACT**, (4.4)]). This is however not true for almost laura algebras, as we will see, and this additional assumption will be very useful in our attempt to describe their AR-quivers. The main result of this section is an analogue to the result of [**RS**, (3.1)] for laura algebras (see also [**LS**, (4.2.5)]), and states that if A is left (or right) supported, then A is almost laura if and only if its AR-quiver admits a generalized standard, convex and faithful component (see (4.9)).

Here, we recall basic features needed in the subsequent developments. For a full account, we refer to [**ACT, ACLST**]. By [**AS3**], a full subcategory $\mathcal{C}$ of $\mathrm{mod}\,A$ is **contravariantly finite** if for any $N\in\mathrm{mod}\,A$, there exists a morphism $f_{\mathcal{C}}:M_{\mathcal{C}}\longrightarrow N$, with $M_{\mathcal{C}}\in\mathcal{C}$, such that any morphism $f:M\longrightarrow N$, with $M\in\mathcal{C}$, factors through $f_{\mathcal{C}}$. The dual notion is that of a **covariantly finite** subcategory. Following [**ACT**], an artin algebra A is called **left supported** in case $\mathrm{add}\,\mathcal{L}_A$ is contravariantly finite in $\mathrm{mod}\,A$. We define dually the **right supported** algebras. In what follows, the dual statements for right supported algebras hold as well.

In order to have a better description of left supported algebras, we define, following [**ACT**], two subclasses of $\mathcal{L}_A$:

$\varepsilon_1=\{\ M\in\mathcal{L}_A\mid$ there exists an injective I and a path of irreducible morphisms $I\rightsquigarrow M\}$, and
$\varepsilon_2=\{\ M\in\mathcal{L}_A\setminus\varepsilon_1\mid$ there exists a projective $P\notin\mathcal{L}_A$ and a path of irreducible morphisms $P\rightsquigarrow\tau^{-1}M\}$.

Moreover, we set $\varepsilon=\varepsilon_1\cup\varepsilon_2$. We also denote by E the direct sum of all indecomposable A-modules lying in ε and by F the direct sum of a full set of representatives of the isomorphism classes of indecomposable projective A-modules not lying in $\mathcal{L}_A$. Finally, we set $T=E\oplus F$. The following summarizes some characterizations of left supported algebras, as stated and proved in [**ACT**, (Theorem A)] and [**A**].

THEOREM 4.1. *Let A be an algebra. The following are equivalent:*

(a) *A is left supported.*
(b) $\mathrm{add}\,\mathcal{L}_A$ *coincides with the set* $\mathrm{Cogen}E$ *of A-modules cogenerated by E.*
(c) *$T=E\oplus F$ is a tilting A-module.*
(d) *Every morphism $f:M\longrightarrow N$ in* $\mathrm{ind}\,A$, *with $M\in\mathcal{L}_A$ and $N\notin\mathcal{L}_A$ factors through* $\mathrm{add}\,E$. □

REMARK 4.2. Strict almost laura algebras are not left supported in general. Indeed, for the almost laura algebra of (2.5)(b), it is easily verified that

$T = \begin{smallmatrix} 44 \\ 3 \end{smallmatrix} \oplus 4 \oplus \begin{smallmatrix} & 5 & \\ 2 & & 4 \\ & 1 & \end{smallmatrix}$. Since T admits less indecomposable direct summands than the number of non-isomorphic simple modules, T is not a tilting module. So A is not left supported by the above theorem.

We begin the study of left supported almost laura algebras with the following lemma. In the sequel, we write $M \Subset N$ to express that an A-module M is a direct summand of an A-module N.

LEMMA 4.3. *Let A be an almost laura algebra. If $M \Subset T$, then the component containing M also contains injective modules.*

Proof. If $M \in \varepsilon_1$, this follows from the definition of ε_1. If $M \in \varepsilon_2$, then there is a projective module $P \notin \mathcal{L}_A$ and a path of irreducible morphism $P \rightsquigarrow \tau^{-1}M$. Since $P \notin \mathcal{L}_A$, it follows from (1.1) that there is an injective module I and a path $I \rightsquigarrow P$. Since A is almost laura, I, P and N belong to the same component of $\Gamma(\operatorname{mod} A)$ by (2.2). Finally, if $M \Subset F$, then M is a projective module not in $\mathcal{L}_A$. A repetition of the above argument leads to the result. □

As a consequence, we obtain the following very useful result.

PROPOSITION 4.4. *Let A be a left supported almost laura algebra.*

(a) *If A is quasitilted, then A is tilted and there exists a connecting component Γ of $\Gamma(\operatorname{mod} A)$ containing every indecomposable direct summand of T. In particular, Γ is faithful.*
(b) *If A is not quasitilted, then $\Gamma(\operatorname{mod} A)$ has a unique non-semiregular component Γ. Moreover, Γ contains every indecomposable direct summand of T and is faithful.*

Proof. (a). If A is quasitilted, then A is tilted having ε as complete slice by [**Sm**, (3.8)]. Since $F = 0$ in this case, the result follows at once.
(b). If A is not quasitilted, let Γ be a non-semiregular component as in (3.5). Then, T admits an indecomposable direct summand in Γ. Indeed, let P be a projective module in Γ. If $P \notin \mathcal{L}_A$, then $P \Subset F$, and we are done. Otherwise, $P \in \mathcal{L}_A$, and since Γ contains injective modules, it follows from [**ACT**, (3.5)] that $\Gamma \cap \varepsilon \neq \emptyset$. We now show that Γ contains every indecomposable direct summand of T. Indeed, if this is not the case, then there exists an indecomposable direct summand T' of T such that $\operatorname{rad}^\infty(\Gamma, T') \neq 0$ or $\operatorname{rad}^\infty(T', \Gamma) \neq 0$ (since $\operatorname{End}_A T$ is connected). Since the component containing T' contains injective modules by (4.3), we have $\operatorname{rad}^\infty(T', \Gamma) = 0$ by (3.1). So $\operatorname{rad}^\infty(\Gamma, T') \neq 0$. Applying (3.7), we get $T' \in \mathcal{R}_A \backslash \mathcal{L}_A$, and so $T' \Subset F$. But then T' is projective and we get a contradiction to (3.1). This proves our claim. Finally, since T is a faithful module, then so is Γ. □

COROLLARY 4.5. *Let A be a left supported almost laura algebra. Assume that Γ is a non-semiregular component of $\Gamma(\operatorname{mod} A)$ and $M \in \operatorname{ind} A$.*

(a) *$\mathcal{L}_A \cap \mathcal{R}_A$ is finite and lies in Γ.*
(b) *If $M \notin \mathcal{L}_A \cup \mathcal{R}_A$, then $M \in \Gamma$.*
(c) *If $M \notin \Gamma$, then $M \in \mathcal{L}_A \setminus \mathcal{R}_A$ or $M \in \mathcal{R}_A \setminus \mathcal{L}_A$.*

Proof. (a). Let $M \in \mathcal{L}_A \cap \mathcal{R}_A$, and assume that $M \notin \Gamma$. Since $M \in \operatorname{Cogen} E$ and $\varepsilon \subseteq \Gamma$, we have $\operatorname{Hom}_A(M, \Gamma) \neq 0$. By (3.7), we obtain $M \notin \mathcal{R}_A$, a contradiction. Now, assume to the contrary that $\mathcal{L}_A \cap \mathcal{R}_A$ is infinite. Since Γ has only finitely many non-periodic τ-orbits by [**Sko1**, (2.3)], there exists a τ-orbit $\mathcal{O}$ of Γ such that $|\mathcal{O} \cap (\mathcal{L}_A \cap \mathcal{R}_A)| = \infty$. Let $M \in \mathcal{O}$ and assume, without loss of generality, that $\tau^m M \in \mathcal{L}_A \cap \mathcal{R}_A$ for infinitely many $m \leq 0$. Then, M is right stable and it follows from [**CL**, (1.1)] that there exists $n \leq 0$ such that $\tau^n M$ is a successor of an injective

module in Γ. By (1.1), we have $\tau^{n-1}M \notin \mathcal{L}_A$. But this contradicts our assumption on M. So $\mathcal{L}_A \cap \mathcal{R}_A$ is finite.
(b). This follows from (3.8)(a).
(c). This follows from (a) and (b). □

This yields the following structure results.

LEMMA 4.6. *Let A be a left supported almost laura algebra. Assume that Γ is a non-semiregular component of $\Gamma(\operatorname{mod} A)$. Let $M \in \operatorname{ind} A$. If $M \notin \Gamma$, then*

(a) $\operatorname{Hom}_A(M, \Gamma) \neq 0$ *if and only if* $M \in \mathcal{L}_A \setminus \mathcal{R}_A$.
(b) $\operatorname{Hom}_A(\Gamma, M) \neq 0$ *if and only if* $M \in \mathcal{R}_A \setminus \mathcal{L}_A$.
(c) *Either* $\operatorname{Hom}_A(M, \Gamma) \neq 0$ *and* $\operatorname{Hom}_A(\Gamma, M) = 0$, *or* $\operatorname{Hom}_A(M, \Gamma) = 0$ *and* $\operatorname{Hom}_A(\Gamma, M) \neq 0$.

Proof. (a). Since the necessity follows from (3.7), assume that $M \in \mathcal{L}_A \setminus \mathcal{R}_A$. Since $M \in \mathcal{L}_A \subseteq \operatorname{Cogen} E$ and $\varepsilon \subseteq \Gamma$, we have $\operatorname{Hom}_A(M, \Gamma) \neq 0$.
(b). Since the necessity follows from (3.7), assume that $M \in \mathcal{R}_A \setminus \mathcal{L}_A$. Let P be an indecomposable projective module such that there exists a non-zero morphism $\pi : P \longrightarrow M$. If $P \in \mathcal{L}_A$, then π factors through $\operatorname{add} E$ by (4.1) and so $\operatorname{Hom}_A(\Gamma, M) \neq 0$ since $\varepsilon \subseteq \Gamma$ by (4.4). Otherwise, $P \in F$, and so $P \in \Gamma$. Consequently, $\operatorname{Hom}_A(\Gamma, M) \neq 0$.
(c). By (4.5), we have $M \in \mathcal{L}_A \setminus \mathcal{R}_A$ or $M \in \mathcal{R}_A \setminus \mathcal{L}_A$. The result then follows from (a) and (b). □

THEOREM 4.7. *Let A be a left supported almost laura algebra. Assume that Γ is a non-semiregular component of $\Gamma(\operatorname{mod} A)$. Let $\Gamma' \neq \Gamma$ be a component of $\Gamma(\operatorname{mod} A)$.*

(a) $\operatorname{Hom}_A(\Gamma', \Gamma) \neq 0$ *if and only if* $\Gamma' \subseteq \mathcal{L}_A \setminus \mathcal{R}_A$.
(b) $\operatorname{Hom}_A(\Gamma, \Gamma') \neq 0$ *if and only if* $\Gamma' \subseteq \mathcal{R}_A \setminus \mathcal{L}_A$.
(c) *Either* $\operatorname{Hom}_A(\Gamma', \Gamma) \neq 0$ *and* $\operatorname{Hom}_A(\Gamma, \Gamma') = 0$, *or* $\operatorname{Hom}_A(\Gamma', \Gamma) = 0$ *and* $\operatorname{Hom}_A(\Gamma, \Gamma') \neq 0$.

In particular, Γ is the unique faithful component of $\Gamma(\operatorname{mod} A)$.

Proof. (a). Since the necessity follows from (3.7), assume that $\Gamma' \subseteq \mathcal{L}_A \setminus \mathcal{R}_A$. Let $M \in \Gamma'$. By (4.6), we have $\operatorname{Hom}_A(M, \Gamma) \neq 0$ and so $\operatorname{Hom}_A(\Gamma', \Gamma) \neq 0$.
(b). The proof is similar to that of (a) and is left to the reader.
(c). Let $M \in \Gamma'$. By (4.6), we have $\operatorname{Hom}_A(\Gamma', \Gamma) \neq 0$ or $\operatorname{Hom}_A(\Gamma, \Gamma') \neq 0$. The result then follows from (a) and (b).
Finally, observe that Γ is faithful by (4.4) and that if Γ' was another faithful component, then we would have $\operatorname{Hom}_A(\Gamma, \Gamma') \neq 0$ and $\operatorname{Hom}_A(\Gamma', \Gamma) \neq 0$. □

REMARK 4.8. Under the assumptions of (4.7) the component Γ induces a trisection in the family of AR-components (in the sense of [**dlPnR**]) : there are the components lying in $\mathcal{L}_A \setminus \mathcal{R}_A$, those lying in $\mathcal{R}_A \setminus \mathcal{L}_A$ and Γ. Also, any component Γ' in $\mathcal{L}_A \setminus \mathcal{R}_A$ contains no injective and maps non-trivially to Γ, and Γ maps non-trivially to any component Γ'' in $\mathcal{R}_A \setminus \mathcal{L}_A$, and these contains no projective. In addition, with those notations, it follows from (4.1)(d) and (4.4) that any morphism from Γ' to Γ'' factors through Γ. Moreover, by [**ACT**, (5.5)], any component lying in $\mathcal{L}_A \setminus \mathcal{R}_A$ is either a postprojective component, a semiregular tube without injectives, a component of the form $\mathbb{Z}A_\infty$ or a ray extension of $\mathbb{Z}A_\infty$. Numerous important families of algebras accept a trisection of its module category, notably the tilted algebras, the quasitilted algebras, the weakly shod algebras and the laura algebras.

We can now prove the main result of this section, which is a characterization of left supported almost laura algebras.

THEOREM 4.9. *Let A be a left supported algebra. Then A is almost laura if and only if $\Gamma(\operatorname{mod} A)$ contains a generalized standard, convex and faithful component.*

Proof. The necessity follows from (4.4), (3.3) and the fact that any connecting component is generalized standard and convex. Conversely, assume that Γ is a generalized standard, convex and faithful component in $\Gamma(\operatorname{mod} A)$. In addition, let $I \rightsquigarrow P$ be a path in $\operatorname{ind} A$ from an injective module to a projective module. Since Γ is faithful, there exist $M, N \in \Gamma$ and a path of the form $M \longrightarrow I \rightsquigarrow P \longrightarrow N$ Since Γ is convex, then every module on this path belongs to Γ. Now, Γ being generalized standard, this path is finite. So A is almost laura by (2.2). $\square$

At this point, we stress that the assumption of being left supported was unnecessary to prove the sufficiency. We then deduce the following corollary.

COROLLARY 4.10. *Let A be an algebra and assume that Γ is a standard generalized and convex component of $\Gamma(\operatorname{mod} A)$. The algebra $B = A/\operatorname{ann}\Gamma$ is almost laura, where $\operatorname{ann}\Gamma = \{a \in A \mid aM = 0$ for each $M \in \Gamma\}$.*

Proof. Clearly Γ is a faithful component of $\Gamma(\operatorname{mod} B)$. In addition, since $\operatorname{mod} B$ is a full subcategory of $\operatorname{mod} A$, then Γ is generalized standard and convex as a component of $\Gamma(\operatorname{mod} B)$. The result then follows from (4.9). $\square$

The above corollary shows the importance of identifying the generalized standard and convex components. In the vein of [**Sm, LS**], we then state the following result whose proof, left to the reader, easily follows using (1.2).

PROPOSITION 4.11. *Let A be an algebra and Γ be a component in $\Gamma(\operatorname{mod} A)$. Then Γ is generalized standard and convex if and only if any path connecting two modules in Γ is finite. In addition,*

(a) *If Γ is non-semiregular, then this is the case if and only if any path from an injective in Γ to a projective in Γ is finite.*
(b) *If Γ is semiregular, then this is the case if and only if any cycle $M \rightsquigarrow M$, with $M \in \Gamma$, is finite. Moreover,*
    (i) *if Γ contains injectives but no projectives, then this occurs if and only if any path from an injective in Γ to a module in Γ is finite;*
    (ii) *if Γ contains projectives but no injectives, then this occurs if and only if any path from a module in Γ to a projective in Γ is finite.* $\square$

If A is strict almost laura, then the generalized standard, convex and faithful component of (4.9) is non-semiregular. Since, by [**RS**, (3.1)], an algebra A which is not quasitilted is laura if and only if $\Gamma(\operatorname{mod} A)$ has a non-semiregular faithful and quasi-directed component, this motivates the following problem.

PROBLEM 2. *Let A be a left (or right) supported strict almost laura algebra and Γ be the unique non-semiregular component of $\Gamma(\operatorname{mod} A)$. Is Γ almost directed?*

Since any strict laura algebra is left and right supported, a positive answer would show that, for a strict almost laura algebra A, the following are equivalent:

(a) A is left supported.
(b) A is right supported.
(c) A is laura.

We end this section with a discussion of the case where $\mathcal{L}_A$ is finite, that is contains only finitely many objects.

PROPOSITION 4.12. *Let A be an almost laura algebra such that $\mathcal{L}_A$ is finite. Then $\Gamma(\operatorname{mod} A)$ admits a faithful non-semiregular π-component Γ. In particular, $\operatorname{rad}^{\infty}(-,\Gamma) = 0$ and A is left glued.*

Proof. We can clearly assume that A is representation-infinite. Moreover, observe that A is left supported since $\mathcal{L}_A$ is finite, and let Γ be the (faithful) component of (4.4). Since $\mathcal{L}_A$ is finite and Γ is generalized standard, we have $\mathrm{rad}^\infty(-,\Gamma)=0$ by (4.7). In particular, Γ contains projective modules, and so Γ is non-semiregular by (3.8). Then Γ is a π-component by [**L2**, (2.1)-(2.3)]. Hence A is left glued. $\square$

PROPOSITION 4.13. *Let A be an almost laura algebra. Then $\mathcal{L}_A$ and $\mathcal{R}_A$ are finite if and only if A is representation-finite.*

Proof. It clearly suffices to prove the necessity. If A is quasitilted, then there is nothing to show since $\mathrm{ind}\, A = \mathcal{L}_A \cup \mathcal{R}_A$ by [**HRS**, (II.1.13)]. So, let A be a strict almost laura algebra and Γ be a non-semiregular component as in (3.5). By (4.12) and its dual, we have $\mathrm{rad}^\infty(-,\Gamma)=0=\mathrm{rad}^\infty(\Gamma,-)$. So $\mathrm{rad}^\infty(\mathrm{mod}\, A)=0$ and A is representation-finite by [**ARS**, (V.7.7)]. $\square$

5. Full subcategories, split-by-nilpotent extensions and skew group algebras

It is frequent in the representation theory of artin algebras to consider problems in which two algebras A and B are closely related. It is then natural to ask which properties of $\mathrm{mod}\, A$ carry over $\mathrm{mod}\, B$ and conversely. In this final section, we consider three different situations and show that almost laura algebras behave well with respect to those situations.

5.1. Full subcategories. We consider the following problem. Let A, B be artin algebras such that B is a connected full subcategory of A. We choose an idempotent $e \in A$ so that $B = eAe$. Let $P = Ae$ be the corresponding projective A-module. We denote by $\mathrm{pres}\, P$ the full subcategory of $\mathrm{mod}\, A$ formed by the **P-presented modules**, that is the A-modules M for which there exists an exact sequence, of the form $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, with P_0, P_1 in $\mathrm{add}\, P$. By [**ARS**, (II.2.5)], the functor $\mathrm{Hom}_A(P,-) : \mathrm{mod}\, A \rightarrow \mathrm{mod}\, B$ induces an equivalence $\mathrm{pres}\, P \cong \mathrm{mod}\, B$, under which direct summands of P correspond to the projective B-modules. In addition, by [**AC3**, (2.1)], its left inverse is $P \otimes_B - : \mathrm{mod}\, B \longrightarrow \mathrm{pres}\, P \subseteq \mathrm{mod}\, A$, that is if X is a B-module, then the A-module $P \otimes_B X$ is P-presented and $\mathrm{Hom}_A(P, P\otimes_B X) \cong X$, functorially.

It is shown in [**AC3**] that B is laura (or weakly shod, or left glued) whenever so is A. The following enlarges this result to almost laura algebras.

PROPOSITION 5.1. *Let A be an algebra and e be an idempotent in A such that $B = eAe$ is connected. If A is almost laura, then so is B.*

Proof. Assume that $f : X \longrightarrow Y$ is a morphism in $\mathrm{ind}\, B$, with $X, Y \notin \mathcal{L}_B \cup \mathcal{R}_B$. The functor $P \otimes_B -$ gives a morphism $P \otimes_B f : P \otimes_B X \longrightarrow P \otimes_B Y$, where $P \otimes_B X$ and $P \otimes_B Y$ do not lie in $\mathcal{L}_A \cup \mathcal{R}_A$. Indeed, if, for instance, $P \otimes_B X \in \mathcal{L}_A \cup \mathcal{R}_A$, then $X \cong \mathrm{Hom}_A(P, P \otimes_B X) \in \mathcal{L}_B \cup \mathcal{R}_B$ by [**AC3**, (2.3)], a contradiction. Now, since A is almost laura, we have $P \otimes_B f \notin \mathrm{rad}^\infty(\mathrm{mod}\, A)$, and then $f \notin \mathrm{rad}^\infty(\mathrm{mod}\, B)$ since $\mathrm{Hom}_A(P,-) : \mathrm{pres}\, P \longrightarrow \mathrm{mod}\, B$ is an equivalence. So B is almost laura. $\square$

REMARK 5.2. We may ask whether an artin algebra A is almost laura provided eAe is almost laura for any idempotent $e \neq 1$ of A. The answer is negative, and can be easily verified on the algebra of (2.4).

5.2. Split-by-nilpotent extensions. We now consider another construction. Informally, if one can roughly think of taking full subcategories as "deleting points", the construction we now outline can be thought of as "deleting arrows".

Let A and B be artin algebras and let Q be a nilpotent ideal of A (that is, $Q \subseteq \mathrm{rad}A$). Following [**AM**], we say that A is a **split-by-nilpotent extension of** B **by** Q if there exists a split surjective algebra morphism $A \longrightarrow B$ with kernel Q.

In particular, B is a subalgebra of A (and has the same primitive idempotents). For instance, if $Q^2 = 0$, then the above definition coincides with that of the trivial extension of B by Q. Another example is that of one-point extension. For further examples, we refer the reader to [**AZ**].

Clearly, if A and B are as above, and B is a connected algebra, then so is A, but the converse is generally not true. We have the change of rings functors $A \otimes_B - : \mod B \longrightarrow \mod A$ and $B \otimes_A - : \mod A \longrightarrow \mod B$. The image of the functor $A \otimes_B -$ in $\mod A$ is called the category of **induced** modules. We have the obvious natural isomorphism $B \otimes_A A \otimes_B - \cong 1_{\mod B}$.

In is shown in [**AZ**] that if A is laura (or weakly shod, or left glued), then so is B. The same result holds for almost laura algebras.

PROPOSITION 5.3. *Let A be a split-by-nilpotent extension of B by Q. If A is almost laura, then so is B.*

Proof. Assume that $f : X \longrightarrow Y$ is a morphism in $\mathrm{ind}\, B$, with $X, Y \notin \mathcal{L}_B \cup \mathcal{R}_B$. The functor $A \otimes_B -$ gives a morphism of induced indecomposable A-modules $A \otimes_B f : A \otimes_B X \longrightarrow A \otimes_B Y$. Moreover, by [**AZ**, (2.3)], $A \otimes_B X$ and $A \otimes_B Y$ do not lie in $\mathcal{L}_A \cup \mathcal{R}_A$. Now, since A is almost laura, we have $A \otimes_B f \notin \mathrm{rad}^{\infty}(\mod A)$, and then $f \notin \mathrm{rad}^{\infty}(\mod B)$ since $B \otimes_A -$ induces an equivalence between $\mod B$ and the induced modules in $\mod A$. Therefore B is almost laura. $\square$

5.3. Skew group algebras. The final construction we consider is that of skew group algebras. We are mainly motivated by the fact that skew group algebras generally retain most features from the algebras they arise, especially concerning homological properties. The study of the representation theory of skew group algebras was started in [**RR, dlPn**], and more recently pursued in [**FR, ALR, DLS**]. We recall the relevant definitions and refer the reader to [**RR, ARS, ALR**] for details.

Let A be an artin k-algebra and G be a group with identity e. We say that G **acts on** A if there is a function $G \times A \longrightarrow A$, $(\sigma, a) \longmapsto \sigma(a)$, such that:

(a) For each σ in G, the morphism $\sigma : A \longrightarrow A$ is an algebra automorphism;
(b) $(\sigma_1\sigma_2)(a) = \sigma_1(\sigma_2(a))$ for all $\sigma_1, \sigma_2 \in G$ and $a \in A$;
(c) $e(a) = a$ for all $a \in A$.

Such an action induces an action of G on $\mod A$ as follows : for any $M \in \mod A$ and $\sigma \in G$, let ${}^{\sigma}M$ be the A-module with the additive structure of M and with the multiplication $a \cdot m = \sigma^{-1}(a)m$, for $a \in A$ and $m \in M$. This allows to define an automorphism ${}^{\sigma}(-) : \mod A \longrightarrow \mod A$ for each $\sigma \in G$, where ${}^{\sigma}f : {}^{\sigma}M \longrightarrow {}^{\sigma}N$ is defined by $m \longmapsto f(m)$ for $f \in \mathrm{Hom}_A(M, N)$ and $m \in M$ (see [**ALR**, (4.1)]).

Suppose that G acts on A. The **skew group algebra** $A[G]$ has as underlying A-module structure the free left A-module having as basis all elements in G, and is endowed by the multiplication $(a\sigma)(b\varsigma) = a\sigma(b)\sigma\varsigma$ for all $a, b \in A$ and $\sigma, \varsigma \in G$. Observe that $A[G]$ is generally not basic nor connected although A is, but will not play any role in the sequel.

The main aim of this section is to show that if A is an algebra and G is a finite group acting on A and such that its order is invertible in A, then A is almost laura if and only if so is $A[G]$ (see (5.11)). It is well-known that similar results hold for tilted, quasitilted, weakly shod and laura algebras (see [**ALR**, (1.2)]).

As we shall see, the techniques used in the proof will also result in analogue statements for algebras having nilpotent infinite radical and cycle-finite algebras (see (5.12)).

Throughout this section, we assume that G is a finite group acting on A and whose order is invertible in A. Then, the natural inclusion of A in $A[G]$ induces the change of rings functors $F := A[G] \otimes_A - : \operatorname{mod} A \longrightarrow \operatorname{mod} A[G]$ and $H := \operatorname{Hom}_{A[G]}(A[G], -) : \operatorname{mod} A[G] \longrightarrow \operatorname{mod} A$. We recall the following useful result from [**RR**, (1.1)].

THEOREM 5.4. *Let A and G be as above. Then*

(a) *(F, H) and (H, F) are two adjoint pairs of functors.*

(b) (i) *The unit $\varepsilon : \mathrm{id}_{\operatorname{mod} A} \longrightarrow HF$ of the adjoint pair (F, H) is a section of functors.*

(ii) *The counit $\eta : FH \longrightarrow \mathrm{id}_{\operatorname{mod} A[G]}$ of the adjoint pair (F, H) is a retraction of functors.* □

We refer to [**RR**, (1.1)] for the details. Moreover, in the sequel, we shall use the following notations. We denote by $\phi : \operatorname{Hom}_{A[G]}(F(-), ?) \longrightarrow \operatorname{Hom}_A(-, H(?))$ the natural equivalence associated to the adjoint pair (F, H). On the other hand, we denote by $\psi : \operatorname{Hom}_A(H(?), -) \longrightarrow \operatorname{Hom}_{A[G]}(?, F(-))$ the natural equivalence associated to the adjoint pair (H, F). Finally, we let μ and ρ be the unit and counit of this adjoint pair.

With these notations, we have (see [**M**, (p. 118)], for instance) the following useful lemma.

LEMMA 5.5. *Let M be an A-module and X be an $A[G]$-module.*

(a) *If $f \in \operatorname{Hom}_{A[G]}(F(M), X)$, then $\phi(f) = H(f) \circ \varepsilon_M$.*

(b) *If $f \in \operatorname{Hom}_A(M, H(X))$, then $\phi^{-1}(f) = \eta_X \circ F(f)$.*

(c) *If $f \in \operatorname{Hom}_A(H(X), M)$, then $\psi(f) = F(f) \circ \mu_X$.*

(d) *If $f \in \operatorname{Hom}_{A[G]}(X, F(M))$, then $\psi^{-1}(f) = \rho_M \circ H(f)$.* □

We recall the following definition : let $\mathcal{C}$ and $\mathcal{D}$ be two categories, a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is called a **radical functor** if, for any objects M, N in $\mathcal{C}$, we have $\mathcal{F}(\operatorname{rad}_{\mathcal{C}}(M, N)) \subseteq \operatorname{rad}_{\mathcal{D}}(\mathcal{F}(M), \mathcal{F}(N))$. For instance, any full functor is radical. It turns out that F and H are radical functors although they are not full.

PROPOSITION 5.6. *The functors F and H are radical functors.*

Proof. We first show that F is a radical functor. Let M, N be indecomposable A-modules and let $f \in \operatorname{rad}_A(M, N)$. Now, assume to the contrary that $F(f) \notin \operatorname{rad}_{A[G]}(F(M), F(N))$. So, there exist an indecomposable $A[G]$-module X together with a section $\iota : X \longrightarrow F(M)$ and a retraction $\pi : F(N) \longrightarrow X$ such that the composition $\pi \circ F(f) \circ \iota$ is an isomorphism. Denote by ω the left inverse of ι. Applying H gives a commutative diagram:

$$\begin{tikzcd}
H(X) \arrow[r, shift left, "H(\iota)"] & H(F(M)) \arrow[l, shift left, "H(\omega)"] \arrow[r, "H(F(f))"] & H(F(N)) \arrow[r, "H(\pi)"] & H(X) \\
& M \arrow[u, "\varepsilon_M"'] \arrow[ul, "\phi(\omega)"] \arrow[r, "f"] & N \arrow[u, "\varepsilon_N"'] \arrow[ur, "\phi(\pi)"'] &
\end{tikzcd}$$

where the first row is an isomorphism, $H(\omega) \circ \varepsilon_M = \phi(\omega)$ and $H(\pi) \circ \varepsilon_N = \phi(\pi)$ by (5.5)(a) and $\varepsilon_N \circ f = H(F(f)) \circ \varepsilon_N$ by (5.4)(b). Since ϕ is a bijection and $\omega \neq 0$, we have $\phi(\omega) \neq 0$. It then follows from the indecomposability of M that $H(\iota) \circ \phi(\omega) = \varepsilon_M$ and so $\phi(\omega)$ is a section. Therefore, we have

$$\phi(\pi) \circ f = H(\pi) \circ \varepsilon_N \circ f = H(\pi) \circ H(F(f)) \circ \varepsilon_M = H(\pi) \circ H(F(f)) \circ H(\iota) \circ \phi(\omega).$$

Since $H(\pi) \circ H(F(f)) \circ H(\iota)$ is an isomorphism and $\phi(\omega)$ is a section, then $\phi(\pi) \circ f$ is a section. In particular, f is a section, a contradiction since N is indecomposable. So $F(f) \in \operatorname{rad}_{A[G]}(F(M), F(N))$ and F is a radical functor. Using (5.5)(b) and the fact that η is a retraction of functors, one can show in a similar way that H is also a radical functor. We leave the verification to the reader. □

Since almost laura algebras are defined in terms of the behavior of their infinite radicals, the knowledge of each power of the radical is rather important. As a consequence of the above proposition, we now show that the maps ϕ and ψ can be used to relate the different powers of the radicals of $\operatorname{mod} A$ and $\operatorname{mod} A[G]$.

PROPOSITION 5.7. *Let A and G be as above. Let $n \geq 1$, M be an A-module and X be an $A[G]$-module. Then,*

(a) $\phi(\operatorname{rad}^n_{A[G]}(F(M), X)) = \operatorname{rad}^n_A(M, H(X))$;
(b) $\psi(\operatorname{rad}^n_A(H(X), M)) = \operatorname{rad}^n_{A[G]}(X, F(M))$.

Proof. We only prove (a) since the proof of (b) is similar.
(a). Assume that $f \in \operatorname{rad}^n_{A[G]}(F(M), X)$. Then, there exist $F(M) = Y_0, Y_1, \ldots, Y_n = X$ and $f_i \in \operatorname{rad}_{A[G]}(Y_{i-1}, Y_i)$, with $1 \leq i \leq n-1$ such that $f = f_n f_{n-1} \cdots f_1$. Then, by (5.5)(a), we have

$$\phi(f) = H(f) \circ \varepsilon_M = H(f_n) \circ \cdots \circ H(f_1) \circ \varepsilon_M.$$

Since H is a radical functor by (5.6), we have $H(f_i) \in \operatorname{rad}_A(H(Y_{i-1}), H(Y_i))$ for each i. So $\phi(f) \in \operatorname{rad}^n_A(M, H(X))$. Similarly, if $h \in \operatorname{rad}^n_A(M, H(X))$, then $\phi^{-1}(h) \in \operatorname{rad}^n_{A[G]}(F(M), X)$. The result follows. □

The following two corollaries are generalizations of [**ALR**, (4.4)] and [**ALR**, (4.6)] respectively. But first, we need to recall from [**ALR**, (4.3)] that if X is an indecomposable $A[G]$-module, then there exists an indecomposable A-module M such that $M \Subset H(X)$ and $X \Subset F(M)$.

COROLLARY 5.8. *Let $n \geq 1$ and M, N be indecomposable A-modules such that $\operatorname{rad}^n_A(M, N) \neq 0$.*

(a) *For any direct summand X of $F(M)$, we have $\operatorname{rad}^n_{A[G]}(X, F(N)) \neq 0$;*
(b) *For any direct summand Y of $F(N)$, we have $\operatorname{rad}^n_{A[G]}(F(M), Y) \neq 0$.*

Proof. We only prove (a) since the proof of (b) is similar.
(a). By [**RR**, (1.8)], we have an indecomposable decomposition $F(M) \cong \oplus_{i=1}^m X_i$ in $\operatorname{mod} A[G]$ such that $H(X_i) \cong \oplus_{\sigma \in G_i}\, {}^{\sigma}M$ for some $G_i \subseteq G$. In addition, for each i, and each $\gamma \in G$, there exists $\sigma \in G_i$ with ${}^{\gamma}M \cong {}^{\sigma}M$. In particular, we can assume that $M \Subset H(X_i)$ for each i. We need to show that $\operatorname{rad}^n_A(X_i, F(N)) \neq 0$ for each i and, by (5.7)(b), it is sufficient to show that $\operatorname{rad}^n_A(H(X_i), N) \neq 0$. Since M is a direct summand of $H(X_i)$ for each i, this is clearly the case. □

COROLLARY 5.9. *Let $n \geq 1$ and X, Y be indecomposable $A[G]$-modules such that $\operatorname{rad}^n_{A[G]}(X, Y) \neq 0$. Then, for all indecomposable A-modules M, N such that $X \Subset F(M)$ and $Y \Subset F(N)$, there exists $\sigma \in G$ such that $\operatorname{rad}^n_A(M, {}^{\sigma}N) \neq 0$.*

Proof. Let M and N be as in the statement. Then, by hypothesis, we have $\operatorname{rad}^n_{A[G]}(F(M), F(N)) \neq 0$, and thus $\operatorname{rad}^n_A(M, H(F(N))) \neq 0$ by (5.7). Since on the other hand we have $H(F(N)) \cong \oplus_{\sigma \in G}\, {}^{\sigma}N$ by [**RR**, (1.8)], there exists $\sigma \in G$ with $\operatorname{rad}^n_A(M, {}^{\sigma}N) \neq 0$. □

We also get the following corollary, which complements [**ALR**, (4.5)(4.7)].

COROLLARY 5.10. (a) *Let $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t$ be a path in $\operatorname{ind} A$, with $f_i \in \operatorname{rad}^{n_i}_A(M_{i-1}, M_i)$ for each i. For any indecomposable $X_0 \Subset F(M_0)$, there exists a path $X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \cdots \xrightarrow{g_t} X_t$ in $\operatorname{ind} A[G]$ with $X_i \Subset F(M_i)$, $M_i \Subset H(X_i)$ and $g_i \in \operatorname{rad}^{n_i}_{A[G]}(X_{i-1}, X_i)$ for each i.*

(b) *Let* $X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \cdots \xrightarrow{g_t} X_t$ *be a path in* $\operatorname{ind} A[G]$*, with* $g_i \in \operatorname{rad}^{n_i}_{A[G]}(X_{i-1}, X_i)$ *for each* i*. For any indecomposable* M_0 *such that* $X_0 \Subset F(M_0)$*, there exist* $\sigma_1, \sigma_2, \ldots, \sigma_t \in G$ *and a path* $M_0 \xrightarrow{f_1} {}^{\sigma_1}M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} {}^{\sigma_t}M_t$ *in* $\operatorname{ind} A$ *with* $M_i \Subset H(X_i)$*,* $X_i \Subset F(M_i)$ *and* $f_i \in \operatorname{rad}^{n_i}_A({}^{\sigma_{i-1}}M_{i-1}, {}^{\sigma_i}M_i)$ *for each* i*.*

Proof. (a). Since $X_0 \Subset F(M_0)$ and $\operatorname{rad}^{n_1}_A(M_0, M_1) \neq 0$, it follows from (5.8) that $\operatorname{rad}^{n_1}_A(X_0, F(M_1)) \neq 0$. Hence there exists an indecomposable $X_1 \Subset F(M_1)$ with $\operatorname{rad}^{n_1}_A(X_0, X_1) \neq 0$. The result follows from an obvious induction. Observe that $M_i \Subset H(X_i)$ for each i by the proof of (5.8).
(b). Let M_0, M_1 be indecomposable A-modules such that $X_i \Subset F(M_i)$, for $i = 1, 2$. By (5.9), there exists $\sigma_1 \in G$ such that $\operatorname{rad}^{n_1}_A(M_0, {}^{\sigma_1}M_1) \neq 0$. Similarly, there exists an indecomposable M_2 such that $X_2 \Subset F(M_2)$ together with an element $\sigma_2' \in G$ such that $\operatorname{rad}^{n_2}_A(M_1, {}^{\sigma_2'}M_2) \neq 0$. Applying the automorphism ${}^{\sigma_1}(-) : \operatorname{mod} A \longrightarrow \operatorname{mod} A$ we obtain $\operatorname{rad}^{n_2}_A({}^{\sigma_1}M_1, {}^{\sigma_2}M_2) \neq 0$, where $\sigma_2 = \sigma_1\sigma_2'$. The result now follows from an obvious induction. Observe that $M_i \Subset H(X_i)$ for each i by the proof of (5.8). □

We are now ready to prove the main result of this section.

THEOREM 5.11. *Let* A *be an algebra and* G *be a finite group acting on* A *and whose order is invertible in* A*.*

(a) *A is almost laura if and only if so is* $A[G]$*.*
(b) *A is strict almost laura if and only if so is* $A[G]$*.*

Proof. (a). Assume that A is almost laura and let $X \xrightarrow{g} Y$ be a morphism in $\operatorname{ind} A[G]$, with $X, Y \notin \mathcal{L}_{A[G]} \cup \mathcal{R}_{A[G]}$ and $g \in \operatorname{rad}^n_{A[G]}(X, Y)$. By (5.10)(b), there exist $\sigma \in G$ and a morphism $M \xrightarrow{f} {}^{\sigma}N$ in $\operatorname{ind} A$ with $f \in \operatorname{rad}^n_A(M, {}^{\sigma}N)$. In addition, by [**ALR**, (5.1)(5.3)], we have $M, {}^{\sigma}N \notin \mathcal{L}_A \cup \mathcal{R}_A$. Since A is almost laura, f does not belong to $\operatorname{rad}^{\infty}(\operatorname{mod} A)$, and so g does not belong to $\operatorname{rad}^{\infty}(\operatorname{mod} A[G])$. Hence $A[G]$ is almost laura. The converse is proven in the same way, using (5.10)(a) instead of (5.10)(b).
(b). This follows from (a) and [**HRS**, (III.1.6)]. □

Our work on the infinite radical carries consequences on other classes of algebras, for instance on cycle-finite algebras and algebras having nilpotent infinite radical. Recall from [**AS1**] that an algebra A is **cycle-finite** if no cycle in $\operatorname{ind} A$ contains morphisms in $\operatorname{rad}^{\infty}(\operatorname{mod} A)$. Examples of cycle-finite algebras are all representation-finite algebras, tame tilted algebras [**K, R**], tubular algebras [**R**], iterated tubular algebras [**dlPnT**], and multicoil algebras [**AS2**]. It is known (see [**AS1**]) that every cycle-finite algebra is of tame representation type.

On the other hand, given an algebra A, it is important to study the nilpotency of the infinite radical of $\operatorname{mod} A$ in order to understand the complexity of $\operatorname{mod} A$. This has been considered, for instance, in [**CMMS1, CMMS2, KS, Sc, AC2**]. More precisely, we say that $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ is **nilpotent** if there exists an integer $n \geq 1$ such that $(\operatorname{rad}^{\infty}(\operatorname{mod} A))^n = 0$. Such a minimal integer n is then called the **index of nilpotency** of $\operatorname{rad}^{\infty}(\operatorname{mod} A)$.

We have the following result.

PROPOSITION 5.12. *Let* A *be an algebra and* G *be a finite group acting on* A *and whose order is invertible in* A*.*

(a) *The infinite radical of* $\operatorname{mod} A$ *is nilpotent if and only if so is the infinite radical of* $\operatorname{mod} A[G]$ *and, in this case, they have the same index of nilpotency.*
(b) *A is cycle-finite if and only if so is* $A[G]$*.*
Moreover, in this case, A is domestic if and only if so is $A[G]$*.*

Proof. (a). Assume that there exists an integer $n \geq 1$ such that $(\mathrm{rad}^\infty(\mathrm{mod}\, A))^n = 0$ but $(\mathrm{rad}^\infty(\mathrm{mod}\, A[G]))^n \neq 0$ Thus, there exists a path $X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} X_n$ in $\mathrm{ind}\, A[G]$ such that $g_i \in \mathrm{rad}^\infty_{A[G]}(\mathrm{mod}\, A[G])$ for each i and $g = g_n \cdots g_2 g_1 \neq 0$. Now, since H is faithful by (5.4)(b)(ii) and a radical functor by (5.6), we have $0 \neq H(g) \in (\mathrm{rad}^\infty_A(\mathrm{mod}\, A))^n$, a contradiction. So $(\mathrm{rad}^\infty_{A[G]}(\mathrm{mod}\, A[G]))^n = 0$. The converse is proven in the same way, using F and invoking (5.4)(b)(i) instead of (5.4)(b)(ii).

(b). Assume that A is cycle-finite and let $X = X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \cdots \xrightarrow{g_t} X_t = X$ be a cycle in $\mathrm{ind}\, A[G]$, with $g_i \in \mathrm{rad}^{n_i}_{A[G]}(X_{i-1}, X_i)$ for each i. By (5.10)(b), there exist $\sigma_1, \sigma_2, \ldots, \sigma_t \in G$ and a path in $\mathrm{ind}\, A$ of the form $\delta : M_0 \xrightarrow{f_1} {}^{\sigma_1}M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} {}^{\sigma_t}M_t$, with $f_i \in \mathrm{rad}^{n_i}_{A[G]}({}^{\sigma_{i-1}}M_{i-1}, {}^{\sigma_i}M_i)$ for each i. Moreover, by [**RR**, (1.8)], we have ${}^{\sigma}M_0 \cong M_t$ for some $\sigma \in G$ and thus ${}^{\sigma\sigma_t}M_0 \cong {}^{\sigma_t}M_t$. Let $\tau = \sigma\sigma_t$ and m be the order of τ in G. Applying repeatedly the functor ${}^{\tau}(-) : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A$ on δ yields a cycle

$$M_0 \overset{\delta}{\rightsquigarrow} {}^{\tau}M_0 \overset{{}^{\tau}\delta}{\rightsquigarrow} {}^{\tau^2}M_0 \overset{{}^{\tau^2}\delta}{\rightsquigarrow} \cdots \overset{{}^{\tau^m}\delta}{\rightsquigarrow} {}^{\tau^m}M_0 = M_0$$

Since A is cycle-finite, no morphism in δ belongs to $\mathrm{rad}^\infty(\mathrm{mod}\, A)$, and so no g_i belongs to $\mathrm{rad}^\infty(\mathrm{mod}\, A[G])$. Hence $A[G]$ is cycle-finite.
On the other hand, assume that $A[G]$ is cycle-finite and let

$$M = M_0 \xrightarrow{g_1} M_1 \xrightarrow{g_2} \cdots \xrightarrow{g_t} M_t = M$$

be a cycle in $\mathrm{ind}\, A$, with $g_i \in \mathrm{rad}^{n_i}_A(M_{i-1}, M_i)$ for each i. Let $F(M) = \oplus_{j=1}^m X_j$ be an indecomposable decomposition in $\mathrm{mod}\, A[G]$. Then, for each j, there exists by (5.10)(a) a path in $\mathrm{ind}\, A[G]$ of the form $\delta_j : X_j \rightsquigarrow X_{s_j}$ with $1 \leq s_j \leq m$ containing at least one morphism in $\mathrm{rad}^{n_i}_{A[G]}(\mathrm{mod}\, A[G])$ for each $1 \leq i \leq t$. Let $s : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, m\}$ be the application defined by $s(j) = s_j$. Then, there exist j and q such that $j = s^q(j)$. Consequently, there is a cycle of the form

$$X_j \overset{\delta_j}{\rightsquigarrow} X_{s(j)} \overset{\delta_{s(j)}}{\rightsquigarrow} X_{s^2(j)} \overset{\delta_{s^2(j)}}{\rightsquigarrow} \cdots \overset{\delta_{s^q(j)}}{\rightsquigarrow} X_{s^q(j)} = X_j$$

containing morphisms in $\mathrm{rad}^{n_i}_{A[G]}(\mathrm{mod}\, A[G])$ for each $1 \leq i \leq t$. Since $A[G]$ is cycle-finite, no morphism in this path belongs to the infinite radical, and so A is cycle-finite. The latter part directly follows from (a) and [**Sko4**, (5.1)]. $\square$

REMARK 5.13. Recall from [**KS**] that $\mathrm{rad}^\infty(\mathrm{mod}\, A)$ is called **left (or right) T-nilpotent** if for each sequence $(f_i)_{i \in \mathbb{N}}$ in $\mathrm{rad}^\infty(\mathrm{mod}\, A)$, there exists a natural number m such that $f_m \cdots f_1 = 0$ (or $f_1 \cdots f_m = 0$, respectively). It is easily seen that the proof of (5.12)(a) can be adapted to show that $\mathrm{rad}^\infty(\mathrm{mod}\, A)$ is left (or right) T-nilpotent if and only if so is $\mathrm{rad}^\infty(\mathrm{mod}\, A[G])$.

ACKNOWLEDGEMENTS. Most of the results in this paper are part of the author's Ph.D. thesis, completed at the University of Sherbrooke (Canada). The author thanks his PhD advisor Ibrahim Assem for many helpful comments, as well as Flávio Coelho for his assistance regarding Theorem 3.15. He also acknowledges financial support from NSERC of Canada and FQRNT of Québec.

References

[A] I. Assem. Left sections and the left part of an artin algebra. In preparation.

[AC1] I. Assem and F.U. Coelho. Glueings of tilted algebras. *J. Pure Appl. Algebra*, 96(3):225–243, 1994.

[AC2] I. Assem and F.U. Coelho. Two-sided gluings of tilted algebras. *J. Algebra*, 269(2):456–479, 2003.

[AC3] I. Assem and F.U. Coelho. Endomorphism algebras of projective modules over laura algebras. *J. Algebra Appl.*, 3(1):49–60, 2004.

[ACLST] I. Assem, F.U. Coelho, M. Lanzilotta, D. Smith, and S. Trepode. Algebras determined by their left and right parts. In *Algebraic structures and their representations*, number 376 in Contemp. Math., pages 13–47, Providence, RI, 2005. Amer. Math. Soc.

[ACT] I. Assem, F.U. Coelho, and S. Trepode. The left and the right parts of a module category. *J. Algebra*, 281(2):518–534, 2004.

[ALR] I. Assem, M. Lanzilotta, and M.J. Redondo. Laura skew group algebras. *Comm. Algebra*, to appear.

[AM] I. Assem and N. Marmaridis. Tilting modules over split-by-nilpotent extensions. *Comm. Algebra*, 26(5):1547–1555, 1998.

[AS1] I. Assem and A. Skowroński. Minimal representation-infinite coil algebras. *Manuscripta Math.*, 67(3):305–331, 1990.

[AS2] I. Assem and A. Skowroński. Indecomposable modules over multicoil algebras. *Math. Scand.*, 71(1):31–61, 1992.

[AZ] I. Assem and D. Zacharia. On split-by-nilpotent extensions. *Colloq. Math.*, 98(2):259–275, 2003.

[ARS] M. Auslander, I. Reiten, and S.O. Smalø. *Representation Theory of Artin Algebras*. Number 36 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997.

[AS3] M. Auslander and S.O. Smalø. Preprojective modules over artin algebras. *J. Algebra*, 66(1):61–122, 1980.

[BS] R. Bautista and S.O. Smalø. Non-existent cycles. *Comm. Algebra*, 11(6):1755–1767, 1983.

[B] K. Bongartz. On a result of Bautista and Smalø on cycles. *Comm. Algebra*, 11(6):2123–2124, 1983.

[C] F.U. Coelho. Components of Auslander-Reiten quivers with only preprojective modules. *J. Algebra*, 157(2):472–488, 1993.

[CL] F.U. Coelho and M. Lanzilotta. On non-semiregular components containing paths from injective to projective modules. *Comm. Algebra*, 30(10):4837–4849, 2002.

[CMMS1] F.U. Coelho, E.N. Marcos, H.A. Merklen, and A. Skowroński. Module categories with infinite radical square zero are of finite type. *Comm. Algebra*, 22(11):4511–4517, 1994.

[CMMS2] F.U. Coelho, E.N. Marcos, H.A. Merklen, and A. Skowroński. Module categories with infinite radical cube zero. *J. Algebra*, 183(1):1–23, 1996.

[CS] F.U. Coelho and A. Skowroński. On Auslander-Reiten components for quasitilted algebras. *Fund. Math.*, 149(1):67–82, 1996.

[DLS] J. Dionne, M. Lanzilotta, and D. Smith. Piecewise hereditary skew group algebras. In preparation.

[DS] J. Dionne and D. Smith. Articulations of algebras and their homological properties. *J. Algebra Appl.*, 5(3):1–15, 2006.

[FR] O. Funes and M. J. Redondo. Skew group algebras of simply connected algebras. *Ann. Sci. Math. Québec*, 26(2):171–180, 2002.

[HRS] D. Happel, I. Reiten, and S.O. Smalø. Tilting in abelian categories and quasi-tilted algebras. *Mem. Am. Math. Soc.*, 120(575):viii+ 88 pp., 1996.

[IT] K. Igusa and G. Todorov. A characterization of finite Auslander-Reiten quivers. *J. Algebra*, 89(1):148–177, 1984.

[K] O. Kerner. Tilting wild algebras. *J. London Math. Soc. (2)*, 39(1):29–47, 1989.

[KS] O. Kerner and A. Skowroński. On module categories with nilpotent infinite radical. *Compositio Math.*, 77(3):313–333, 1991.

[LS] M. Lanzilotta and D. Smith. Laura algebras and quasi-directed components. *Colloq. Math.*, 105(2):179–196, 2006.

[L1] S. Liu. Semi-stable components of an Auslander-Reiten quiver. *J. London Math. Soc. (2)*, 47(3):405–416, 1993.

[L2] S. Liu. Preprojective modules and Auslander-Reiten components. *Comm. Algebra*, 31(12):6051–6061, 2003.

[M] B. Mitchell. *Theory of categories*, volume XVII of *Pure and Applied in Mathematics*. Academic Press, New York - London, 1965.

[dlPn] J.A. de la Peña. Automorfismos, álgebras torcidas y cubiertas. Ph.D. thesis, Universidad Nacional Autonóma de México, 1983.

[dlPnR] J.A. de la Peña and I. Reiten. Trisections of module categories. *Colloq. Math.*, 107(2):191–219, 2007.

[dlPnT] J. A. de la Peña and Tomé. Iterated tubular algebras. *J. Pure Appl. Algebra*, 64(3):303–314, 1990.

[RR] I. Reiten and C. Riedtmann. Skew group algebras in the representation theory of artin algebras. *J. Algebra*, 92(1):224–282, 1985.

[RS] I. Reiten and A. Skowroński. Generalized double tilted algebras. *J. Math. Soc. Japan*, 56(1):269–288, 2004.

[R] C.M. Ringel. *Tame algebras and integral quadratic forms.* Number 1099 in Lecture Notes in Mathematics. Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.

[Sc] J. Schröer. On the infinite radical of a module category. *Proc. London Math. Soc. (3)*, 81(3):651–674, 2000.

[Sko1] A. Skowroński. Generalized standard Auslander-Reiten components. *J. Math. Soc. Japan*, 46(3):517–543, 1994.

[Sko2] A. Skowroński. Minimal representation-infinite artin algebras. *Math. Proc. Cam. Phi. Soc.*, 116(2):229–243, 1994.

[Sko3] A. Skowroński. Regular Auslander-Reiten components containing directing modules. *Proc. Amer. Math. Soc.*, 120(1):19–26, 1994.

[Sko4] A. Skowroński. Cycle-finite algebras. *J. Pure Appl. Algebra*, 103(1):105–116, 1995.

[Sko5] A. Skowroński. On artin algebras with almost all indecomposable modules of projective or injective dimension at most one. *Cent. Eur. J. Math.*, 1(1):108–122, 2003.

[Sm] D. Smith. On generalized standard Auslander-Reiten components having only finitely many non-directing modules. *J. Algebra*, 279(2):493–513, 2004.

D. Smith; Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway
E-mail address: david.smith@math.ntnu.no