NAKAYAMA ORIENTED PULLBACKS AND STABLY HEREDITARY ALGEBRAS

JESSICA LÉVESQUE

Abstract. We introduce a new class of algebras, the Nakayama oriented pullbacks, obtained from pullbacks of surjective morphisms of algebras $A \twoheadrightarrow C$ and $B \twoheadrightarrow C$. We prove that such a pullback is tilted when A and B are hereditary. We also show that stably hereditary algebras respecting the clock condition are Nakayama oriented pullbacks, and we use results about these pullbacks to show when is a stably hereditary algebra tilted or iterated tilted.

Introduction.

Pullbacks of rings and algebras have been studied from many points of view (see, for instance, [1, 8, 16, 17, 22, 25]), but not from the tilting point of view. In this paper, we consider a particular class of pullbacks of algebras over an algebraically closed field K, the Nakayama oriented pullbacks. These are pullbacks of surjective morphisms of algebras $A \twoheadrightarrow C$ and $B \twoheadrightarrow C$ with C a hereditary Nakayama algebra. We construct a tilting module T over this kind of pullback, and we compute the endomorphism algebra $\operatorname{End} T$ of this module when B is hereditary (see 2.4.6). A first consequence of this result is the principal theorem of this paper:

Theorem . *Let R be a Nakayama oriented pullback of K-algebra surjective morphisms $A \twoheadrightarrow C$ and $B \twoheadrightarrow C$. Suppose that A and B are hereditary. Then R is tilted.*

On the other hand, we show that a stably hereditary algebra respecting the clock condition, that is, the number of clockwise oriented relations on each cycle of its bound quiver equals the number of counterclockwise oriented relations, can be expressed as a Nakayama oriented pullback. As a second consequence of 2.4.6 we give a new proof of theorem 2.6 of [19].

This paper consists of three sections. The first is devoted to preliminaries, the second to Nakayama oriented pullbacks, and the third to stably hereditary algebras.

1. Preliminaries

1.1. **Notation.** All algebras in this paper are basic, associative, finite dimensional algebras with identities over a fixed algebraically closed field K, and all modules are finitely generated right modules. For an algebra A, we denote by $\operatorname{mod} A$ its module category, by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of a complete set of representatives of the isomorphism classes of indecomposable objects in $\operatorname{mod} A$, and by $\operatorname{proj} A$ the full subcategory of $\operatorname{ind} A$ consisting of the projective objects.

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Given an A-module M, we denote by $\operatorname{pd} M$ its projective dimension and by $\operatorname{id} M$ its injective dimension.

We recall that a **quiver** Q is defined by a set of points Q_0 and a set of arrows Q_1. A **relation** from $x \in Q_0$ to $y \in Q_0$ is a linear combination of paths from x to y of length at least two. Let KQ be the path algebra of Q, and let I denote the ideal of KQ generated by a set of relations; then the pair (Q, I) is called a **bound quiver**. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i$ in I (where the λ_i are non-zero scalars and the w_i are paths) with $m \geq 2$ is called **minimal** if there is no proper non-empty subset $J \subset \{1, ..., m\}$ such that $\sum_J \lambda_i w_i$ is also a relation in I, and is called **monomial** or **zero-relation** if it equals a path ($m = 1$). We say that a monomial relation is of **minimal length** if it does not contain a proper subpath which is also in I. It is well-known that if A is a basic and connected finite dimensional K-algebra, then there exists a connected bound quiver (Q_A, I) such that $A \cong KQ_A/I$ (see [5]). For a point a in the quiver of A, we denote by $P(a)$ the corresponding indecomposable projective A-module, and by $I(a)$ the corresponding indecomposable injective A-module. Given an A-module M, we denote by $\operatorname{Supp} M$ the full bound subquiver of Q_A generated by the points a such that $\operatorname{Hom}_A(P(a), M) \neq 0$. We say that A is *triangular* whenever its quiver Q_A has no oriented cycles.

For an arrow α of Q, we denote by $s(\alpha)$ its source, by $t(\alpha)$ its target and by α^{-1} its formal inverse of source $s(\alpha^{-1}) = t(\alpha)$ and of target $t(\alpha^{-1}) = s(\alpha)$. A **walk** in Q is a sequence $w = c_1 ... c_n$ with c_i an arrow or the formal inverse of an arrow such that $t(c_i) = s(c_{i+1})$ for all i such that $1 \leq i < n$. A walk w in Q is called *reduced* if $w = c_1 ... c_n$ with $c_i \neq c_{i+1}^{-1}$ for all i such that $1 \leq i < n$. It is called a *non-zero walk* if it does not contain any zero-relation. Finally, a reduced walk is called a **double-zero** if it contains exactly two zero-relations, and moreover, they point in the same direction in w. The double-zero has been used for the classification of tilted and quasi-tilted special biserial algebras, string algebras and gentle algebras [2, 7, 13, 14, 15, 18].

For general properties of the category $\operatorname{mod} A$ of finitely generated right A-modules, we refer the reader to [5, 23]. For tilting theory, tilted and iterated tilted algebras, we refer the reader to [3, 10, 12, 23]. Throughout this paper, we assume all the tilting modules to be multiplicity-free.

1.2. **The Bound Quiver of a Pullback.** Let $A = KQ_A/I_A$ and $B = KQ_B/I_B$ be bound quiver algebras, and let Q_C be a subquiver of Q_A and Q_B such that
i) every connected component of Q_C is full and convex in Q_A and Q_B,
ii) the restrictions of I_A and I_B to KQ_C are the same (that is, $I_A \cap KQ_C = I_B \cap KQ_C$).
We denote $I_A \cap KQ_C = I_B \cap KQ_C$ by I_C. Let $e_C = \sum_{i \in (Q_C)_0} e_i$ be the identity of C. Then $C = KQ_C/I_C \cong e_C A e_C \cong e_C B e_C$ is a common quotient of A and B.

We have the canonical projections

$$\begin{array}{cccc} f_A : & A & \twoheadrightarrow & C \\ & a & \mapsto & e_C a e_C \end{array} \quad , \quad \begin{array}{cccc} f_B : & B & \twoheadrightarrow & C \\ & b & \mapsto & e_C b e_C \end{array}$$

which are well defined since every connected component of Q_C is full and convex in Q_A and Q_B. The algebra $R = \{(a,\ b) \in A \times B \mid e_C a e_C = e_C b e_C\}$ is the pullback of f_A and f_B. The following lemma describes the bound quiver of R in terms of the bound quivers of A, B and C. Similar descriptions are done in [16] and [17].

Lemma 1.2.1. *Let R be the pullback of f_A and f_B. Let $Q_R = Q_A \amalg_{Q_C} Q_B$ be the pushout of the inclusion morphisms $Q_C \hookrightarrow Q_A$ and $Q_C \hookrightarrow Q_B$, and $I_R = I_A + I_B + <\bar{\rho}>$ with $\bar{\rho}$ the set of paths v of Q_R such that $v = \beta_1 v' \beta_2$ with v' a path, and*

$$\beta_1 \in (Q_A)_1 \backslash (Q_C)_1 \quad \text{and} \quad \beta_2 \in (Q_B)_1 \backslash (Q_C)_1$$

or

$$\beta_1 \in (Q_B)_1 \backslash (Q_C)_1 \quad \text{and} \quad \beta_2 \in (Q_A)_1 \backslash (Q_C)_1.$$

Then $R \cong KQ_R/I_R$.

Moreover, Q_A, Q_B and Q_C are convex in Q_R.

Proof. Let $R' = KQ_R/I_R$. Let $e_A = \sum_{a \in (Q_A)_0} e_a$ and $e_B = \sum_{b \in (Q_B)_0} e_b$ be the identities of A and B respectively.

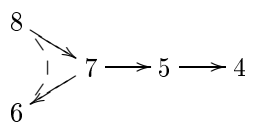
Consider the following map:

$$\begin{array}{rccc} \Phi : & R' & \to & R \\ & x & \mapsto & (e_A x e_A,\ e_B x e_B)\ . \end{array}$$

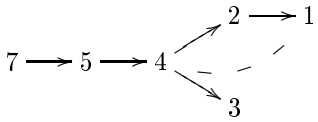
Then Φ is well defined. Indeed, $e_C(e_A x e_A)e_C = e_C x e_C = e_C(e_B x e_B)e_C$, and $\Phi(I_R) = 0$ since by hypothesis no path of $\bar{\rho}$ is a path of Q_A or of Q_B.

It is straightforward to verify that Φ is a K-algebra isomorphism. The last statement follows easily from the structure of $(Q_R,\ I_R)$ and the convexity of the connected components of Q_C in Q_A and Q_B. $\square$

Example 1.2.2. Suppose that we have

$(Q_A,\ I_A) =$ 

, $(Q_C,\ I_C) =$ 

and $(Q_B,\ I_B) =$ 

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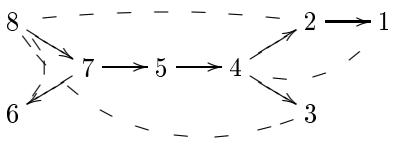
Then the bound quiver of the pullback R of the canonical projections of KQ_A/I_A and KQ_B/I_B over KQ_C/I_C is

$(Q_R,\ I_R) =$ 

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Since Q_A, Q_B and Q_C are full and convex in Q_R, it is easily seen that every indecomposable A-module, B-module or C-module has a natural R-module structure. Therefore, from now on, when we have an R-module that is isomorphic to a A, B or C-module, we will refer to it as a A, B or C-module. We can assume without loss of generality that $\operatorname{ind} C$ is contained in $\operatorname{ind} A$ and in $\operatorname{ind} B$ and similarly that $\operatorname{ind} A$ and $\operatorname{ind} B$ are contained in $\operatorname{ind} R$.

2. Nakayama Oriented Pullbacks

In this section we study a particular type of pullback R of K-algebras morphisms $A \twoheadrightarrow C$ and $B \twoheadrightarrow C$. We show that the Auslander-Reiten quiver of this kind of pullback has very nice properties, and we construct a tilting R-module from tilting modules over A, B and C. Finally, we conclude that R is tilted when A and B are hereditary.

2.1. **Bound Quiver.**

Definition 2.1.1. *Let $R \cong KQ_R/I_R$ be the pullback of the K-algebra surjective morphisms $f_A : A \twoheadrightarrow C$ and $f_B : B \twoheadrightarrow C$ as seen in 1.2. We say that R is a* **Nakayama oriented pullback** *if its bound quiver $(Q_R,\ I_R)$ satisfies the following conditions:*

- **(i)** *There is no path from $(Q_C)_0$ to $(Q_A)_0 \backslash (Q_C)_0$ and from $(Q_B)_0 \backslash (Q_C)_0$ to $(Q_C)_0$ in Q_R.*
- **(ii)** *C is a hereditary Nakayama algebra and the connected components Q_{C_1}, Q_{C_2}, ..., Q_{C_r} of Q_C are of the form $Q_{C_i} = a_{i,\,t_i} \to a_{i,\,t_i\,-\,1} \to \cdots \to a_{i,\,3} \to a_{i,\,2} \to a_{i,\,1}$ with $1 \leq i \leq r$ and $t_i \geq 1$.*
- **(iii)** *In Q_C, only sources are target of arrows of $(Q_A)_1 \backslash (Q_C)_1$, and only sinks are sources of arrows of $(Q_B)_1 \backslash (Q_C)_1$.*
- **(iv)** *No minimal (or of minimal length) relation of R has its origin in $(Q_C)_0$.*

The last condition ensures that the projective dimension of C as a B-module (and so as an R-module) is at most one. This is a necessary condition for the existence of a tilting C-module that is also a partial tilting B-module. We need such a module to construct a tilting R-module in 2.3. (The first three conditions imply that C is always a projective A-module.)

For the remainder of this section, we suppose that R is a Nakayama oriented pullback of the surjective morphisms $f_A : A \twoheadrightarrow C$ and $f_B : B \twoheadrightarrow C$, and we use the notations of the definition 2.1.1.

Remarks 2.1.2. These conditions and 1.2.1 imply that:

- **(i)** For all $i \in (Q_A)_0$, $I(i)e_A = I(i)$,
- **(ii)** For all $j \in (Q_B)_0$, $P(j)e_B = P(j)$,
- **(iii)** For all $i \in (Q_A)_0 \backslash (Q_C)_0$, $P(i)e_A = P(i)$,
- **(iv)** For all $j \in (Q_B)_0 \backslash (Q_C)_0$, $I(j)e_B = I(j)$.

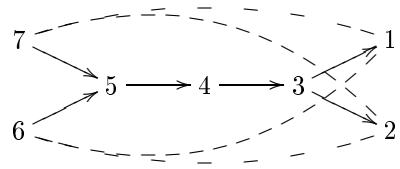
Examples 2.1.3. **(1)** In 1.2.2, condition (i) of 2.1.1 is not satisfied since there is an arrow from 7 to 6, whereas $7 \in (Q_C)_0$ and $6 \in (Q_A)_0 \backslash (Q_C)_0$.

(2) All conditions of 2.1.1 are satisfied in the following case:

$$Q_A = \begin{tikzcd} 7 \arrow[dr] & & & \\ & 5 \arrow[r] & 4 \arrow[r] & 3 \\ 6 \arrow[ur] & & & \end{tikzcd}, \qquad Q_B = \begin{tikzcd} & & & 1 \\ 5 \arrow[r] & 4 \arrow[r] & 3 \arrow[ur] \arrow[dr] & \\ & & & 2 \end{tikzcd},$$

$$Q_C = \quad 5 \longrightarrow 4 \longrightarrow 3$$

and $(Q_R, I_R) =$



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2.2. **Auslander-Reiten Quiver.** The category of modules $\operatorname{mod} R$ of a Nakayama oriented pullback R has a particular structure: every indecomposable R-module is isomorphic to an A- or to a B-module, and Γ_R is the pushout of the inclusion morphisms $\Gamma_C \hookrightarrow \Gamma_A$ and $\Gamma_C \hookrightarrow \Gamma_B$ (see 2.2.4 below).

This first lemma is essentially due to Igusa, Platzeck, Todorov and Zacharia (see [16], Lemma 3.5). This result is given in a context where A, B and C are connected, but this condition is not necessary. So the result and the proof are the same in our context.

Lemma 2.2.1. [16] $\operatorname{ind} R = \operatorname{ind} A \cup \operatorname{ind} B$ *and* $\operatorname{ind} A \cap \operatorname{ind} B = \operatorname{ind} C$. $\square$

The next result follows easily from the proof of 2.2.1 (see [16]).

Corollary 2.2.2. *Let* $M \in \operatorname{ind} A \backslash \operatorname{ind} C$. *Then* $\operatorname{top} M \in \operatorname{mod} A \backslash \operatorname{mod} C$. *Dually, let* $N \in \operatorname{ind} B \backslash \operatorname{ind} C$. *Then* $\operatorname{soc} M \in \operatorname{mod} B \backslash \operatorname{mod} C$. $\square$

The following results allow us to describe the structure of the Auslander-Reiten quiver of R. We recall that ν denotes the Nakayama functor and τ denotes the Auslander-Reiten translation.

Lemma 2.2.3.

(i) *Let* $M \in \operatorname{ind} A$ *be an* A*-module that is not* A*-projective. Then* $\tau_R M \cong \tau_A M$.
(ii) *Let* $M \in \operatorname{ind} B$ *be a* B*-module that is not* B*-injective. Then* $\tau_R^{-1} M \cong \tau_B^{-1} M$.
(iii) *Let* $M \in \operatorname{ind} C$ *be a* C*-module that is not* C*-projective. Then* $\tau_B M \cong \tau_C M$.
(iv) *Let* $M \in \operatorname{ind} C$ *be a* C*-module that is not* C*-injective. Then* $\tau_A^{-1} M \cong \tau_C^{-1} M$.
(v) *Let* $M \in \operatorname{ind} A$ *be a projective* A*-module that is not* R*-projective. Then* $\tau_R M \cong \tau_B M$.
(vi) *Let* $M \in \operatorname{ind} B$ *be an injective* B*-module that is not* R*-injective. Then* $\tau_R^{-1} M \cong \tau_A^{-1} M$.

Proof. **(i)** Since M is not A-projective, it cannot be R-projective.

Let $P_1 \to P_0 \to M \to 0$ be a minimal projective presentation of M in $\operatorname{mod} R$. By 2.2.2 and condition (i) of 2.1.1, we have $P_0 = \bigoplus_{i \in \mathcal{I}} (e_i R)^{m_i}$ and $P_1 = \bigoplus_{j \in \mathcal{J}} (e_j R)^{n_j}$ with $\mathcal{I} \subseteq (Q_A)_0$ and $\mathcal{J} \subseteq (Q_A)_0$ (if $M \in \operatorname{ind} C$, this follows from the fact that since M is not A-projective, it is not C-projective, and so the top of the kernel of $P_0 \to M \to 0$ is in $\operatorname{mod} C$).

Then

$$P_1' = \bigoplus_{j \in \mathcal{J}} (e_j R e_A)^{n_j} \to P_0' = \bigoplus_{i \in \mathcal{I}} (e_i R e_A)^{m_i} \to M_A \to 0$$

is a minimal projective presentation of M in $\operatorname{mod} A$. So we have the following exact sequences:

$$\begin{array}{rccccccccl} & 0 & \to & \tau_R M & \to & \nu_R P_1 & \to & \nu_R P_0 & \text{in } \operatorname{mod} R \\ \text{and} & 0 & \to & \tau_A M & \to & \nu_A P_1' & \to & \nu_A P_0' & \text{in } \operatorname{mod} A. \end{array}$$

By 2.1.2 (i) we have

$$\begin{array}{rrcccccccl} & \nu_R P_0 & = & \nu_R(\bigoplus_{i\in\mathcal{I}} e_i R^{m_i}) & \cong & \bigoplus_{i\in\mathcal{I}} I(i)_R^{m_i} & \cong & \bigoplus_{i\in\mathcal{I}} I(i)_A^{m_i} & \cong & \nu_A P_0' \\ \text{and} & \nu_R P_1 & = & \nu_R(\bigoplus_{j\in\mathcal{J}} e_j R^{n_j}) & \cong & \bigoplus_{j\in\mathcal{J}} I(j)_R^{n_j} & \cong & \bigoplus_{j\in\mathcal{J}} I(j)_A^{n_j} & \cong & \nu_A P_1' \,. \end{array}$$

Since these isomorphisms are functorial, we have $\tau_R M \cong \tau_A M$.
(ii) Dual of the proof of **(i)**, using 2.1.2 (ii).
(iii) Follows from **(i)**, by setting $Q_A = Q_C$ (and so $B = R$).
(iv) Follows from **(ii)**, by setting $Q_B = Q_C$ (and so $A = R$).
(v) Let M be an indecomposable A-module which is A-projective but not R-projective. Then $M \cong P(i)e_A$ with $i \in (Q_C)_0$. Therefore M has a non-projective B-module structure. Therefore there exists an indecomposable B-module M' such that $M \cong \tau_B^{-1}M'$. But it follows from **(ii)** that $\tau_B^{-1}M' \cong \tau_R^{-1}M'$. So $\tau_R M \cong \tau_R(\tau_R^{-1}M') \cong M' \cong \tau_B(\tau_B^{-1}M') \cong \tau_B M$.
(vi) Similar to the proof of **(v)**. □

It follows from this lemma that τ_A and τ_B are defined in the same way over the common objects of $\operatorname{ind} A$ and $\operatorname{ind} B$, which are the modules in $\operatorname{ind} C$. Also, we see that τ_R is defined for all modules in $\operatorname{ind} A$ and $\operatorname{ind} B$. Since $\operatorname{ind} R = \operatorname{ind} A \cup \operatorname{ind} B$ and $\operatorname{ind} C = \operatorname{ind} A \cap \operatorname{ind} B$, we naturally have the following corollary.

Corollary 2.2.4. *The Auslander-Reiten quiver Γ_R is the pushout of the quiver inclusions $\Gamma_C \hookrightarrow \Gamma_A$ and $\Gamma_C \hookrightarrow \Gamma_B$ with τ_R completely determined by τ_A and τ_B.*

Proof. By 2.2.1, we know that $\operatorname{ind} R = \operatorname{ind} A \cup \operatorname{ind} B$ and $\operatorname{ind} A \cap \operatorname{ind} B = \operatorname{ind} C$. It follows from 2.2.3 (iii) and (iv) that Γ_C is a full subquiver of Γ_A and Γ_B, and by 2.2.3 (i) and (ii) that Γ_A and Γ_B are full subquivers of Γ_R.

The definition of τ_R follows directly from 2.2.3.

It is now easy to see that $\Gamma_R = \Gamma_A \amalg_{\Gamma_C} \Gamma_B$, which completes the proof. □

2.3. **Tilting Module.** In this subsection, we construct a tilting module over R using tilting modules over C, A and B. More precisely, we first take a tilting C-module T'', which we complete to a tilting A-module $T = N \oplus T''$ and to a tilting B-module $T' = N' \oplus T''$. Here, we see the importance of the condition stating that there is no minimal relation of R that has its origin in Q_C (2.1.1 (iv)); without this condition the existence of a module like T' is not ensured.

So let $T = N \oplus T''$ and $T' = N' \oplus T''$ be tilting modules over A and B respectively, with T'' tilting over C.

Lemma 2.3.1. *Every indecomposable direct summand of N is in $\operatorname{ind} A \setminus \operatorname{ind} C$, and every indecomposable direct summand of N' is in $\operatorname{ind} B \setminus \operatorname{ind} C$.*

Proof. Since T'' is tilting over C, it has t isomorphism classes of indecomposable summands, and their dimension vectors form a basis of $K_0(C)$ (see [12], Proposition 3.2). If N has an indecomposable summand in $\operatorname{ind} C$, its dimension vector is a linear combination of the t dimension vectors of the indecomposable summands of T''; it gives $t+1$ linearly dependent dimension vectors in $K_0(C)$, and so in $K_0(R)$, a contradiction. Therefore every indecomposable summand of N is in $\operatorname{ind} A \setminus \operatorname{ind} C$. Similarly, every indecomposable summand of N' is in $\operatorname{ind} B \setminus \operatorname{ind} C$. □

Recall that Q_C has r connected components of type $\mathbb{A}_n$, and KQ_C is a hereditary Nakayama algebra. So for each of these components there is an indecomposable

C-module which is both injective and projective. We denote by Q_1, Q_2, ..., Q_r these projective-injective C-modules ($Q_i = P(a_{i,t_i})e_C$). For each i between 1 and r, the module Q_i is necessarily a summand of T'' (see [3]).

Lemma 2.3.2. *Let M be an indecomposable direct summand of N. Then* $\operatorname{pd} M_R \leq 1$. *Moreover, if $f : P \twoheadrightarrow M$ is a projective cover of M in* $\operatorname{mod} R$, *then, for all $j \in (Q_C)_0$, $e_j R e_A$ is not a direct summand of the kernel L of f.*

Proof. Since M is a summand of N, we have $M \in \operatorname{ind} A \backslash \operatorname{ind} C$ (2.3.1). Hence, by 2.2.2, $\operatorname{top} M$ is in $\operatorname{mod} A \backslash \operatorname{mod} C$, and so by 2.1.2 (iii) we have $Pe_A = P$. Therefore the kernel L of $f : P \twoheadrightarrow M$ is an A-module, and so $Le_A = L$.

Suppose that $\operatorname{pd} M_R > 1$. Since T is tilting over A, we have $\operatorname{pd} M_A \leq 1$. So L has an indecomposable summand which is A-projective but not R-projective. Therefore there exists $j \in (Q_C)_0$ such that $e_j R e_A$ is not an R-projective summand of L. Let $L = L' \oplus e_j R e_A$.

We have the following exact sequence in $\operatorname{mod} A$:

$$0 \longrightarrow \tau_A M \xrightarrow{f} \nu_A L \xrightarrow{g} \nu_A P$$

with $\nu_A L \cong (\nu_A L' \oplus I(j))$ and $\nu_A P \cong \bigoplus_{l \in \mathcal{J}} I(l)$ (here, $\mathcal{J} \subseteq (Q_A)_0 \backslash (Q_C)_0$ since $\operatorname{top} M$ is in $\operatorname{mod} A \backslash \operatorname{mod} C$).

Since $j \in (Q_C)_0$ and $\mathcal{J} \subseteq (Q_A)_0 \backslash (Q_C)_0$, the C-submodule $I(j)e_C$ of $I(j)e_A$ is a submodule of $\operatorname{Ker} g = \operatorname{Im} f$. So $I(j)e_C$ is a submodule of $\tau_A M$.

However, there exists i between 1 and r such that we have an epimorphism $Q_i \twoheadrightarrow I(j)e_C$ (where Q_i is the module $P(a_{i,t_i})e_C$ which is both C-projectice and C-injective). Hence $\operatorname{Hom}_A(Q_i,\ \tau_A M) \neq 0$; this contradicts the fact that T is tilting over A (since Q_i and M are both summands of T). So $\operatorname{pd} M_R \leq 1$.

Therefore, L is R-projective. Since for all $j \in (Q_C)_0$ we have $e_j R e_A \neq e_i R$ (with i the index of Q_i), $e_j R e_A$ cannot be a summand of L. □

Lemma 2.3.3. *We have* $\operatorname{Hom}_R(N', \tau_R N) = 0$.

Proof. Let M be an indecomposable summand of N. If M is R-projective, we have $\operatorname{Hom}_R(N', \tau_R M) = 0$. Otherwise, it follows from 2.3.2 that $\operatorname{pd} M_R = 1$. Let P be the projective cover of M, and L be the kernel of a projective cover $P \twoheadrightarrow M$. Then L is R-projective, and so $L = \bigoplus_{j \in \mathcal{J}} e_j R$ with $\mathcal{J} \subseteq (Q_A)_0 \backslash (Q_C)_0$ (see 2.3.2).

Therefore we have the following exact sequence in $\operatorname{mod} R$:

$$0 \longrightarrow \tau_R M \longrightarrow \nu_R L \cong \oplus_{j \in \mathcal{J}} I(j) \longrightarrow \nu_R P\ .$$

Since $\mathcal{J} \subseteq (Q_A)_0 \backslash (Q_C)_0$, the support of $\nu_R L$ is completely contained in $Q_A \backslash Q_C$, and so is the support of $\tau_R M$. As $N' \in \operatorname{ind} B \backslash \operatorname{ind} C$, we have $\operatorname{Hom}_R(N', \tau_R M) = 0$. Therefore $\operatorname{Hom}_R(N', \tau_R N) = 0$. □

It is now possible to prove the principal result of this subsection.

Theorem 2.3.4. *Let $R \cong KQ_R/I_R$ be a Nakayama oriented pullback of K-algebras surjections $f_A : A \twoheadrightarrow C$ and $f_B : B \twoheadrightarrow C$. Let $T = N \oplus T''$ and $T' = N' \oplus T''$ be tilting modules over A and B respectively, with T'' tilting over C. Then $T''' = N \oplus T'' \oplus N'$ is tilting over R.*

Proof. Since T, T' and T'' are tilting over A, B and C respectively, it follows from 2.3.1 that T''' has $|(Q_R)_0|$ isomorphism classes of indecomposable summands, which is exactly the number of isomorphism classes of simple R-modules.

By 2.3.2, we have $\operatorname{pd} N_R \leq 1$. It follows from 2.1.2 (ii) and from the fact that $T' = T'' \oplus N'$ is tilting over B that $\operatorname{pd} N'_R \leq 1$ and $\operatorname{pd} T''_R \leq 1$.

We have

$$\begin{aligned} \operatorname{Ext}^1_R(T''', T''') &= \operatorname{Ext}^1_R(N \oplus T'' \oplus N', N \oplus T'' \oplus N') \\ &\cong D \operatorname{Hom}_R(N \oplus T'' \oplus N', \tau_R(N \oplus T'' \oplus N')). \end{aligned}$$

By 2.3.3 we have $\operatorname{Hom}_R(N', \tau_R N) = 0$. By 2.2.4 we have $\operatorname{Hom}_R(N, \tau_R N') = 0$ and $\operatorname{Hom}_R(N, \tau_R T'') = 0$.

Since $N \in \operatorname{ind} A \setminus \operatorname{ind} C$, the module N is A-projective if and only if it is R-projective (2.1.2 (iii)). So by 2.2.3 and 2.2.4 we have $D \operatorname{Hom}_R(N, \tau_R N) \cong D \operatorname{Hom}_A(N, \tau_A N) = 0$ since T_A is tilting over A.

Similarly, we find $D \operatorname{Hom}_R(T'', \tau_R N) \cong D \operatorname{Hom}_A(T'', \tau_A N) = 0$.

The other steps to show that $\operatorname{Ext}^1_R(T''', T''') = 0$ are done similarly using the fact that T'_B is tilting over B. $\square$

Examples 2.3.5. (1) Let T'' be a tilting module over C which is partial tilting over A and over B. By Bongartz' lemma, there exist an A-module N and a B-module N' such that $T'' \oplus N$ and $T'' \oplus N'$ are tilting over A and B respectively. Then it follows from 2.3.4 that $N \oplus T'' \oplus N'$ is tilting over R.

(2) Let A, B and C be hereditary algebras whose ordinary quivers are the following:

$$Q_A = \begin{tikzcd} & 3 \arrow[r] & 2 & \\ 7 \arrow[ur] \arrow[dr] & & & \\ & 6 \arrow[r] & 5 \arrow[r] & 4 \end{tikzcd}, \qquad Q_B = \begin{tikzcd} 3 \arrow[r] & 2 \arrow[dr] & & \\ & & & 1 \\ 6 \arrow[r] & 5 \arrow[r] & 4 \arrow[ur] & \end{tikzcd}$$

$$\text{and } Q_C = \begin{tikzcd} 3 \arrow[r] & 2 & \\ 6 \arrow[r] & 5 \arrow[r] & 4 \end{tikzcd}.$$

We easily see that $T'' = I(2)e_C \oplus S(3) \oplus I(4)e_C \oplus I(5)e_C \oplus S(6)$ is tilting over C, $T'' \oplus P(7)$ is tilting over A and $T'' \oplus I(1)$ is tilting over B. Therefore by 2.3.4 the module $P(7) \oplus T'' \oplus I(1)$ is tilting over R, and the bound quiver of R is the following:

$$\begin{tikzcd} & 3 \arrow[r] & 2 \arrow[dr] & \\ 7 \arrow[ur] \arrow[dr] \arrow[urr, dashed] \arrow[drr, dashed] & & & 1 \\ & 6 \arrow[r] & 5 \arrow[r] & 4 \arrow[ul] \end{tikzcd}.$$

(3) Let A be an algebra, x be a source of Q_A such that $S(x)$ is partial tilting over A, and T be a tilting A-module that has $S(x)$ as a summand (Bongartz' lemma ensures the existence of such a module T). Consider the one-point extension $R = A[S(x)]$ of A by $S(x)$, and let y be the new vertex added to Q_A to obtain Q_R. It is clear that R is a Nakayama oriented pullback, and it follows from 2.3.4 that $P(y) \oplus T$ is tilting over R (since $P(y) \oplus S(x)$ is tilting over the hereditary algebra whose ordinary quiver is $y \to x$).

2.4. **The case where B is hereditary.** We now consider the particular case where B is hereditary. We construct an R-module using tilting modules over C, A and B, and we show that it is tilting over R using 2.3.4. The particularity of this module is that the minimal relations of its endomorphism algebra are the minimal relations of A (we "remove" the other relations). We then conclude that R is tilted when A is also hereditary.

So let

$$T'' = \bigoplus_{i\in(Q_C)_0} P(i)e_C, \quad N = \bigoplus_{i\in(Q_A)_0\setminus(Q_C)_0} P(i) \text{ and } N' = \bigoplus_{i\in(Q_B)_0\setminus(Q_C)_0} I(i).$$

We clearly see that T'' is a C-module, N an A-module and N' a B-module.

Lemma 2.4.1. *T'' is tilting over C and partial tilting over A and B.*

Proof. We have $T''_C \cong C$ as a C-module, so it is tilting over C. Moreover, T'' is A-projective, and so is partial tilting over A.

Since B is hereditary, the projective dimension of T'' over B is at most one. As T'' is tilting over C, we have $\operatorname{Ext}^1_C(T'', T'') = 0$. Since C is full and convex in B, we also have $\operatorname{Ext}^1_B(T'', T'') = 0$. □

Lemma 2.4.2. *$T'' \oplus N$ is tilting over A.*

Proof. This follows from the fact that $T'' \oplus N \cong A$ as an A-module. □

Lemma 2.4.3. *$T'' \oplus N'$ is tilting over B.*

Proof. Let $T' = T'' \oplus N'$. Since B is hereditary, we have $\operatorname{pd} T'_B \leq 1$. It is clear that the number of isomorphism classes of indecomposable summands of T' is equal to the rank of $K_0(B)$.

Using the B-injectivity of N' and 2.4.1, we have

$$\begin{aligned}
\operatorname{Ext}^1_B(T'' \oplus N', T'' \oplus N') &\cong \operatorname{Ext}^1_B(T'' \oplus N', T'') \\
&\cong \operatorname{Ext}^1_B(T'', T'') \oplus \operatorname{Ext}^1_B(N', T'') \\
&= \operatorname{Ext}^1_B(N', T'') \\
&\cong D\operatorname{Hom}_B(T'', \tau_B N').
\end{aligned}$$

Let M be an indecomposable summand of N'. Suppose that $\tau_B M$ is a C-module. Then $\tau_B M$ cannot be C-injective because C-injective modules are also B-injective. Therefore, it follows from 2.2.3 (iv) that $\tau_A^{-1}(\tau_B M) \cong \tau_C^{-1}(\tau_B M)$. Moreover, by 2.2.4, we have $\tau_B M \cong \tau_R M$ and $\tau_A^{-1}(\tau_B M) \cong \tau_R^{-1}(\tau_B M)$. So

$$\begin{aligned}
\tau_C^{-1}(\tau_B M) &\cong \tau_A^{-1}(\tau_B M) \\
&\cong \tau_R^{-1}(\tau_R M) \\
&\cong M.
\end{aligned}$$

However, $\tau_C^{-1}(\tau_B M)$ is a C-module, which is a contradiction since M is in $\operatorname{ind} B \setminus \operatorname{ind} C$. Hence $\tau_B M$ cannot be a C-module, and neither can every indecomposable summand of $\tau_B N'$. Therefore, it follows from 2.2.2 that $\operatorname{soc}(\tau_B N')$ is in $\operatorname{mod} B \setminus \operatorname{mod} C$. So since T'', as a C-module, does not have any composition factor in $\operatorname{mod} B \setminus \operatorname{mod} C$, we have $\operatorname{Hom}_B(T'', \tau_B N') = 0$.

Hence $\operatorname{Ext}^1_B(T', T') = 0$, and T' is tilting over B. □

Proposition 2.4.4. *Let $T = N \oplus T'' \oplus N'$. Then T is tilting over R.*

Proof. This follows from 2.4.1, 2.4.2, 2.4.3, and 2.3.4. □

Now, let us consider a particular quiver construction, which we need in order to describe $Q_{\operatorname{End} T}$.

Definition 2.4.5. *Let Q be a quiver and Q'' a full subquiver of Q. The* **difference of quivers** *$Q\backslash Q''$ is the subquiver of Q such that $(Q\backslash Q'')_1 = Q_1\backslash Q''_1$ and*

$$(Q\backslash Q'')_0 = (Q_0\backslash Q''_0) \cup \left\{ x \in Q''_0 \;\middle|\; \begin{array}{c} \text{there exists } \alpha \in (Q\backslash Q'')_1 \text{ such that} \\ x = s(\alpha) \text{ or } x = t(\alpha) \end{array} \right\}.$$

Now, consider the difference $Q_B\backslash Q_C$. Then, in $Q_B\backslash Q_C$, we identify the vertices that are sinks in Q_C to the sources t_1, t_2, ..., t_r of Q_C, and we denote this new quiver by Q_{BC}.

For instance, if

$$Q_B = \begin{tikzcd} 7 \arrow[r] & 6 \arrow[r] & 5 \arrow[dr] & & \\ & & & 2 \arrow[r] & 1 \\ & 4 \arrow[r] & 3 \arrow[ur] & & \end{tikzcd} \quad \text{and} \quad Q_C = \begin{tikzcd} 7 \arrow[r] & 6 \arrow[r] & 5 \\ & 4 \arrow[r] & 3 \end{tikzcd},$$

then

$$Q_B\backslash Q_C = \begin{tikzcd} 5 \arrow[dr] & & \\ & 2 \arrow[r] & 1 \\ 3 \arrow[ur] & & \end{tikzcd} \quad \text{and} \quad Q_{BC} = \begin{tikzcd} 7 \arrow[dr] & & \\ & 2 \arrow[r] & 1 \\ 4 \arrow[ur] & & \end{tikzcd}.$$

Finally, we consider the quiver $Q_A \amalg_{Q_{C_S}} Q_{BC}$ where $Q_{C_S} = \bullet_{t_1} \; \bullet_{t_2} \; \cdots \; \bullet_{t_r}$ (the sources of the connected components of Q_C).

If we continue the previous example with

$$Q_A = \begin{tikzcd} & 7 \arrow[r] & 6 \arrow[r] & 5 \\ 9 \arrow[ur] \arrow[dr] & & & \\ & 8 \arrow[r] & 4 \arrow[r] & 3 \end{tikzcd},$$

we obtain

$$Q_A \amalg_{Q_{C_S}} Q_{BC} = \begin{tikzcd} & 7 \arrow[r] \arrow[ddr] & 6 \arrow[r] & 5 & \\ 9 \arrow[ur] \arrow[dr] & & & 2 \arrow[r] & 1 \\ & 8 \arrow[r] & 4 \arrow[r] & 3 \arrow[ur] & \end{tikzcd}.$$

We easily see that $|(Q_R)_0| = |(Q_A \amalg_{Q_{C_S}} Q_{BC})_0|$. Actually, from now on, we identify the vertices of Q_A to those which naturally correspond to them in the quiver $Q_A \amalg_{Q_{C_S}} Q_{BC}$, and we do the same for the vertices of Q_B that are not in Q_C.

Theorem 2.4.6. *Let $T = N \oplus T'' \oplus N'$. The endomorphism algebra* $\operatorname{End} T$ *admits a presentation $(Q_{\operatorname{End} T},\ I_{\operatorname{End} T})$ with $Q_{\operatorname{End} T} = Q_A \amalg_{Q_{C_S}} Q_{BC}$ and $I_{\operatorname{End} T} = I_A$.*

Proof. Similar to the proof of [19] (2.3), and done in [20] (3.4.5). □

As a direct consequence, we obtain the main result of this paper.

Theorem 2.4.7. *Let $R \cong KQ_R/I_R$ be a Nakayama oriented pullback of K-algebra surjective morphisms $A \twoheadrightarrow C$ and $B \twoheadrightarrow C$. Suppose that A and B are hereditary. Then R is tilted.* □

Example 2.4.8. Consider the bound quiver of 2.3.5 (2). By 2.4.4, the module $T = I(1) \oplus S(2) \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus S(4) \oplus \begin{smallmatrix}5\\4\end{smallmatrix} \oplus \begin{smallmatrix}6\\5\\4\end{smallmatrix} \oplus P(7)$ is tilting over R, and by 2.4.6 and 2.4.7, $\operatorname{End} T$ is hereditary with

$$Q_{\operatorname{End} T} = \begin{tikzcd} & 3 \arrow[r] \arrow[dr] & 2 & \\ 7 \arrow[ur] \arrow[dr] & & 1 & \\ & 6 \arrow[r] \arrow[ur] & 5 \arrow[r] & 4 \end{tikzcd} .$$

3. Iterated Tilted and Tilted Stably hereditary algebras

3.1. **The bound quiver of a stably hereditary algebra.** In our main results, we use some properties of the bound quiver of a stably hereditary algebra that are easy to identify. This subsection is therefore devoted to the bound quiver of a stably hereditary algebra.

Let $R = KQ/I$ be a stably hereditary algebra. Then, by [6], we have $I = I_{\Sigma_R}$ with

$$\Sigma_R = \left\{ \ x \in Q_0 \ \middle| \ S(x) \text{ is a non-projective submodule of } R \ \right\}$$

and

$$I_{\Sigma_R} = \left\langle \ \alpha\beta \ \middle| \ t(\alpha) \in \Sigma_R \ \right\rangle.$$

That is, I_{Σ_R} is the ideal generated by all paths $\alpha\beta$ with $t(\alpha) = s(\beta) \in \Sigma_R$. In particular, R is a monomial algebra (that is, I_{Σ_R} is generated by monomial relations).

Definitions 3.1.1. *A cycle C in a bound quiver (Q, I) satisfies the* **clock condition** *if the number of clockwise oriented relations on C equals the number of counterclockwise oriented relations. We say that (Q, I) satisfies the* **clock condition** *if all cycles in (Q, I) satisfy the clock condition.*

The following theorem, due to Skowroński, allows us to characterize the bound quiver of a stably hereditary algebra using the clock condition. For the notion of special cycle in G_{eRe}, we refer the reader to [24].

Theorem 3.1.2. [24] *Let S be an algebra tilting equivalent to a hereditary or to a canonical algebra. Then for any idempotent e of S, any special cycle C in G_{eSe} satisfies the clock condition. In particular, Q_S has no oriented cycles.* □

Actually, in our context of stably hereditary algebras, the above statement can be reformulated to say that if R is tilting equivalent to a hereditary algebra, then any cycle of (Q, I_{Σ_R}) satisfies the clock condition. Therefore, if (Q, I_{Σ_R}) does not satisfy the clock condition, R is not iterated tilted. Hence, from now on, we suppose that R is a stably hereditary algebra such that its bound quiver (Q, I_{Σ_R}) satisfies the clock condition. In particular, R is a triangular algebra.

We want to decompose Q into maximal subquivers which do not contain any relation, that is, the ordinary quivers of the algebras $R_1, ..., R_n$ such that R is stably equivalent to $R_1 \times R_2 \times ... \times R_n$ (see [6]).

Let $\alpha \in Q_1$, and let Q_α be the subquiver of Q such that

$$(Q_\alpha)_1 = \left\{ \ \beta \in Q_1 \ \middle| \ \begin{array}{c} \text{there exists a non-zero walk } w \\ \text{such that } w = \alpha^* w' \beta^* \text{ where} \\ \alpha^* \in \{\alpha, \alpha^{-1}\}, \ \ \beta^* \in \{\beta, \beta^{-1}\} \end{array} \ \right\}$$

and

$$(Q_\alpha)_0 = \left\{ \; s(\beta), t(\beta) \in Q_0 \quad \middle| \quad \beta \in (Q_\alpha)_1 \; \right\}.$$

Remarks 3.1.3.

(i) Since R is a stably hereditary algebra, we easily see that Q_α is a full subquiver of Q. Moreover, since (Q, I_{Σ_R}) satisfies the clock condition, Q_α is convex in Q.

(ii) Since (Q, I_{Σ_R}) satisfies the clock condition, every walk w containing zero-relations that all point in the same direction in w (and these are the only relations on w)

$$w = \bullet \longrightarrow p_1 \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow p_2 \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow p_s \longrightarrow \bullet$$

is such that $p_1 \neq p_2 \neq ... \neq p_s$. In particular, every double-zero in (Q, I_{Σ_R}) contains two distinct vertices of Σ_R.

(iii) Let $\alpha \in Q_1$, and $\beta \in (Q_\alpha)_1$. Then $Q_\alpha = Q_\beta$.

(iv) If $Q_\alpha \neq Q_\beta$, then $(Q_\alpha)_1 \cap (Q_\beta)_1 = \emptyset$ and $(Q_\alpha)_0 \cap (Q_\beta)_0 \subseteq \Sigma_R$. On the other hand, for all $x \in \Sigma_R$, there exist α, $\beta \in Q_1$ such that $Q_\alpha \neq Q_\beta$ and $x \in (Q_\alpha)_0 \cap (Q_\beta)_0$.

(v) For all $\alpha \in Q_1$ and $x \in (Q_\alpha)_0 \backslash \Sigma_R$, we have $I(x) \in \operatorname{ind} KQ_\alpha$ and $P(x) \in \operatorname{ind} KQ_\alpha$.

Let $\alpha_1, \alpha_2, ..., \alpha_t \in Q_1$ be such that $Q_{\alpha_i} \neq Q_{\alpha_j}$ when $i \neq j$, and such that for all $\beta \in Q_1$, there exists $i \in \{1, ..., t\}$ such that $\beta \in (Q_{\alpha_i})_1$. We set $\mathcal{I}(R) = \{\alpha_1, \alpha_2, ..., \alpha_t\}$.

Then $Q_1 = \bigcup_{i=1}^t (Q_{\alpha_i})_1$ and $Q_0 = \bigcup_{i=1}^t (Q_{\alpha_i})_0$. It follows from the remarks above and from [6] that R is stably equivalent to $KQ_{\alpha_1} \times KQ_{\alpha_2} \times ... \times KQ_{\alpha_t}$.

Example 3.1.4. Let R be a stably hereditary algebra whose bound quiver $(Q,\ I_{\Sigma_R})$ is the following:

$$\begin{tikzcd}
 & & 5 \arrow[r, "\beta_5"] & 3 & & & & \\
10 \arrow[r, "\beta_{10}"] & 9 \arrow[ur, "\beta_9"] \arrow[dr, "\beta_8"'] & & & 4 \arrow[ul, "\beta_3"'] & 6 \arrow[l, "\beta_6"'] \arrow[r, "\beta_2"'] & 2 \arrow[r, "\beta_1"'] & 1 \\
 & & 8 \arrow[r, "\beta_7"] & 7 \arrow[ur, "\beta_4"'] & & & &
\end{tikzcd}$$

A possible choice for $\mathcal{I}(R)$ is $\mathcal{I}(R) = \{\beta_1, \beta_2, \beta_3, \beta_{10}\}$, and then we have

$$Q_{\beta_{10}} = \quad 10 \xrightarrow{\beta_{10}} 9 \;, \qquad Q_{\beta_3} = \quad \begin{tikzcd} 5 \arrow[r, "\beta_5"] & 3 & \\ & & 4 \arrow[ul, "\beta_3"'] \end{tikzcd} \;,$$

$$Q_{\beta 2} = \quad \begin{tikzcd}
 & 5 & & & \\
9 \arrow[ur, "\beta_9"] \arrow[dr, "\beta_8"'] & & & 4 & 6 \arrow[l, "\beta_6"'] \arrow[r, "\beta_2"'] & 2 \\
 & 8 \arrow[r, "\beta_7"] & 7 \arrow[ur, "\beta_4"'] & & &
\end{tikzcd} \quad \text{and} \quad Q_{\beta_1} = \quad 2 \xrightarrow[\beta_1]{} 1 \;.$$

In the following lemma, we show that when a stably hereditary algebra respects the clock condition, it is always possible to describe it as a pullback of surjective algebra morphisms.

Lemma 3.1.5. *Let $R \cong KQ/I_{\Sigma_R}$. There exist K-algebras A and C and arrows α_{i1}, α_{i2}, ..., $\alpha_{is} \in \mathcal{I}(R)$ such that R is the pullback of surjective morphisms of K-algebras $A \twoheadrightarrow C$ and $KQ_{\alpha_{i1}} \times KQ_{\alpha_{i2}} \times ... \times KQ_{\alpha_{is}} \twoheadrightarrow C$.*

Proof. Let α_{i1}, α_{i2}, ..., $\alpha_{is} \in \mathcal{I}(R)$ be such that $(Q_{\alpha_{ij}})_0 \cap (Q_{\alpha_{ij'}})_0 = \emptyset$ when $j \neq j'$. Let $B = KQ_{\alpha_{i1}} \times KQ_{\alpha_{i2}} \times ... \times KQ_{\alpha_{is}}$, and let Q_C be the quiver such that $(Q_C)_0 = \Sigma_R \cap (Q_B)_0$ (and $(Q_C)_1 = \emptyset$). Let $\mathcal{I}(R)^* = \mathcal{I}(R) \backslash \{\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{is}\}$ (it follows from 3.1.3 (iv) that $\mathcal{I}(R)^* \neq \emptyset$). Let $C = KQ_C$, and $A = KQ_A/I_A$ with Q_A the full subquiver of Q generated by $\bigcup_{\alpha_j \in \mathcal{I}(R)^*} (Q_{\alpha_j})_0$ and $I_A =< \alpha\beta \mid t(\alpha) \in \Sigma_R \backslash (Q_B)_0 >$.

We easily see that the connected components of Q_C are full and convex in Q_A and Q_B, and that C is a common quotient of A and B (see 3.1.3 (iv)). Let R' be the pullback of the canonical projections $A \twoheadrightarrow C$ and $B \twoheadrightarrow C$. Then $Q_{R'} = Q_A \amalg_{Q_C} Q_B$ (see 1.2.1), and $R' = KQ_{R'}/I_{R'}$ with $I_{R'}$ generated by I_A and by the set $\bar{\rho}$ (as described in 1.2.1).

But it follows from 3.1.3 that the quiver of R is exactly $Q_A \amalg_{Q_C} Q_B$, and

$$\begin{aligned} I_{\Sigma_R} &= < \alpha\beta \mid t(\alpha) \in \Sigma_R > \\ &= < \{\alpha\beta \mid t(\alpha) \in \Sigma_R \backslash (Q_B)_0\} \cup \{\alpha\beta \mid t(\alpha) \in \Sigma_R \cap (Q_B)_0\} > \\ &= < \{\alpha\beta \mid t(\alpha) \in \Sigma_R \backslash (Q_B)_0\} \cup \bar{\rho} > \\ &= I_{R'}. \end{aligned}$$

So $R \cong R'$ and R is the pullback of the canonical projections $A \twoheadrightarrow C$ and $B \twoheadrightarrow C$. □

In this case, we say that $R = A \amalg_C B$ is the **pullback associated to** $KQ_{\alpha_{i1}} \times KQ_{\alpha_{i2}} \times ... \times KQ_{\alpha_{is}}$.

We now describe $(Q,\ I_{\Sigma_R})$ in a way showing that there always exist α_{i1}, α_{i2}, ..., $\alpha_{is} \in \mathcal{I}(R)$ such that the pullback associated to $KQ_{\alpha_{i1}} \times KQ_{\alpha_{i2}} \times ... \times KQ_{\alpha_{is}}$ is a Nakayama oriented pullback which respects all the conditions of 2.4 (with $B = KQ_{\alpha_{i1}} \times KQ_{\alpha_{i2}} \times ... \times KQ_{\alpha_{is}}$). This description also allows us to state a sufficient and necessary condition for $(Q,\ I_{\Sigma_R})$ to contain a double-zero.

So, first, let

$$V_1 = \left\{ \ \alpha_i \in \mathcal{I}(R) \ \middle| \ \text{there exists } \beta \in (Q_{\alpha_i})_1 \text{ such that } t(\beta) \in \Sigma_R \ \right\}$$

and

$$V_2 = \left\{ \ \alpha_i \in \mathcal{I}(R) \ \middle| \ \text{there exists } \beta \in (Q_{\alpha_i})_1 \text{ such that } s(\beta) \in \Sigma_R \ \right\}.$$

In 3.1.4, we have $V_1 = \{\beta_2,\ \beta_{10}\}$ and $V_2 = \{\beta_1,\ \beta_2,\ \beta_3\}$.

Remark 3.1.6. Let α_i, $\alpha_j \in V_2 \backslash V_1$ with $\alpha_i \neq \alpha_j$. Then $(Q_{\alpha_i})_0 \cap (Q_{\alpha_j})_0 = \emptyset$. Indeed, if the intersection is non-empty, it follows from 3.1.3 (iv) that there exists $x \in \Sigma_R$ which is contained by both Q_{α_i} and Q_{α_j}. But this contradicts the fact that α_i and α_j are both in $V_2 \backslash V_1$.

Lemma 3.1.7. *Let $R \cong KQ/I_{\Sigma_R}$. Then there exists $\alpha_i \in V_2 \backslash V_1$.*

Proof. Let $\alpha_{i1} \in \mathcal{I}(R)$. If $\alpha_{i1} \in V_2 \backslash V_1$, we are done.

If this is not the case, then $Q_{\alpha_{i1}}$ contains an arrow β_{i1} such that $t(\beta_{i1}) \in \Sigma_R$.

$$\begin{tikzcd} \bullet \arrow[r, "\beta_{i1}"'] \arrow[rr, bend left, dashed] & p_{i1} \arrow[r, "\beta'_{i1}"'] & \bullet \end{tikzcd}$$

Let $\alpha_{i2} \in \mathcal{I}(R)\backslash\{\alpha_{i1}\}$ such that $\beta'_{i1} \in (Q_{\alpha_{i2}})_1$. If $\alpha_{i2} \in V_2\backslash V_1$, we are done. Otherwise, $Q_{\alpha_{i2}}$ contains an arrow β_{i2} such that $t(\beta_{i2}) \in \Sigma_R$, and so there is a double-zero of the form

$$\begin{tikzcd} \bullet \arrow[r, "\beta_{i1}"'] \arrow[rr, bend left, dashed] & p_{i1} \arrow[r, "\beta'_{i1}"'] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[r, "\beta_{i2}"'] \arrow[rr, bend left, dashed] & p_{i2} \arrow[r, "\beta' i2"'] & \bullet \end{tikzcd} \qquad (p_{i1} \neq p_{i2} \text{ by } 3.1.3).$$

We then consider $\alpha_{i3} \in \mathcal{I}(R)\backslash\{\alpha_{i1}, \alpha_{i2}\}$ such that $\beta'_{i2} \in (Q_{\alpha_{i3}})_1$. We repeat the argument, and since $|\mathcal{I}(R)|$ and $|\Sigma_R|$ are finite, the statement follows by induction. □

Lemma 3.1.8. *The bound quiver $(Q,\ I_{\Sigma_R})$ contains no double-zero if and only if $V_1 \cap V_2 = \emptyset$.*

Proof. If $V_1 \cap V_2 \neq \emptyset$, there exists $\alpha_i \in \mathcal{I}(R)$ such that Q_{α_i} contains an arrow whose target is in Σ_R, and an arrow whose source is in Σ_R. This implies the existence of a double-zero in (Q, I_{Σ_R}).

On the other hand, if (Q, I_{Σ_R}) contains a double-zero,

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left, dashed] & p_1 \arrow[r, "\beta_1"'] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[r, "\beta_2"'] \arrow[rr, bend left, dashed] & p_2 \arrow[r] & \bullet \end{tikzcd}$$

we see that there exists a non-zero walk containing the arrows β_1 and β_2. Hence there exists $\alpha_i \in \mathcal{I}(R)$ such that $\beta_1, \beta_2 \in (Q_{\alpha_i})_1$, and so $\alpha_i \in V_1 \cap V_2$. □

Lemma 3.1.9. *Let α_{i1}, α_{i2}, ..., $\alpha_{is} \in V_2\backslash V_1$. Then there exist K-algebras A and C such that R is the pullback of morphisms $A \twoheadrightarrow C$, $KQ_{\alpha_{i1}} \times KQ_{\alpha_{i2}} \times \cdots \times KQ_{\alpha_{is}} \twoheadrightarrow C$, and this pullback is Nakayama oriented.*

Proof. The first statement was proven in 3.1.5.

We now have to verify that R, as the pullback associated to $KQ_{\alpha_{i1}} \times KQ_{\alpha_{i2}} \times \cdots \times KQ_{\alpha_{is}}$, satisfies the conditions of 2.1.1.

The condition saying that there is no path from $(Q_C)_0$ to $(Q_A)_0\backslash(Q_C)_0$ nor from $(Q_B)_0\backslash(Q_C)_0$ to $(Q_C)_0$ is satisfied since α_{i1}, α_{i2}, ..., $\alpha_{is} \in V_2\backslash V_1$.

We easily see that the conditions (ii) and (iii) of 2.1.1 are satisfied.

The last condition (which says that there is no minimal relation of R having its origin in $(Q_C)_0$) is satisfied since α_{i1}, α_{i2}, ..., $\alpha_{is} \in V_2\backslash V_1$ and B is hereditary. □

Hence, in the context of this lemma, R satisfies the conditions of 2.4 since $B = KQ_{\alpha_{i1}} \times KQ_{\alpha_{i2}} \times \cdots \times KQ_{\alpha_{is}}$ is hereditary. So we can use the results of 2.4 to construct a tilting R-module in the next subsection.

3.2. **Tilting module.** To show that a stably hereditary algebra which respects the clock condition is iterated tilted, we need a particular tilting module T, and we give its construction in 3.2.1. We see in 3.2.2 that $\operatorname{End} T$ is also a stably hereditary algebra which respects the clock condition. Moreover, the bound quiver of $\operatorname{End} T$ contains fewer relations. This is the key to the proof that R is iterated tilted.

Lemma 3.2.1. *Let $J \subseteq \{1, ..., t\}$ be such that for all $j \in J$, $\alpha_j \in V_2\backslash V_1$. Let $Q_{J_0} = \cup_{j\in J}(Q_{\alpha_j})_0$. Then*

$$T = \Big(\bigoplus_{i\in Q_0\backslash Q_{J_0}} P(i)\Big) \oplus \Big(\bigoplus_{p\in\Sigma_R\cap Q_{J_0}} S(p)\Big) \oplus \Big(\bigoplus_{k\in Q_{J_0}\backslash\Sigma_R} I(k)\Big)$$

is a tilting R-module.

Proof. Follows from 3.1.9 and 2.4.4. □

Lemma 3.2.2. *Let $J \subseteq \{1, ..., t\}$ be such that for all $j \in J$, $\alpha_j \in V_2 \backslash V_1$. Let $Q_{J_0} = \cup_{j \in J} (Q_{\alpha_j})_0$, and*

$$T = \Big(\bigoplus_{i \in Q_0 \backslash Q_{J_0}} P(i) \Big) \oplus \Big(\bigoplus_{p \in \Sigma_R \cap Q_{J_0}} S(p) \Big) \oplus \Big(\bigoplus_{k \in Q_{J_0} \backslash \Sigma_R} I(k) \Big).$$

Then $\operatorname{End} T \cong KQ/I_{\Sigma_{\operatorname{End} T}}$ with $\Sigma_{\operatorname{End} T} = \Sigma_R \backslash Q_{J_0}$, and $(Q, I_{\Sigma_{\operatorname{End} T}})$ respects the clock condition.

Proof. The first statement follows from 3.1.9 and 2.4.6. Actually, we have $\operatorname{End} T \cong KQ_{\operatorname{End} T}/I_{\Sigma_{\operatorname{End} T}}$ with $Q_{\operatorname{End} T} = Q_A \amalg_{Q_{C_S}} (Q_{BC})$ and $I_{\Sigma_{\operatorname{End} T}} = I_A$ with A, B and C as described in the proof of 3.1.9. But when R is stably hereditary, we easily see that $Q_A \amalg_{Q_{C_S}} (Q_{BC}) = Q$, and so $Q = Q_{\operatorname{End} T}$.

Finally, the construction of $(Q,\ I_{\Sigma_R})$ and the fact that for all $j \in J$, $\alpha_j \in V_2 \backslash V_1$, imply that the bound quiver $(Q,\ I_{\Sigma_{\operatorname{End} T}})$ respects the clock condition. □

Corollary 3.2.3. *If $(Q,\ I_{\Sigma_R})$ contains no double-zero, then R is tilted of type Q.*

Proof. If $(Q,\ I_{\Sigma_R})$ contains no double-zero, then it follows from 3.1.8 that $V_1 \cap V_2 = \emptyset$.

Let $x \in \Sigma_R$. It follows from 3.1.3 (iv) that there exist $\alpha_i,\ \alpha_j \in \mathcal{I}(R)$ such that x is contained in both Q_{α_i} and Q_{α_j}. So, by 3.1.6, for all $x \in \Sigma_R$, there exists $\alpha_j \in V_2$ such that $x \in (Q_{\alpha_j})_0 \cap \Sigma_R$. Therefore, for the construction of the module T as seen in 3.2.2, we take all the α_j contained in V_2 (since $V_1 \cap V_2 = \emptyset$), and so $Q_{J_0} = \Sigma_R$. Hence $\operatorname{End} T$ is hereditary since $\Sigma_{\operatorname{End} T} = \Sigma_R \backslash Q_{J_0}$. So R is tilted. □

The proof of the following lemma is similar to those of [18] (2.3) and [14] (2.6), which are done in the contexts of gentle and special biserial algebras respectively.

Lemma 3.2.4. *If $(Q,\ I_{\Sigma_R})$ contains a double-zero, then R is not tilted.*

Proof. Suppose that $(Q,\ I_{\Sigma_R})$ contains a double-zero of the form

$$1 \longrightarrow 2 \longrightarrow 3 \text{ —— } \bullet \cdots\cdots \bullet \text{ —— } t-2 \longrightarrow t-1 \longrightarrow t$$

with $t \geq 4$.

Since R is monomial, if $t = 4$, then, by [9](1.2), $\operatorname{gl.dim} R > 2$, hence R is not tilted.

Thus, suppose that $t \geq 5$, and let M be the indecomposable R-module of support

$$3 \text{ —— } \bullet \cdots\cdots \bullet \text{ —— } t-2$$

such that $M(x) = K$ for all x such that $3 \leq x \leq t-2$ (this indecomposable module exists since R is monomial).

Let s be the source of $\operatorname{Supp} M$ such that there exists a path from s to $t-2$ in $\operatorname{Supp} M$. Since R is monomial, we see that the kernel of the canonical morphism $P(s) \to M$ has a non-projective direct summand and hence $\operatorname{pd} M > 1$.

Similarly, one proves that $\operatorname{id} N > 1$. Thus, by [11](III 2.3), R is not quasi-tilted, and therefore is not tilted. □

It is now possible to prove the main result of this section:

Theorem 3.2.5. *Let $R = KQ/I_{\Sigma_R}$ be a stably hereditary algebra. Then*

(i) *R is iterated tilted if and only if $(Q,\ I_{\Sigma_R})$ satisfies the clock condition. In this case the type of R is Q.*
(ii) *R is tilted if and only if $(Q,\ I_{\Sigma_R})$ satisfies the clock condition and does not contain any double-zero.*

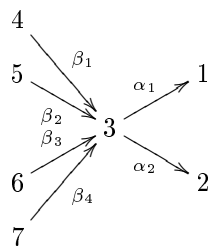
Proof. The first statement follows from 3.1.7, 3.2.1, 3.2.2, from [24](cor.1) and from the fact that $|\Sigma_R|$ is finite. The second statement follows from 3.2.3 and 3.2.4. □

We easily obtain the following corollary, which, in particular, proves a conjecture of Dieter Happel saying that an algebra $R = KQ/I$ with Q a tree and such that $\operatorname{rad}^2 R = 0$ is iterated tilted.

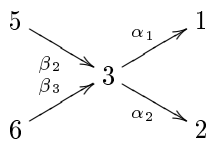
Corollary 3.2.6. *Let $R = KQ/I_{\Sigma_R}$ be a stably hereditary algebra with Q a tree. Then R is iterated tilted of type Q, and is tilted if and only if $(Q,\ I_{\Sigma_R})$ does not contain any double-zero.* □

Let $R = KQ/I$ be an algebra with Q a tree and such that I is generated by paths of length two, and such that $(Q,\ I)$ does not contain any double-zero. If R is not stably hereditary, then it is not tilted in general, as is shown in the following example.

Example 3.2.7. Let $(Q,\ I)$ be the following quiver:

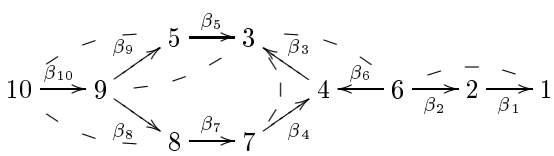


bound by $\beta_1\alpha_1 = 0$, $\beta_1\alpha_2 = 0$, $\beta_4\alpha_1 = 0$ and $\beta_4\alpha_2 = 0$. Then $R = KQ/I$ is isomorphic to $R_1 = H[S(3)][S(3)]$ with H the hereditary algebra with ordinary quiver



By the proof of [4](3.2), the component of Γ_{A_1} containing $S(3)$ is not directed. Hence, it follows from [21](3.7) that for A_1 to be tilted, this component should be quasi-serial or obtained from a quasi-serial translation quiver by ray or coray insertions, which is not the case. Therefore $A_1 = H[S(3)][S(3)]$ is not tilted.

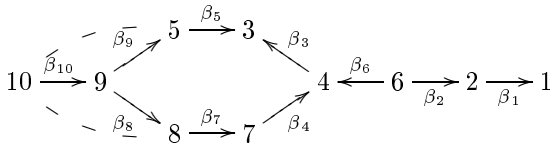
Example 3.2.8. Let $R = KQ/I_{\Sigma_R}$ be the stably hereditary algebra seen in 3.1.4:



We have $\Sigma_R = \{2,4,5,9\}$, and $\{\beta_1,\beta_2,\beta_3,\beta_{10}\}$ is a possible choice for $\mathcal{I}(R)$. Hence $V_1 = \{\beta_2,\beta_{10}\}$, $V_2 = \{\beta_1,\beta_2,\beta_3\}$ and $V_2 \backslash V_1 = \{\beta_1,\beta_3\}$, $Q_{J_0} = (Q_{\beta_1})_0 \cup (Q_{\beta_3})_0 = \{1,2,3,4,5\}$, $\Sigma_R \cap Q_{J_0} = \{2,4,5\}$ and thus by 3.2.1

$$T_1 = \Big(\bigoplus_{i=6}^{10} P(i)\Big) \oplus S(5) \oplus S(4) \oplus S(2) \oplus I(3) \oplus I(1)$$

is a tilting R-module. Moreover, by 3.2.2 we have $\operatorname{End} T_1 \cong KQ/I_{\Sigma_{\operatorname{End} T_1}}$ with $\Sigma_{\operatorname{End} T_1} = \Sigma_R \backslash (\Sigma_R \cap (Q_{\beta_1})_0) = \{9\}$. Therefore $(Q, I_{\Sigma_{\operatorname{End} T_1}})$ is the following bound quiver:



We see that $\operatorname{End} T_1$ is stably hereditary. A possible choice for $\mathcal{I}(R)$ is $\{\beta_1,\beta_{10}\}$, and then $V_1 = \{\beta_{10}\}$, $V_2 = \{\beta_1\}$ and thus $V_1 \cap V_2 = \emptyset$. Hence by 3.1.8 $(Q, I_{\Sigma_{\operatorname{End} T_1}})$ does not contain any double-zero, and therefore it follows from 3.2.3 that $\operatorname{End} T_1$ is a tilted algebra of type Q, and so R is iterated tilted of type Q.

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References

[1] Arnold, D. M.: *Finitely generated modules over pullback rings and algebras*, Advances in algebra and model theory (Essen, 1994; Dresden, 1995), Algebra Logic Appl., Vol. 9, 401-414, Gordon and Breach, Amsterdam (1997).

[2] Assem, I.: *Tilted algebras of type* $\mathbb{A}_n$, Comm. Algebra 10(19), 2121-2139 (1982).

[3] Assem, I.: *Tilting theory - an introduction*, Topics in algebra, Banach Center Publ. 26, Vol. 1, 127-180 (PWN. Warsaw, 1990).

[4] Assem, I. and Skowroński, A.: *Algebras with cycle-finite derived categories*, Math. Ann., Vol. 280, 441-463 (1988).

[5] Auslander, M., Reiten, I. and Smalø, S.O.: *Representation theory of Artin Algebra*, Cambridge University Press, 1995.

[6] Bongartz, K. and Riedtmann, C.: *Algèbres stablement héréditaires*, C. R. Acad. Sci. Paris Sér. A-B 288, No 15, 703-706 (1988).

[7] Bélanger, J. andt Tosar, C.: *Shod String Algebras*, Comm. Algebra, Vol. 33, No 8 (2005).

[8] Facchini, A., Vámos, P.: *Injective modules over pullbacks*, J. London Math. Soc. (2) 31, No 3, 425-438 (1985).

[9] Green, E., Happel, D. and Zacharia, D.: *Projective resolutions over artin algebras with zero relations*, Ill. J. Math., Vol. 29, No 1, 180-190 (1985).

[10] Happel, D.: *Triangulated categories in the representation theory of finite dimensional algebras*, London Math. Soc. Lecture Note Series 119, Cambridge Univ. Press (1988).

[11] Happel, D., Reiten, I. and Smalø, S.O.: *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc., Vol. 120, No 575 (1996).

[12] Happel, D. and Ringel, C.M.: *Tilted algebras*, Trans. Amer. Math. Soc., Vol. 274, No 2, 399-443 (1982).

[13] Huard, F.: *Tilted gentle algebras*, Comm. Algebra, Vol. 26, No 1, 63-72 (1998).

[14] Huard, F. and Liu, S.: *Tilted string algebras*, Journal of Pure and Applied Algebra, Vol. 153, No 2, 151-164 (2000).

[15] Huard, F. and Liu, S.: *Tilted special biserial algebras*, J. Algebra, Vol. 217, No 2, 679-700 (1999).

[16] Igusa, K., Platzeck, M. I., Todorov, G. and Zacharia, D.: *Auslander algebras of finite reprensentation type*, Comm. Algebra, Vol. 15, No 1-2, 377-424 (1987).

[17] Jagadeeshan, S., and Kleiner, M.: *Projectively stable Artin algebras: representation theory and quivers with relations*, Comm. Algebra, Vol. 27, No 5, 2277-2316 (1999).

[18] Lévesque, J.: *Tilted algebras of type* $\widetilde{\mathbb{A}}_n$, Comm. Algebra, Vol. 31, No 2, 545-552 (2003).

[19] Lévesque, J.: *Iterated tilted and tilted stably hereditary algebras*, Colloq. Math. 98, No 1, 49-62 (2003).

[20] Lévesque, J.: *Produits fibrés d'algèbres et inclinaison*, Ph.D. thesis, Sherbrooke (2004).

[21] Liu, S.: *The connected components of the Auslander-Reiten quiver of a tilted algebra*, J. Algebra, Vol. 161, No 2, 505-523 (1993).

[22] Milnor, J. W.: *Introduction to algebraic K-theory*, Annals of Mathematics Studies, Vol. 72, Princeton Univ. Press, Princeton, NJ (1971).

[23] Ringel, C.M.: *Tame algebras and integral quadratic forms*, Springer Lecture Notes 1099 (1984).

[24] Skowroński, A.: *Generalization of Yamagata's theorem on trivial extensions*, Arch. Math., Vol. 48, 68-76 (1987).

[25] Wiseman, A. N.: *Projective modules over pullback rings*, Math. Proc. Cambridge Philos. Soc., Vol. 97, No 3, 399-406 (1985).

Jessica Lévesque
Département de SIMQG
Université de Sherbrooke
Sherbrooke, Québec
J1K 2R1, Canada
E-mail: jlevesque@adm.usherbrooke.ca