Bayes Estimation Under a General Class of Balanced Loss Functions

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ABSTRACT

For estimating an unknown parameter θ, we introduce and motivate the use of balanced loss functions of the form $L_{\rho,\omega,\delta_0}(\theta,\delta) = \omega\rho(\delta_0,\delta) + (1-\omega)\rho(\theta,\delta)$, as well as weighted versions $q(\theta)L_{\rho,\omega,\delta_0}(\theta,\delta)$, with $q(\cdot)$ being a positive weight function, where $\rho(\theta,\delta)$ is an arbitrary loss function, δ_0 is a chosen a priori "target" estimator of θ, and the weight ω takes values in $[0,1)$. A general development with regards to Bayesian estimators under L_{ρ,ω,δ_0} is given, namely by relating such estimators to Bayesian solutions for the unbalanced case, i.e., L_{ρ,ω,δ_0} with $\omega = 0$. Illustrations are given for various choices of ρ, such as for absolute value, entropy, linex, intrinsic (i.e., model based), and a generalization of squared error losses.

Keywords and phrases: Balanced loss function; Bayes estimator; entropy loss; intrinsic loss; Linex loss

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1 Introduction

This paper concerns Bayesian point estimation of an unknown parameter θ of a model $X = (X_1, \ldots, X_n) \sim F_\theta$, under "Balanced" loss functions of the form:

$$L_{\rho,\omega,\delta_0}(\theta, \delta) = \omega\rho(\delta_0, \delta) + (1 - \omega)\rho(\theta, \delta), \tag{1}$$

as well as weighted versions $q(\theta)L_{\rho,\omega,\delta_0}(\theta, \delta)$, with $q(\cdot)$ being a positive weight function (see Remark 2). Here $\rho(\theta, \delta)$ is an arbitrary loss function, while δ_0 is a chosen a priori "target" estimator of θ, obtained for instance from the criterion of maximum likelihood estimator, least-squares, or unbiasedness among others. Loss L_{ρ,ω,δ_0}, which depends on the observed value of $\delta_0(X)$, reflects a desire of closeness of δ to both: **(i)** the target estimator δ_0, and **(ii)** the unknown parameter θ; with the relative importance of these criteria governed by the choice of $\omega \in [0, 1)$.

Loss L_{ρ,ω,δ_0} may be viewed as a natural extension to Zellner's (1994) balanced loss function (in one dimension), the lather being specific to a squared error loss ρ and a least-squares δ_0. Subsequent findings, due to Dey, Ghosh and Strawderman (1999), as well as Jafari Jozani, Marchand and Parsian (2006); who also study the use of more general target estimators δ_0; involve the issues of Bayesianity, admissibility, dominance, and minimaxity and demonstrate how results for unbalanced ($\omega = 0$) squared error loss serve directly (and necessarily) in obtaining results for the balanced case with $\omega > 0$.

Here we give a general Bayesian estimation connection (Lemma 1) between the cases $\omega = 0$ and $\omega > 0$. This useful connection is quite general, especially with respect to the choices of ρ and δ_0. Illustrations are given for various choices of ρ, such as for absolute value, entropy, linex, intrinsic (i.e., model based), and a generalization of squared error losses. We conclude with a discussion of ongoing related work.

2 Bayes estimation under balanced loss

We consider here Bayes estimation under balanced loss function L_{ρ,ω,δ_0} as in (1). When $\omega = 0$, we simply use L_0 instead of $L_{\rho,0,\delta_0}$ unless we want to emphasize the role of ρ . We show how the Bayes estimator of θ under balanced loss function L_{ρ,ω,δ_0} can be derived or expressed in terms of a Bayes estimator of θ under the loss L_0. We assume throughout necessary conditions for finite expected posterior loss. In particular, we work with ρ and δ_0 such that $\rho(\delta, \delta_0) < \infty$ for at least one $\delta \neq \delta_0$.

Lemma 1 *For estimating θ under balanced loss function L_{ρ,ω,δ_0} in (1) and for a prior $\pi(\theta)$, the* Bayes estimator $\delta_{\pi,\omega}(X)$ *corresponds to the* Bayes solution $\delta^*(X)$ *with respect to $(\pi^*(\theta|x), L_0)$, for all x, where*

$$\pi^*(\theta|x) = \omega 1_{\{\delta_0(x)\}}(\theta) + (1-\omega)\pi(\theta|x),$$

i.e., a mixture of a point mass at $\delta_0(x)$ and the posterior $\pi(\theta|x)$.

Proof: Denoting $\mu_X(\cdot)$ and $\nu_X(\cdot)$ as dominating measures of $\pi(\theta|x)$ and $\pi^*(\theta|x)$ respectively, we have by definition of $\delta_{\omega,\pi}(x)$, $\delta^*(x)$, and L_0

$$\begin{aligned}\delta_{\omega,\pi}(x) &= argmin_{\delta} \int_{\Theta} \{\omega\rho(\delta_0, \delta) + (1-\omega)\rho(\theta, \delta)\}\pi(\theta|x) d\mu_X(\theta) \\ &= argmin_{\delta} \int_{\Theta \cup \{\delta_0(x)\}} L_0(\theta, \delta)\pi^*(\theta|x) d\nu_X(\theta) = \delta^*(x). \qquad \square\end{aligned}$$

Remark 1 *Strictly speaking, $\delta^*(X)$ is not a Bayes estimator under loss L_0 (because $\pi^*(\theta|x)$ is not the posterior associated with π), while $\delta_{\omega,\pi}(X)$ is indeed Bayes under loss L_{ρ,ω,δ_0} and prior π.*

Remark 2 *Lemma 1 also applies for weighted versions of balanced loss function (1), i.e., $L^q_{\rho,\omega,\delta_0}(\theta, \delta) = q(\theta) L_{\rho,\omega,\delta_0}(\theta, \delta)$, with $q(\theta)$ being a positive weight function. Indeed, it is easy to see that the Bayes estimator of θ under loss $L^q_{\rho,\omega,\delta_0}$ using the prior π_0 is equivalent to the Bayes estimator of θ under loss L_{ρ,ω,δ_0} using prior $\pi \equiv q \times \pi_0$.*

We pursue with various applications of Lemma 1.

Example 1 *(A generalization of squared error loss). In (1), the choice $\rho(\theta, \delta) = \rho_1(\theta, \delta) = \tau(\theta)(\delta - \theta)^2$, with $\tau(\cdot) > 0$, leads to loss $\omega\tau(\delta_0)(\delta - \delta_0)^2 + (1-\omega)\tau(\theta)(\delta - \theta)^2$. Since under L_0 and a prior π, the Bayes estimator is given by $\delta_{0,\pi}(x) = \frac{E_\pi(\theta\tau(\theta)|x)}{E_\pi(\tau(\theta)|x)}$ (subject to the finiteness conditions $E_\pi(\theta^i\tau(\theta)|x) < \infty$; $i = 0, 1$; for all x), Lemma 1 tells us that*

$$\delta_{\omega,\pi}(x) = \frac{E_{\pi^*}(\theta\tau(\theta)|x)}{E_{\pi^*}(\tau(\theta)|x)} = \frac{\omega\delta_0(x)\tau(\delta_0(x)) + (1-\omega)E_\pi(\theta\tau(\theta)|x)}{\omega\tau(\delta_0(x)) + (1-\omega)E_\pi(\tau(\theta)|x)}.$$

The case ($\tau(\theta) = 1$, δ_0 = least square estimator) leads to balanced squared error loss (and $\delta_{\omega,\pi}(x) = \omega\delta_0(x) + (1-\omega)E_\pi(\theta|x)$) as introduced by Zellner (1994), and as further analyzed by Dey, Ghosh and Strawderman (1999). The case $L_{\rho_1,\omega,\delta_0}(\theta, \delta)$ with $\tau(\theta) = 1$ and *arbitrary δ_0, as well as its weighted version $q(\theta)L_{\rho 1,\omega,\delta_0}(\theta,\delta)$, were investigated by Jafari Jozani, Marchand and Parsian (2006) with respect to classical decision theory criteria such as Bayesianity, dominance, admissibility, and minimaxity. Gómez-Déniz (2006) investigates the use of $q(\theta)L_{\rho 1,\omega,\delta_0}(\theta,\delta)$ with $\tau(\theta)=1$ to credibility premiums.*

Example 2 *(Entropy balanced loss). The choice $\rho(\theta,\delta)=\frac{\theta}{\delta}-log\frac{\theta}{\delta}-1$ in (1) leads to loss:*

$$\omega(\frac{\delta_0}{\delta}-log(\frac{\delta_0}{\delta})-1)+(1-\omega)(\frac{\theta}{\delta}-log(\frac{\theta}{\delta})-1). \tag{2}$$

Since under L_0 (i.e., $\omega=0$ in (2), and a prior π, the Bayes solution is given by $\delta_{0,\pi}(x)=E_\pi(\theta|x)$ (subject to the finiteness conditions: $E_\pi(\theta|x)<\infty$ and $E_\pi(log(\theta)|x)<\infty$, for all x). Lemma 1 tells us that $\delta_{\omega,\pi}(x)=E_{\pi^*}(\theta|x)=\omega\delta_0(x)+(1-\omega)E_\pi(\theta|x)$. As a special example, take the model $X\sim Gamma(\alpha,\theta)$; α known; $\theta>0$; (density proportional to $x^{\alpha-1}e^{-x/\theta}$); and consider estimating θ under loss (2) with arbitrary δ_0 (e.g., $\delta_0(x)=\delta_{mle}(x)=\frac{x}{\alpha}$). With conjugate prior π such that $\lambda=\theta^{-1}\sim Gamma(\gamma,\frac{1}{h})$; $\gamma>-\alpha,h\geq 0$; and with the posterior distribution of θ^{-1} being $Gamma(\alpha+\gamma,\frac{1}{h+x})$; the unique Bayes estimator of θ under entropy balanced loss (2) is:

$$\delta_{\omega,\pi}(x)=\omega\delta_0(x)+(1-\omega)\frac{h+x}{\alpha+\gamma-1}.$$

Example 3 *(Stein loss with $\rho(\theta,\delta)=(\frac{\delta}{\theta})^\beta-\beta log\frac{\delta}{\theta}-1$; $\beta\neq 0$). This class of losses, which includes Example 2's loss (i.e., $\beta=1$), as well as Stein's loss (i.e., $\beta=-1$) illustrates well Lemma 1. Under loss L_0, we have the Bayes solution $\delta_{0,\pi}(x)=(E_\pi(\frac{1}{\theta^\beta}|x))^{-\frac{1}{\beta}}$ (subject to risk finiteness conditions). Lemma 1 tells us that $\delta_{\omega,\pi}(x)=(E_{\pi^*}(\frac{1}{\theta^\beta}|x))^{-\frac{1}{\beta}}=\{\frac{\omega}{(\delta_0(x))^\beta}+(1-\omega)E_\pi(\frac{1}{\theta^\beta}|x)\}^{-\frac{1}{\beta}}$.*

Example 4 *(Case with $\rho(\theta,\delta)=(\frac{\theta}{\delta}-1)^2$). Under loss L_0, for a prior π supported on $(0,\infty)$,we have the Bayes solution $\delta_{0,\pi}(x)=\frac{E_\pi(\theta^2|x)}{E_\pi(\theta|x)}$ (subject to the finiteness condition $E_\pi(\theta^2|x)<\infty$, for all x). Lemma 1 tells us that $\delta_{\omega,\pi}(x)=\frac{E_{\pi^*}(\theta^2|x)}{E_{\pi^*}(\theta|x)}=\frac{\omega\delta_0^2(x)+(1-\omega)E_\pi(\theta^2|x)}{\omega\delta_0(x)+(1-\omega)E_\pi(\theta|x)}$. Rodrigues and Zellner (1994) introduced and studied Bayesian estimators under such a loss function for a least square δ_0 and an exponential model.*

Example 5 *(Linex loss with $\rho(\theta,\delta)=e^{a(\delta-\theta)}-a(\delta-\theta)-1$; $a\neq 0$; (e.g., Zellner, 1986; Parsian and Kirmani, 2002)). Under loss L_0, we have the Bayes solution $\delta_{0,\pi}(x)=-\frac{1}{a}logE_\pi(e^{-a\theta}|x)$ (subject to risk finiteness conditions). Hence, using Lemma 1, we have $\delta_{\omega,\pi}(x)=-\frac{1}{a}logE_{\pi^*}(e^{-a\theta}|x)=-\frac{1}{a}log(\omega e^{-a\delta_0(x)}+(1-\omega)E_\pi(e^{-a\theta}|x))=-\frac{1}{a}log(\omega e^{-a\delta_0(x)}+(1-\omega)e^{-a\delta_{0,\pi}(x)})$.*

The following development concerns intrinsic balanced loss functions, where ρ in (1) is derived in some automated manner from the model. For a general reference on intrinsic losses and additional details we refer to Robert (1996).

Example 6 *(Intrinsic balanced loss functions). For a model $X|\theta \sim f(x|\theta)$, the choice $\rho(\theta,\delta) = d(f(\cdot|\theta), f(\cdot|\delta))$ in (1); i.e., the distance between $f(\cdot|\theta)$ and $f(\cdot|\delta)$; where $d(\cdot,\cdot)$ is a suitable distance function, leads to intrinsic balanced loss functions of the form,*

$$\omega d\left(f(\cdot|\delta_0), f(\cdot|\delta)\right) + (1-\omega) d\left(f(\cdot|\theta), f(\cdot|\delta)\right). \tag{3}$$

Two candidates of interest for d in (3) are Hellinger and Kullback-Leibler distance leading to:

- *Kullback-Leibler Balanced Loss,*

$$L^{KL}_{\omega,\delta_0}(\theta,\delta) = \omega E_{\delta_0}[Ln\frac{f(X|\delta_0)}{f(X|\delta)}] + (1-\omega)E_\theta[Ln\frac{f(X|\theta)}{f(X|\delta)}]; \tag{4}$$

- *Hellinger Balanced Loss,*

$$L^{H}_{\omega,\delta_0}(\theta,\delta) = \omega\frac{1}{2}E_{\delta_0}[(\sqrt{\frac{f(X|\delta)}{f(X|\delta_0)}} - 1)^2] + (1-\omega)\frac{1}{2}E_\theta[(\sqrt{\frac{f(X|\delta)}{f(X|\theta)}} - 1)^2]; \tag{5}$$

  or equivalently,

$$L^{H}_{\omega,\delta_0}(\theta,\delta) = 1 - \omega E_{\delta_0}[\sqrt{\frac{f(X|\delta)}{f(X|\delta_0)}}] - (1-\omega)E_\theta[\sqrt{\frac{f(X|\delta)}{f(X|\theta)}}]. \tag{6}$$

Hence, with known Bayesian representations in the unbalanced case ; i.e., $\omega = 0$; (e.g., Brown, 1986; Robert, 1996) and with Lemma 1, we can build a catalog of Bayesian representations for losses of the type L^{KL} and L^H. For instance, natural parameter exponential family of distributions with densities $f(x|\theta) = e^{\theta T(x) - \psi(\theta)}h(x)$ (with respect to a σ-finite measure ν on $\mathcal{X}$), and unknown natural parameter θ, lead to Kullback-Leibler balanced loss functions:

$$\omega\left[(\delta_0 - \delta)\psi'(\delta_0) + \psi(\delta) - \psi(\delta_0)\right] + (1-\omega)\left[(\theta-\delta)\psi'(\theta) + \psi(\delta) - \psi(\theta)\right]. \tag{7}$$

Furthermore, under L_0 (i.e., $\omega = 0$ in (7)), and for prior π, the Bayes estimator of θ is given as a solution of $\psi'(\delta_{0,\pi}(x)) = E_\pi[\psi'(\theta)|x]$ in $\delta_{0,\pi}(x)$. Hence, together with Lemma 1, we have that the Bayes estimator $\delta_{\omega,\pi}(x)$ admits the following representation

$$\psi'(\delta_{\omega,\pi}(x)) = \omega\psi'(\delta_0(x)) + (1-\omega)\psi'(\delta_{0,\pi}(x)),$$

with the special case of estimating $E_\theta[T(X)] = \psi'(\theta)$ *(as for* $X \sim N(\theta, 1)$*,* $T(X) = X$*), yielding* $\delta_{\omega,\pi}(x) = \omega\delta_0(x) + (1-\omega)\delta_{0,\pi}(x)$*; i.e., a convex linear combination of* δ_0 *and the Bayes estimator* $\delta_{0,\pi}(x)$*. Also, it is easy to see that for natural exponential family of distributions as above, Hellinger balanced loss in (6) reduces to:*

$$1 - \omega e^{\left\{\psi(\frac{\delta_0+\delta}{2}) - \frac{\psi(\delta_0)+\psi(\delta)}{2}\right\}} - (1-\omega)e^{\left\{\psi(\frac{\theta+\delta}{2}) - \frac{\psi(\theta)+\psi(\delta)}{2}\right\}}. \tag{8}$$

For instance, a $N(\theta, 1)$ *model leads to Hellinger balanced loss in (8) given by*

$$1 - \omega e^{-\frac{(\delta_0-\delta)^2}{8}} - (1-\omega)e^{-\frac{(\theta-\delta)^2}{8}},$$

and which we can refer to as a Balanced bounded reflected normal loss (e.g., Spiring (1993) for the case $\omega = 0$*).*

Remark 3 *In Example 2, loss (2) is equivalent to Kullback-Leibler balanced loss (7) for a Gamma*(α, θ) *model with known* $\alpha > 0$*, and unknown scale parameter* θ *(see also Parsian and Nematollahi, 1996; Robert, 1996).*

Example 7 *(Balanced absolute loss or* L_1 *loss). The choice* $\rho(\theta, \delta) = |\delta - \theta|$ *in (1) leads to balanced* L_1 *loss:*

$$\omega|\delta - \delta_0| + (1-\omega)|\delta - \theta|. \tag{9}$$

Since Bayesian solutions associated with L_1 *losses are posterior medians, Lemma 1 tells us that the Bayes estimate of* θ*,* $\delta_{\omega,\pi}(X)$*, with respect to loss (9) and priors* π *is a median of* $\pi^*(\theta|x)$*, for all* x*. Working now with priors* π *and models* $f(\cdot|\theta)$ *leading to unique posterior medians, setting* $\alpha(x) = (1-\omega)P_\pi(\theta \leq \delta_0(x)|x)$ *and* $F^{-1}_{\theta|x}$ *as the inverse posterior cdf, it is easy to see that for* $\omega \geq \frac{1}{2}$*,* $\delta_{\omega,\pi}(x) = \delta_0(x)$*, (i.e., the point mass of* $\delta_0(x)$ *has large enough probability to force the median to be* $\delta_0(x)$*), while for* $\omega < \frac{1}{2}$ *a straightforward evaluation leads to:*

$$\delta_{\omega,\pi}(x) = F^{-1}_{\theta|x}(\frac{1-2\omega}{2(1-\omega)})1_{[0,\frac{1}{2}-\omega)}(\alpha(x)) + \delta_0(x)1_{[\frac{1}{2}-\omega,\frac{1}{2})}(\alpha(x)) + F^{-1}_{\theta|x}(\frac{1}{2(1-\omega)})1_{[\frac{1}{2},1)}(\alpha(x)), \tag{10}$$

for all x*. Now, we give some applications of (10),*

(A) (Case of a positive normal mean). Let $X|\theta \sim N(\theta, \sigma^2)$ *with* $\theta \geq 0$*,* σ^2 *known, and cdf* F_0*. Decision theoretic elements concerning the estimation of* θ *have a long history (e.g.,*

see Marchand and Strawderman, 2004, for a review). Here, take $\delta_0(x) = \delta_{mle}(x) = max(0,x)$ *and* $\omega < \frac{1}{2}$ *in (9), and consider the estimator* $\delta_{\omega,\pi}(x)$ *associated with the flat prior* $\pi(\theta) = 1_{(0,\infty)}(\theta)$. *In evaluating (10), we have:* (**i**) $P_\pi(\theta \geq y|x) = \frac{F_0(x-y)}{F_0(x)}$ *for* $y \geq 0$; (**ii**) $F^{-1}_{\theta|x}(z) = x - F_0^{-1}((1-z)F_0(x))$, *and* (**iii**) $\alpha(x) = (1-\omega)P_\pi(\theta \leq \delta_0(x)|x) = (1-\omega)(1-\frac{1}{2F_0(x)})1_{(0,\infty)}(x)$. *Now, since* $\alpha(x) < \frac{1}{2}$, *and* $x \leq F_0^{-1}(1-\omega)$, *we obtain*

$$\begin{aligned}
\delta_{\omega,\pi}(x) &= F^{-1}_{\theta|x}(1-\frac{1}{2(1-\omega)})1_{(-\infty,F_0^{-1}(1-\omega))}(x) + \delta_0(x)1_{[F_0^{-1}(1-\omega),\infty)}(x) \\
&= \{x - F_0^{-1}(\frac{F_0(x)}{2(1-\omega)})\}1_{(-\infty,F_0^{-1}(1-\omega))}(x) + x1_{[F_0^{-1}(1-\omega),\infty)}(x) \\
&= x - \{F_0^{-1}(\frac{F_0(x)}{2(1-\omega)})1_{(-\infty,F_0^{-1}(1-\omega))}(x)\}.
\end{aligned}$$

Note that the expression holds for the unbalanced case as well (i.e., $\omega = 0$*), while a similar development holds for models* $X|\theta \sim f_0(x-\theta)$ *with* $\theta \geq 0$, *and known and symmetric* f_0.

(B) Let $X|\theta \sim N(\theta,\sigma^2)$; $\theta \in \Re$, σ^2 *known; with cdf* $\Phi(\frac{x-\theta}{\sigma})$. *Using conjugate prior* $\theta \sim \pi(\theta) = N(\mu,\tau^2)$; *known* μ *and* τ^2; *and balanced* L_1 *loss in (9) with* $\delta_0(X) = X$, $\omega < \frac{1}{2}$, *and since the posterior* $\theta|x \sim N(\frac{\tau^2}{\tau^2+\sigma^2}x + \frac{\sigma^2}{\tau^2+\sigma^2}\mu, \frac{\tau^2\sigma^2}{\tau^2+\sigma^2})$, *we have* $\alpha(x) = (1-\omega)P_\pi(\theta \leq x|x) = (1-\omega)\Phi(d(x-\mu))$; *with* $d = \frac{\sigma}{\tau\sqrt{\tau^2+\sigma^2}}$. *With* $F^{-1}_{\theta|x}(\alpha) = \frac{\tau^2}{\tau^2+\sigma^2}x + \frac{\sigma^2}{\tau^2+\sigma^2}\mu + \sqrt{\frac{\tau^2\sigma^2}{\tau^2+\sigma^2}}\Phi^{-1}(\alpha)$, *an evaluation of (10) leads to the piecewise linear estimators*

$$\delta_{\omega,\pi}(x) = \begin{cases} \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2} + \sqrt{\frac{\tau^2\sigma^2}{\tau^2+\sigma^2}}\Phi^{-1}(\frac{1-2\omega}{2(1-\omega)}) & \text{if } x \leq \mu + \frac{1}{d}\Phi^{-1}(\frac{1-2\omega}{2(1-\omega)}) \\ x & \text{if } x \in (\mu + \frac{1}{d}\Phi^{-1}(\frac{1-2\omega}{2(1-\omega)}), \mu + \frac{1}{d}\Phi^{-1}(\frac{1}{2(1-\omega)})) \\ \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2} + \sqrt{\frac{\tau^2\sigma^2}{\tau^2+\sigma^2}}\Phi^{-1}(\frac{1}{2(1-\omega)}) & \text{if } x \geq \mu + \frac{1}{d}\Phi^{-1}(\frac{1}{2(1-\omega)}). \end{cases}$$

3 Concluding Remarks

The findings here do permit us in many cases (unique Bayes, proper priors and finite Bayes risk) to draw inferences on the admissibility of resulting Bayesian estimators $\delta_{\omega,\pi}$, as given by Lemma 1 under balanced loss L_{ρ,ω,δ_0}. However, a general admissibility and dominance connection between losses L_0 and L_{ρ,ω,δ_0}, such as the one known for squared error ρ, is lacking and worth further investigation. In this regard, we do wish to point out (following a suggestion by Alexandre Leblanc) that most of the general results in Jafari Jozani, Marchand and Parsian (2006) (i.e, Lemma 1, 2; Corollary 1,2; Theorem 1) hold under the following modification of squared error loss : $\omega q(\delta_0)(\delta-\delta_0)^2 + (1-\omega)q(\delta_0)(\delta-\theta)^2$.

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