# An Invariance Principle for Discontinuous Righthand Sides Dynamical Systems

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**Abstract**

In this paper, we are concerned with discontinuous righthand sides dynamical systems (called Filippov systems). We present an extension of LaSalle invariance principle to locate chaotic attractors of such systems. For this aim, we use locally Lipschitz continuous and regular Lyapunov functions, as well as Filippov theory. The obtained results settled down in the general context of differential inclusions. We use this theoretical result to locate the attractor of a three-dimensional differential system for which numerical evidence of the chaotic behavior is given.

## 1 Introduction

The stability properties of equilibria for continuous differential systems have been developped by Lyapunov theorems and classical LaSalle invariance principle [5] (1960). In [6], an extension of LaSalle's principle to locate chaotic attractors has been done. We have recently adapted this result to find holes within the attractor, that are regions in the phase space in which this chaotic solution does not enter, see [3]. Generalizations of LaSalle's invariance principle for discontinuous systems are presented in [1, 7]. However, these results only provide information about the stability of equilibria. In this paper, we present an extension of LaSalle's invariance principle for discontinuous differential systems that requires less restrictive conditions than those of theorems in [1, 7]. Our result allows to locate, besides trivial solutions, chaotic attractors of discontinuous systems.

To study differential equations with discontinuous righthand sides, we use Filippov theory, see [4]. In this theory, in order to define the notion of solution and the study of equilibria of discontinuous differential equations, the notion of differential inclusions exetnd the differential equations. To do that, the *convex regularization* (or *Filippov regularization*) is used. Indeed, discontinuous differential systems do not need to have classical solutions. The terms of set-valued functions, Clarke generalized directional derivatives and gradients, are defined in [2], and shortly summarized below.

Let a domain $G \subset \mathbb{R}^n$ be separated by a smooth surface $\Sigma$ into domains $\nu_-$ and $\nu_+$, so $G = \nu_- \cup \Sigma \cup \nu_+$. Consider the following differential equation,

$$\frac{dx(t)}{dt} = f(x), \tag{1}$$

where $f \; : G \rightarrow \mathbb{R}^n$ is a continuous function in domains $\nu_-$ and $\nu_+$. The discontinuity hyper-surface $\Sigma$ is defined by an equation $h(x(t)) = 0$. Let $\vec{\eta}$ be the normal of $\Sigma$ directed from $\nu_-$ towards $\nu_+$ : $\vec{\eta} = \vec{\eta}(x(t)) = \text{grad}(h(x(t))$. Then, we have,

$$\nu_- = \{x \in G \; : h(x(t)) < 0\}, \;\; \Sigma = \{x \in G \; : h(x(t)) = 0\}, \;\; \nu_+ = \{x \in G \; : h(x(t)) > 0\}.$$

Let $F \; : G \rightarrow 2^{\mathbb{R}^n}$ be a set-valued function. On $\nu_-$ and $\nu_+$ $f$ is continuous. In this case we define,

$$F(x) = \{f(x)\}, \qquad x \in \nu_- \cup \nu_+.$$

We note $f^-(x)$ (resp. $f^+(x)$), instead of $f(x)$, when $x \in \nu_-$ (resp. $x \in \nu_+$). On $\Sigma$, we define,

$$f^-(x) = \lim_{\substack{x^* \in \nu_- \\ x^* \rightarrow x}} f(x^*), \;\; \text{and} \;\; f^+(x) = \lim_{\substack{x^* \in \nu_+ \\ x^* \rightarrow x}} f(x^*), \qquad x \in \Sigma.$$

Finally, the ***convex regularization*** of (1), given by Filippov in [4], consists in defining $F(x)$ as follows :

$$F(x) = \left\{ \begin{array}{ll} \{f^-(x)\} & \text{si } x \in \nu_- \\ \{\alpha f^-(x) + (1-\alpha) f^+(x), \alpha \in [0,1]\} & \text{if } x \in \Sigma \\ \{f^+(x)\} & \text{si } x \in \nu_+ \end{array} \right. \tag{2}$$

and the ***differential inclusion*** associated to (1) is

$$\frac{dx(t)}{dt} \in F(x). \tag{3}$$

# 2 Some definitions and results, Filippov's guidelines

To state existence and unicity theorems for the solutions of differential inclusions, we need some definitions and reminders.

Let $F : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a set-valued function.

**Definition 2.1 (Basic conditions)** A multivalued function $F$ satisfies the *basic conditions* if, for all $x \in I\!R^n$, $F(x)$ is non-empty, bounded, closed and convex, and if $F$ is upper semi-continuous.

**Definition 2.2 (Filippov solution)** A *Filippov solution* of a discontinuous differential equation $\frac{dx(t)}{dt} = f(x)$ is an absolutely continuous function $x : [0, \tau[ \to I\!R^n$ so that, for almost all $t \in [0, \tau[$, $\frac{dx(t)}{dt} \in F(x)$, whith $F(x)$ defined by (2).

**Theorem 2.1 (Existence of solutions for a differential inclusion)** *If $F$ satisfies the basic conditions, then, for all initial condition $x_0 \in I\!R^n$, there exists a Filippov solution $x$ of the differential inclusion defined on $[0, \tau[$, $\tau > 0$ :*

$$\frac{dx(t)}{dt} \in F(x).$$

We define the projections of $f^+$ and $f^-$ on the normal $\vec{\eta}$, that are, $f_\eta^+ = f^+.\vec{\eta}$ and $f_\eta^- = f^-.\vec{\eta}$ respectively (the dot represents the usual scalar product).

**Theorem 2.2 (Unicity of the solution)** *If the assumptions given in the previous theorem are true and if, for all $x \in \Sigma$, $f_\eta^+(x) < 0$ or $f_\eta^-(x) > 0$, then the solution of the differential inclusion (3) is uniquely determined.*

Let us consider the differential inclusion associated with the autonomous differential equation $\frac{dx(t)}{dt} = f(x)$, that is :

$$\frac{dx(t)}{dt} \in F(x). \tag{4}$$

In the next section, we will formulate and prove an invariance principle. This is why we have decided to define the following notions.

**Definition 2.3** A *Lyapunov function* for (4) is a continuous, positive definite function, $V : I\!R^n \to I\!R$, so that, for all solution $\varphi$ of (4) defined on an interval $I \subset I\!R$ and for all $t_1, t_2 \in I$ :

$$t_1 \leq t_2 \Longrightarrow V(\varphi(t_2)) \leq V(\varphi(t_1)). \tag{5}$$

**Definition 2.4** The *Clarke upper generalized derivative* of a function $V$ at $x$ in the direction of $v$ is :

$$V^0(x,v) = \lim_{y \to x} \sup_{h \downarrow 0} \frac{V(y+hv) - V(y)}{h}$$

**Definition 2.5** A function $V \ : \mathbb{R}^n \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}^n$ if, for all $v \in \mathbb{R}^n$,

(i) the usual right directional derivative $V'_+(x,v)$ exists and

(ii) $V'_+(x,v) = V^0(x,v)$.

**Definition 2.6** The *Clarke generalized gradient* of $V$ in $x$ is the set

$$\partial V(x) = co\left\{ \lim_{i \to +\infty} \nabla V(x_i) \ : (x_i) \to (x), \quad (x_i) \notin \Omega_V \right\},$$

with $\Omega_V$ a set of zero measure in $\mathbb{R}^n$ on which the gradient of $V$ is not defined, and $co(A)$ is the smallest convex and bounded set containing $A$.

**Definition 2.7** The *set-valued derivative of* $V$ with respect to (4) is given by

$$\dot{\overline{V}}(x) = \{a \in \mathbb{R} \ : \exists v \in F(x) \text{ so that } p.v = a, \quad \forall p \in \partial V(x)\}.$$

# 3 Theoretical Results : Invariance Principle for (chaotic) Filippov Systems

In this section, we will present the main result of this paper, an invariance principle for discontinuous differential systems.

The following lemma is needed as the proof of the main theorem of this section, see [1]. We denote by $S_{x_0}$ the set of all solutions starting at $x_0$.

**Lemma 3.1** *Let $\varphi$ be a solution of the differential inclusion (4) and let $V \ : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous and regular function. Then $\frac{d}{dt}V(\varphi(t))$ exists almost everywhere and $\frac{d}{dt}V(\varphi(t)) \in \dot{\overline{V}}(x)$ almost everywhere.*

The following theorem, given in [1], is a version of the LaSalle invariance principle for differential inclusions.

**Theorem 3.2** *Let $V \ : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous and regular function for (4). Let us assume that for some $l > 0$, the connected component $L_l$ of the level set $\{x \in \mathbb{R}^n \ : V(x) \leq l\}$ so that $0 \in L_l$ is bounded. Let $x_0 \in L_l$, $\varphi \in S_{x_0}$ and $Z_V = \{x \in \mathbb{R}^n \ : 0 \in \dot{\overline{V}}(x)\}$ and let $M$ the largest weakly invariant subset of $\overline{Z_V} \cup L_l$.*

*Then* $\operatorname{dist}(\varphi(t), M) \to 0$ *as* $t \to \infty$.

Now, we will present an extension of this theorem that requires less restrictive conditions and that further allows to determine a region in the phase space containing (chaotic) attractors for discontinuous differential systems, that is, an estimate of the (chaotic) attractor of studied dynamical systems.

**Theorem 3.3** *Consider a discontinuous differential system (4). Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a locally Lipschitz continuous and regular function and let* $c : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function so that, for all* $x \in \mathbb{R}^n$, $\max \dot{\overline{V}}(x) \leq -c(x)$. *We define :* $C := \{x \in \mathbb{R}^n : c(x) < 0\}$, $E := \{x \in \mathbb{R}^n : c(x) = 0\}$, *and* $l := \sup_{x \in C} V(x)$. *Assume that* $\overline{\Omega}_l := \{x \in \mathbb{R}^n : V(x) \leq l\}$ *is bounded.*

*If* $x_0 \in \Omega_l$, *then* $\varphi(.) \in \overline{\Omega}_l$, *for all* $\varphi(.) \in S_{x_0}$.

*Moreover, if* $\varphi(.)$ *is a bounded solution and* $M$ *the largest weakly invariant subset of* $\overline{\Omega}_l \cup E$, *then* $\operatorname{dist}(\varphi(t), M) \to 0$ *as* $t \to \infty$.

PROOF : First of all, let us note that if $x \notin \overline{\Omega}_l$, then $x \notin C$ and $c(x) \geq 0$. So, $\max \dot{\overline{V}}(x) \leq 0$, i.e. $v \leq 0, \quad \forall v \in \dot{\overline{V}}(x)$.

• Let us prove the theorem for the first case.
We consider $x_0 \in \overline{\Omega}_l$ and $\varphi \in S_{x_0}$. Then, by definition of $\overline{\Omega}_l$, $V(x_0) = V(\varphi(0)) \leq l$. Assume that there exists $\tilde{t} > 0$ so that $\varphi(\tilde{t}) \notin \overline{\Omega}_l$. Then, $V(\varphi(\tilde{t})) > l$. Since $V$ is continuous, there exists $\overline{t} \in [0, \tilde{t}[$ so that : $V(\varphi(\overline{t})) = l$ and $V(\varphi(t)) > l, \quad \forall t \in ]\overline{t}, \tilde{t}]$. It follows that there exists $]\overline{t}, \overline{t}+\varepsilon]$ on which $\frac{d}{dt}V(\varphi(t))$ is positive, which is impossible, because (since $V$ is locally Lipschitz and regular) $\frac{d}{dt}V(\varphi(t)) \in \dot{\overline{V}}(x)$ (see Lemma (3.1)) and because we are outside of $\overline{\Omega}_l$, $v \leq 0, \quad \forall v \in \dot{\overline{V}}(x)$. It follows that $\varphi(t) \in \overline{\Omega}_l, \quad \forall t \geq 0$.

• Let us prove the theorem for a bounded solution $\varphi$.
We can suppose that $\varphi(t) \notin \overline{\Omega}_l, \quad \forall t \geq 0$ (otherwise, we consider the first part of the proof).
Since the solution is not in $\overline{\Omega}_l$, it follows that $\max \dot{\overline{V}}(x) \leq 0$, and therefore $V \circ \varphi$ is decreasing.
Moreover, $V$ is continuous and $\varphi$ is bounded, so $V \circ \varphi$ is bounded.
We denote by $c$ the value $c = \lim_{t \to \infty} V \circ \varphi(t)$.
Let us consider $y$ in the $\omega$-limit set $\omega(x_0)$ : by definition of $\omega(x_0)$, $\exists (t_k) \nearrow \infty : \varphi(t_k) \to_{k \to \infty} y$, and by continuity of $V$ : $V(\varphi(t_k)) \to V(y) = c, \quad$ for all $y \in \omega(x_0)$.
Let $\psi$ be in the set $S_y$, with $y \in \omega(x_0)$. By definition of the $\omega$-limit set (invariant) $\psi(t) \in \omega(x_0) \quad (\forall t > 0)$, and we have $V(\varphi(t)) = c, \quad \forall t > 0$. So, $\frac{d}{dt}V(\varphi(t)) = 0, \quad \forall t > 0$.
Furthermore, $\frac{d}{dt}V(\varphi(t)) = 0 \in \dot{\overline{V}}(x)$, so $0 \leq \max \dot{\overline{V}}(x) \leq -c(\psi(t))$ and $c(\psi(t)) \leq 0$, for all $\psi \in S_y$ and all $y \in \omega(x_0)$.

Moreover, $\omega(x_0) \notin \overline{\Omega}_l$, so $\psi(t) \notin \overline{\Omega}_l$, $\forall t > 0$ and then $c(\psi(t)) \geq 0$, $\forall t > 0$.
In conclusion, $c(\psi(t)) = 0$, $\forall t > 0$, for all $\psi \in S_y$ and all $y \in \omega(x_0)$.
Since $\omega(x_0)$ is invariant, we have proved that $\omega(x_0) \subset \{x \in I\!R^n : c(x) = 0\} = E$.
□

# 4 Example and Application

Now, let us give an example that use the last theorem.
We consider the new discontinuous righthand sides dynamical system :

$$\left\{ \begin{array}{lcl} \dot{x} & = & -\sigma x + \sigma y \\ \dot{y} & = & rx - y - sgn(y).|x|z \\ \dot{z} & = & -bz + |xy| \end{array} \right. \tag{6}$$

that numerically shows a chaotic attractor for $\sigma = 10$, $r = 28.5$ and $b = 2.5$, see Fig.1.

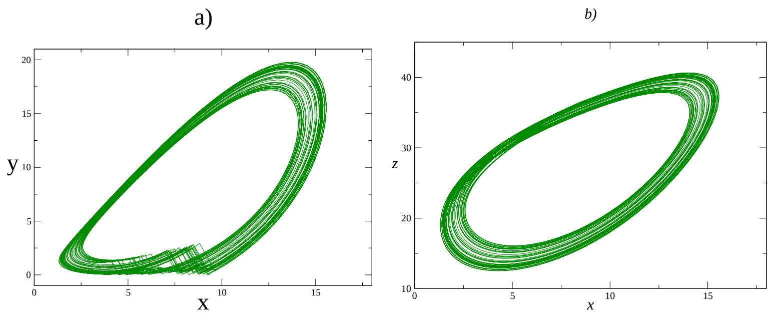


Figure 1: Projection of the attractor of system (6) a) on plane $xy$ and b) on plane $xz$.

The differential inclusion associated with this discontinuous system is $(\dot{x}, \dot{y}, \dot{z})^T \in F(x, y, z)$ where $F(x, y, z)$ is the *convex regularization* of $f(x, y, z)$ given by the set-valued function

$$F(x,y,z) = \left| \begin{array}{ll} \{-\sigma x + \sigma y\} \times \{rx - y - |x|zsgn(y)\} \times \{-bz + |xy|\} & \text{if } \ y \neq 0, \\ \{-\sigma x\} \times \{[rx - |xz|, rx + |xz|])\} \times \{-bz\} & \text{if } \ y = 0. \end{array} \right.$$

To estimate the domain of existence of the chaotic attractor of the previous system (6), we use Theorem 3.3 with

$$V(x, y, z) = \alpha(\delta x + \xi)^2 + \beta(\epsilon y + \rho)^2 + \gamma(\mu z + \tau)^2,$$

where $\alpha > 0, \beta > 0, \gamma > 0, \delta, \epsilon, \mu, \xi, \rho$ and $\tau$ are parameters to determine. The generalized gradient of $V$ is given by

$$\frac{1}{2} \ \partial V(x, y, z) = \Big\{ \alpha\delta\,(\delta x + \xi) \Big\} \times \Big\{ \beta\epsilon\,(\epsilon y + \rho) \Big\} \times \Big\{ \gamma\mu\,(\mu z + \tau) \Big\},$$

and the set-valued deriviative of $V$ if $y \neq 0$ is :

$$\frac{1}{2}\;\dot{\tilde{V}}(x,y,z) = \Big\{\alpha\delta\,(\delta x+\xi)\,(-\sigma x+\sigma y)+\beta\epsilon\,(\epsilon y+\rho)\,(rx-y-|x|z.sgn(y)) +\gamma\mu\,(\mu z+\tau)\,(|xy|-bz)\Big\},$$

and,

$$\frac{1}{2}\;\dot{\tilde{V}}(x,0,z) = \Big\{\alpha\delta\,(\delta x+\xi)\,(-\sigma x)+\beta\epsilon\rho[rx-|xz|,rx+|xz|]+\gamma\mu\,(\mu z+\tau)\,(-bz)\Big\}.$$

So, if $y \neq 0$,

$$\frac{1}{2}\;\dot{\tilde{V}}(x,y,z) = \Big\{-\alpha\sigma\delta^2x^2+\alpha\sigma\delta^2xy-\alpha\sigma\delta\xi x+\alpha\sigma\delta\xi y+\beta\epsilon^2rxy -\beta\epsilon^2y^2-\beta\epsilon^2|xy|z+\beta\epsilon\rho rx-\beta\epsilon\rho y-\beta\epsilon\rho|xz|sgn(y) +\gamma\mu^2|xy|z-\gamma b\mu^2z^2+\gamma\mu\tau|xy|-\gamma\mu b\tau z\Big\},$$

and,

$$\max\frac{1}{2}\;\dot{\tilde{V}}(x,0,z) \leq -\alpha\sigma\delta^2x^2-\alpha\sigma\delta\xi x+\beta\epsilon\rho rx+\beta\epsilon\rho|xz| -\gamma\mu^2bz^2-\gamma\mu b\tau z.$$

In both cases, for all $(x,y,z,)\in I\!R^3$, the result is :

$$\begin{aligned}\max\;\frac{1}{2}\;\dot{\tilde{V}}(x,y,z) &\leq -\alpha\sigma\delta^2x^2-\beta\epsilon^2y^2-\gamma\mu^2bz^2\\ &\quad+\alpha\sigma\delta^2xy+\beta\epsilon^2r|xy|+\gamma\mu\tau|xy|+\beta\epsilon\rho|xz|\\ &\quad+(\gamma\mu^2-\beta\epsilon^2)|xy|z-(\alpha\sigma\delta\xi-\beta r\epsilon\rho)x\\ &\quad-(\beta\epsilon\rho-\alpha\sigma\delta\xi)y-\gamma\mu b\tau z\\ &\leq -\alpha\sigma\delta^2x^2-\beta\epsilon^2y^2-\gamma b\mu^2z^2\\ &\quad+\alpha\sigma\delta^2xy+(\beta\epsilon^2r+\gamma\mu\tau)|xy|+\beta\epsilon\rho|xz|\\ &\quad+(\gamma\mu^2-\beta\epsilon^2)|xy|z-(\alpha\sigma\delta\xi-\beta r\epsilon\rho)x\\ &\quad-(\beta\epsilon\rho-\alpha\sigma\delta\xi)y-\gamma\mu b\tau z\\ &:= -c(x,y,z),\end{aligned}$$

where the last equality obviously defines function $c(x,y,z)$.

We choose $\tau = -\dfrac{\beta\epsilon^2r}{\gamma\mu}$ and $\gamma\mu^2=\beta\epsilon^2$, and then,

$$\max\;\frac{1}{2}\;\dot{\tilde{V}}(x,y,z) \leq -\alpha\sigma\delta^2x^2-\beta\epsilon^2y^2-\gamma b\mu^2z^2+\alpha\sigma\delta^2xy+\beta\epsilon\rho|xz| -(\alpha\sigma\delta\xi-\beta r\epsilon\rho)x-(\beta\epsilon\rho-\alpha\sigma\delta\xi)y-\gamma\mu b\tau z.$$

The function $c(x,y,z)$ of Theorem 3.3 is then :

$$\begin{aligned}\frac{1}{2}\ c(x,y,z) = \quad & \alpha\sigma\delta^2x^2 + \beta\epsilon^2y^2 + \gamma\mu^2bz^2 - \alpha\sigma\delta^2xy - \beta\epsilon\rho|xz| \\ & +(\alpha\sigma\delta\xi - \beta r\epsilon\rho)x + (\beta\epsilon\rho - \alpha\sigma\delta\xi)y + \gamma\mu b\tau z.\end{aligned} \tag{7}$$

We want values for parameters $\alpha, \beta, \gamma, \delta, \epsilon, \mu, \xi, \rho$ and $\tau$ so that the above equation defines a bounded and convex manifold in order to simplify the computation of the sup in Theorem 3.3. To do this, we reduce the equation of $c$ to its canonical form, as follows.

Given the general equation of a quadric,

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$$

$$2a_{10}x + 2a_{20}y + 2a_{30}z + a_{00} = 0,$$

composed of three terms :

- the quadratic part composed of terms of higher degree,

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz,$$

- the linear part composed of term of degree one,

$$2a_{10}x + 2a_{20}y + 2a_{30}z,$$

- and the constant term $a_{00}$,

we follow the following steps :

*i*) we bring back the quadratic part written under matricial form (change of basis) to its canonical form :

$$(x\ \ y\ \ z)\ \Delta \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

with

$$\Delta = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix},$$

*ii*) we include in the new basis the linear part of the quadric,

*iii*) in the new axis directed by the eigenvalues of $\Delta$, the equation of the quadric is :

$$\lambda_1x'^2 + \lambda_2y'^2 + \lambda_3z'^2 + 2\mu_1x' + 2\mu_2y' + 2\mu_3z' + a_{00} = 0,$$

with $\lambda_1, \lambda_2$ and $\lambda_3$ the eigenvalues of $\Delta$.

*iv*) By the translation $x'' = x' + \dfrac{\mu_1}{\lambda_1}$, $\quad y'' = y' + \dfrac{\mu_2}{\lambda_2}$, $\quad z'' = z' + \dfrac{\mu_3}{\lambda_3}$, the equation of the quadric is brought back to its canonical form :

$$\frac{x''^2}{B/\lambda_1} + \frac{y''^2}{B/\lambda_2} + \frac{z''^2}{B/\lambda_3} = 1, \tag{8}$$

with $B = \dfrac{\mu_1^2}{\lambda_1} + \dfrac{\mu_2^2}{\lambda_2} + \dfrac{\mu_3^2}{\lambda_3} - a_{00}$. $B$ is a constant so, to have an ellipsoid for equation (8), it is enough to take $B, \lambda_1, \lambda_2$ and $\lambda_3$ of the same sign.

Let us get back to the equation of $c(x, y, z)$ and define $c_1(x, y, z) = c_{+.+}(x, y, z)$, $c_2(x, y, z) = c_{+.-}(x, y, z)$, $c_3(x, y, z) = c_{-.+}(x, y, z)$, $c_4(x, y, z) = c_{-.-}(x, y, z)$. In that case, the triple index indicates in which quadrant lies $(x, y, z)$, for example, $c_{+.+}(x, y, z) = c(x, y, z)_{|x>0,z>0}$, $c_{+.-}(x, y, z) = c(x, y, z)_{|x>0,z<0}$, etc...

We have to determine values to the parameters, so that $c_1$ is an ellipsoid. Its quadratic part is :

$$\alpha\sigma\delta^2 x^2 + \beta\epsilon^2 y^2 + \gamma b\mu^2 z^2 - \alpha\sigma\delta^2 xy - \beta\epsilon\rho xz,$$

and its canonical form,

$$(x \;\; y \;\; z) \;\; \Delta \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

with

$$\Delta = \begin{pmatrix} A & -A/2 & -B\rho/2\epsilon \\ -A/2 & B & 0 \\ -B\rho/2\epsilon & 0 & C \end{pmatrix}$$

and $A = \alpha\sigma\delta^2 > 0$, $\quad B = \beta\epsilon^2 > 0$ and $C = \gamma b\mu^2 > 0$.

Choosing $\rho = 0$, we get :

$$\Delta = \begin{pmatrix} A & -A/2 & 0 \\ -A/2 & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$

We decide to take values of parameters $A, B$ and $C$ so that the eigenvalues of matrix $\Delta$ are real and positive. That will ensure that equation $c$ defines an ellipsoid.

The eigenvalues of $\Delta$ are positive if :

$$\begin{cases} det(A) = a > 0 \\ det\begin{pmatrix} A & -A/2 \\ -A/2 & B \end{pmatrix} = AB - A^2/4 > 0 \Longleftrightarrow B > A/4 \\ det(\Delta) = ABC - A^2C/4 > 0 \Longleftrightarrow C > 0 \end{cases}$$

$$\Longleftrightarrow A > 0, \quad B > A/4, \quad C > 0.$$

Remember the constraints that we have already given :

$$\begin{array}{ll} \rho = 0 & \text{(simplifies the calculus of the eigenvalues)}, \\ \tau = -\dfrac{\beta\epsilon^2 r}{\gamma\mu} & \text{(annulment of the large terms in } xy), \\ \gamma\mu^2 = \beta\epsilon^2 & \text{(annulment ofthe term in } |xy|z). \end{array}$$

Let us take (for example),

$$A = B = 1, \quad C = b,$$

in order to obtain an ellipsoid for $c_1$. The equation of $c_1$ is now :

$$\frac{x''^2}{B/\lambda_1} + \frac{y''^2}{B/\lambda_2} + \frac{z''^2}{B/\lambda_3} = 1,$$

with $B > 0, \quad \lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_3 > 0.$

Finally :

$$\alpha = \sigma, \ \beta = 1, \gamma = 1, \ \xi = 0, \ \rho = 0, \ \tau = -r,$$
$$\epsilon = 1, \ \delta = 1/\sigma, \ \mu = 1.$$

Then,

$$c_1(x, y, z) = x^2 + y^2 + bz^2 - xy - rbz.$$

To apply Theorem 3.3, we have to determine $\sup\limits_{\{(x,y,z):c(x,y,z)<0\}} V(x, y, z).$

To overcome this difficulty, we use the following lemma which can be found in [6], but to make it clear for all readers, we give its proof here.

**Lemma 4.1** *Let* $V, c, c_i \ : I\!R^n \longrightarrow I\!R, \ i = 1, .., k$ *be continuous functions so that*

$$c(x) \geq \inf\{c_i(x), \quad i = 1, \cdots, k\}, \quad \forall x \in I\!R^n.$$

*Let us consider*

$$C_i := \{x \in I\!R^n \ : c_i(x) < 0\}, \quad \text{and } C := \{x \in I\!R^n \ : c(x) < 0\}.$$

*Then, we have the following result :*

- $C \subset \bigcup_{i=1}^{n} C_i$ *and* $\sup_{x\in C} c(x) \leq \sup_{x\in \bigcup_{i=1}^{k} C_i} c(x)$.
- *Moreover, if $C_i$ is bounded for all $i$, and if there exists a sequence of homeomorphisms $S_i : I\!R^n \longrightarrow I\!R^n$, $i = 1, \cdots, k$ so that $C_j = S_{j-1}(C_{j-1})$, $\forall j = 2, \cdots, k$, $C_1 = S_k(C_k)$, and $V(S_i(x)) = V(x), \forall x \in I\!R^n, \forall i = 1, \cdots, k$. Then*
$$\sup_{x\in C} V(x) \leq \sup_{x\in C_j} V(x), \quad \forall j = 1 \cdots k.$$

PROOF : If $x \in C$, then $\inf\{c_1(x), c_2(x), \cdots, c_n(x)\} \leq c(x) < 0$.
So there exists $j$ so that $c_j(x) < 0$ and then, $x \in C_j \subset \bigcup_{i=1}^{n} C_i$. Consequently, $C \subset \bigcup_{i=1}^{n} C_i$ and $\sup_{x\in C} c(x) \leq \sup_{x\in \bigcup_{i=1}^{k} C_i} c(x)$.

We still need to show that $\sup_{x\in C_i} V(x) \leq \sup_{x\in C_{i+1}} V(x)$. Let $y \in C_{i+1}$, then there exists $z \in C_i$ so that $y = S_i(z)$, so, $V(y) = V(S_i(z)) = V(z) \leq \sup_{x\in C_i} V(x)$ and $\sup_{x\in C_{i+1}} V(x) \leq \sup_{x\in C_i} V(x)$, and consequently :
$$\sup_{x\in C_k} V(x) \leq \sup_{x\in C_{k-1}} V(x) \leq \cdots \leq \sup_{x\in C_2} V(x) \leq \sup_{x\in C_1} V(x).$$
So,
$$\sup_{x\in C} V(x) \leq \sup_{x\in \bigcup_{i=1}^{k} C_i} V(x) = \sup_{x\in C_j} V(x), \quad \forall j = 1, \cdots, k$$

□

Using the same notations of this lemma for the set $C$ and $C_i$, we get
$$\sup_{(x,y,z)\in C} V(x,y,z) \leq \sup_{(x,y,z)\in C_1} V(x,y,z).$$
Here, $C_1$ set (an ellipsoid) is bounded and convex and $V(x,y,z)$ is a convex function. That is why, $\sup_{(x,y,z)\in C_1} V(x,y,z)$ is reached on the boundary of $C$. So, it can be computed using the Lagrange multipliers technique.
To do that, we consider the Lagrange function $\mathcal{L}$,
$$\begin{aligned} \mathcal{L}(x,y,z) = & \; V(x,y,z) + \ell \; c_1(x,y,z) \\ = & \; \frac{1}{\sigma}x^2 + y^2 + (z-r)^2 + \ell\left(x^2 + y^2 + bz^2 - xy - rbz\right), \end{aligned}$$

and we solve the system $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial z} = \frac{\partial \mathcal{L}}{\partial \ell} = 0$, namely,

$$\begin{cases} \frac{2}{\sigma}x + 2\ell x - \ell u & = & 0 \\ 2y + 2\ell y - \ell x & = & 0 \\ 2z + 2b\ell z & = & 2r + \ell rb \\ x^2 + y^2 + bz^2 - xy & = & rbz \end{cases}$$

We obtain two solutions $(0, 0, 0)$ and $(0, 0, r)$. As $V(0, 0, 0) = r^2$ and $V(0, 0, r) = 0$, we can conclude that $\sup\limits_{(x,y,z)\in C} V(x, y, z)$ is reached in $(0, 0, 0)$ and

$$\sup_{(x,y,z)\in C} V(x, y, z) = r^2$$

Consequently, using Theorem 3.3, we have theoretically estimated the domain of existence of the chaotic attractor of system (6) by the manifold defined by $\frac{1}{\sigma}x^2 + y^2 + (z - r)^2 \leq r^2$, i.e.,

$$0.1x^2 + y^2 + (z - 28.5)^2 \leq 28.5^2$$

This result is represented in Fig. 2.

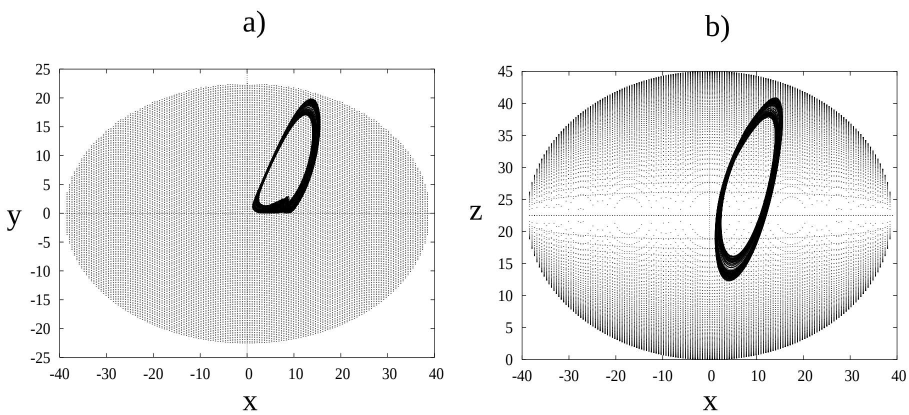


Figure 2: The chaotic attractor of system (6) inside its estimation, in projection a) on plane $xy$ and b) on plane $xz$.

# 5 Conclusion

In this paper, we have studied discontinuous righthand sides differential systems (called Filippov systems), as well as a new theorem to estimate their (chaotic) attractors. This theorem is an extension of the invariance principle written in the case of discontinuous systems. In this study, a numerical evidence of the chaotic behaviour of a new dicontinuous differential system has been given. Our theorem only gives an estimation of these attractors. Our present work (in progress) is to theoretically prove the existence of chaos in such discontinuous systems. One way of doing so is to use the Conley index theory and to compute algebraic invariants of particular spaces to understand the topology and the dynamics of invariant sets and attractors.

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