

Cluster-tilted algebras as trivial extensions

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Abstract

Given a finite dimensional algebra C (over an algebraically closed field) of global dimension at most two, we define its relation-extension algebra to be the trivial extension $C \times \text{Ext}_C^2(DC, C)$ of C by the C - C -bimodule $\text{Ext}_C^2(DC, C)$. We give a construction for the quiver of the relation-extension algebra in case the quiver of C has no oriented cycles. Our main result says that an algebra \tilde{C} is cluster-tilted if and only if there exists a tilted algebra C such that \tilde{C} is isomorphic to the relation-extension of C .

1 Introduction

Cluster categories were introduced in [6], and, for type A_n also in [12], as a means for a better understanding of the cluster algebras of Fomin and Zelevinsky [14, 15]. They are defined as follows: let A be a hereditary algebra, and $\mathcal{D}^b(\text{mod } A)$ be the derived category of bounded complexes of finitely generated A -modules, then the cluster category \mathcal{C}_A is the orbit category of $\mathcal{D}^b(\text{mod } A)$ under the action of the functor $F = \tau^{-1}[1]$, where τ is the Auslander-Reiten translation in $\mathcal{D}^b(\text{mod } A)$ and $[1]$ is the shift.

In [7], Buan, Marsh and Reiten defined the cluster-tilted algebras as follows. Let A be a hereditary algebra, and \tilde{T} be a tilting object in \mathcal{C}_A , that is, an object such that $\text{Ext}_{\mathcal{C}_A}^1(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of \tilde{T} equals the number of isomorphism classes of simple A -modules. Then the endomorphism algebra $\text{End}_{\mathcal{C}_A}(\tilde{T})$ is called cluster-tilted. Since then, these algebras have been the subject of many investigations, see, for instance, [7, 8, 9, 10, 11, 12, 13, 21]. In several particular cases, it was shown that the quiver of a cluster-tilted algebra was obtained from that of a tilted algebra by replacing relations by arrows, see, for instance [10, 11]. Our objective in this paper is to prove this statement in a more general context (not depending on the representation type). This is achieved by looking at cluster-tilted algebras as trivial extensions of tilted algebras by a bimodule which we explicitly describe (compare [3]).

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For this purpose, we let C be a finite dimensional algebra of global dimension two (over an algebraically closed field), and consider the C - C -bimodule $\text{Ext}_C^2(DC, C)$ with the natural action. The trivial extension $C \times \text{Ext}_C^2(DC, C)$ is called the relation-extension algebra of C . Our first main result (Theorem 2.6) describes the quiver of the relation-extension of C in the case where the quiver of C has no oriented cycles: we prove that indeed this quiver is given by replacing each element in a (minimal) system of relations by an arrow (going in the opposite direction to the relation). We then prove the main result of this paper.

Theorem 1.1 *An algebra \tilde{C} is cluster-tilted if and only if there exists a tilted algebra C such that \tilde{C} is the relation-extension of C .*

We note that several tilted algebras may correspond to the same cluster-tilted algebra, so this mapping is not bijective. On the other hand, there clearly exist relation-extension algebras which are not cluster-tilted.

Combining the above theorem with Theorem 2.6 we deduce the construction of the quiver of a cluster-tilted algebra. This allows, for instance, as done in [11], to relate the list of tame concealed algebras of Happel and Vossieck [18] with Seven's list of minimal infinite cluster quivers [23].

This paper consists of two sections. The first one describes relation-extension algebras and their quivers, and the second is devoted to the cluster-tilted algebras. Moreover, we give several examples.

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2 Relation-extension algebras

2.1 The definition

Throughout this paper, algebras are basic and connected finite dimensional algebras over a fixed algebraically closed field k . For an algebra C , we denote by $\text{mod } C$ the category of finitely generated right C -modules and by $\mathcal{D}^b(\text{mod } C)$ the derived category of bounded complexes over $\text{mod } C$. The functor $D = \text{Hom}_k(-, k)$ is the standard duality between $\text{mod } C$ and $\text{mod } C^{op}$. For facts about $\text{mod } C$ or $\mathcal{D}^b(\text{mod } C)$, we refer to [2, 22, 17].

Let C be an algebra. We recall that the *trivial extension* of C by a C - C -bimodule M is the algebra $C \times M$ with underlying k -vector space

$$C \oplus M = \{(c, m) \mid c \in C, m \in M\}$$

and the multiplication defined by

$$(c, m)(c', m') = (cc', cm' + mc')$$

for $c, c' \in C$ and $m, m' \in M$. For trivial extension algebras, we refer to [16, 1].

In this section, we introduce a particular class of trivial extension algebras which are useful for studying the cluster-tilted algebras.

Definition 2.1 *Let C be a finite dimensional algebra of global dimension at most two, and consider the C - C -bimodule $\text{Ext}_C^2(DC, C)$ (with the natural action). The trivial extension*

$$C \times \text{Ext}_C^2(DC, C)$$

is called the relation-extension of C .

Clearly, any hereditary algebra is (trivially) the relation-extension of itself. On the other hand, if C is of global dimension equal to two (thus not hereditary) there exist two simple C -modules S and S' such that $\text{Ext}_C^2(S, S') \neq 0$. Denoting by I the injective envelope of S and by P' the projective cover of S' , the short exact sequences

$$0 \longrightarrow S \longrightarrow I \longrightarrow I/S \longrightarrow 0$$

$$\text{and } 0 \longrightarrow \text{rad } P' \longrightarrow P' \longrightarrow S' \longrightarrow 0$$

induce an epimorphism $\text{Ext}_C^2(I, P') \rightarrow \text{Ext}_C^2(S, S')$. Thus $\text{Ext}_C^2(I, P') \neq 0$ and consequently $\text{Ext}_C^2(DC, C) \neq 0$.

2.2 A system of relations

We wish to describe the bound quiver of a relation-extension algebra. Let C be an algebra. It is well-known that there exists a (uniquely determined) quiver Q_C and an admissible ideal I of the path algebra kQ_C of Q_C such that $C \cong kQ_C/I$, see, for instance, [5]. We denote by $(Q_C)_0$ the set of points of Q_C and by $(Q_C)_1$ its set of arrows. For each point $x \in (Q_C)_0$, we let e_x denote the corresponding primitive idempotent of C , and by S_x, P_x, I_x respectively, the corresponding simple, indecomposable projective and indecomposable injective C -module.

Following [4], we define a *system of relations* for $C \cong kQ_C/I$ to be a subset R of $\bigcup_{x,y \in (Q_C)_0} e_x I e_y$ such that R , but no proper subset of R , generates I as a two-sided ideal of kQ_C . Thus, for any $x, y \in (Q_C)_0$, the elements of $R \cap (e_x I e_y)$ are linear combinations of paths (of length at least two) from x to y . We need the following result.

Lemma 2.2 (*([4, 1.2])*) *Let $C \cong kQ_C/I$ be such that Q_C has no oriented cycles and R be a system of relations for C . Then, for each $x, y \in (Q_C)_0$, the cardinality of the set $R \cap (e_x I e_y)$ is independent of the chosen system of relations for C , and equals $\dim_k \text{Ext}_C^2(S_x, S_y)$.*

2.3 The quiver of a trivial extension

We start with the following easy lemma.

Lemma 2.3 *Let C be an algebra, and M be a C - C -bimodule. The quiver $Q_{C \times M}$ of the trivial extension of C by M is constructed as follows:*

1. $(Q_{C \times M})_0 = (Q_C)_0$

2. For $x, y \in (Q_C)_0$, the set of arrows in $Q_{C \times M}$ from x to y equals the set of arrows in Q_C from x to y plus

$$\dim_k \frac{e_x M e_y}{e_x M (\text{rad } C) e_y + e_x (\text{rad } C) M e_y}$$

additional arrows from x to y .

Proof. Since $M \subset \text{rad}(C \times M)$, the quivers of $C \times M$ and of C have the same points. The arrows in the quiver of $C \times M$ correspond to a k -basis of the vector space

$$\text{rad}(C \times M) / \text{rad}^2(C \times M).$$

Now, as a vector space

$$\text{rad}(C \times M) = \text{rad } C \oplus M$$

and since $M^2 = 0$ in $C \times M$,

$$\text{rad}^2(C \times M) = \text{rad}^2 C \oplus [M(\text{rad } C) + (\text{rad } C)M].$$

Since $\text{rad}^2 C \subset \text{rad } C$ and $M(\text{rad } C) + (\text{rad } C)M \subset M$ and since the arrows of Q_C correspond to a basis of $\text{rad } C / \text{rad}^2 C$, the additional arrows of $Q_{C \times M}$ correspond to a k -basis of $M/[M(\text{rad } C) + (\text{rad } C)M]$. The arrows from x to y are obtained upon multiplying by e_x on the left and by e_y on the right.

2.4 The top of $\text{Ext}_C^2(DC, C)$

In the situation of section 2.3, the C - C -bimodule $M(\text{rad } C) + (\text{rad } C)M$ is the radical of M , and the quotient $M/[M(\text{rad } C) + (\text{rad } C)M]$ is its top. In the case of relation-extension algebras, we are interested in the top of $\text{Ext}_C^2(DC, C)$.

Lemma 2.4 *Let C be an algebra of global dimension two. The top of the C - C -bimodule $\text{Ext}_C^2(DC, C)$ is isomorphic to $\text{Ext}_C^2(\text{soc } DC, \text{top } C)$.*

Proof. The short exact sequences

$$0 \longrightarrow \text{rad } C \xrightarrow{i} C \longrightarrow \text{top } C \longrightarrow 0$$

$$0 \longrightarrow \text{soc } DC \xrightarrow{j} DC \longrightarrow DC / \text{soc } DC \longrightarrow 0$$

where i, j are the inclusions, induce a commutative diagram with exact rows and columns (the zeros are obtained from the condition that the global dimension

of C is two).

$$\begin{array}{ccccccc}
\text{Ext}_C^2(DC/\text{soc } DC, \text{rad } C) & \longrightarrow & \text{Ext}_C^2(DC/\text{soc } DC, C) & \longrightarrow & \text{Ext}_C^2(DC/\text{soc } DC, \text{top } C) & \longrightarrow & 0 \\
\downarrow & & \downarrow j^* & & \downarrow & & \\
\text{Ext}_C^2(DC, \text{rad } C) & \xrightarrow{i_*} & \text{Ext}_C^2(DC, C) & \xrightarrow{\quad} & \text{Ext}_C^2(DC, \text{top } C) & \longrightarrow & 0 \\
\downarrow & & \downarrow & \searrow p & \downarrow & & \\
\text{Ext}_C^2(\text{soc } DC, \text{rad } C) & \longrightarrow & \text{Ext}_C^2(\text{soc } DC, C) & \longrightarrow & \text{Ext}_C^2(\text{soc } DC, \text{top } C) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

By the commutativity of the lower-right square, there exists an epimorphism $p : \text{Ext}_C^2(DC, C) \rightarrow \text{Ext}_C^2(\text{soc } DC, \text{top } C)$. We thus only need to show that the kernel of p is isomorphic to the radical

$$\text{Ext}_C^2(DC, C) (\text{rad } C) + (\text{rad } C) \text{Ext}_C^2(DC, C)$$

of the C - C -bimodule $\text{Ext}_C^2(DC, C)$. Now an easy diagram chasing yields

$$\text{Ker } p = \text{Im } j^* + \text{Im } i_*.$$

Thus, it suffices to prove that

$$\text{Im } i_* = (\text{rad } C) \text{Ext}_C^2(DC, C) \quad \text{and} \quad \text{Im } j^* = \text{Ext}_C^2(DC, C) (\text{rad } C).$$

We only show the first equality, the second is shown similarly. Let

$$0 \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} DC \longrightarrow 0$$

be a projective resolution of DC . By definition

$$\text{Ext}_C^2(DC, C) = \text{Hom}_C(P_2, C) / \text{Im } \text{Hom}_C(d_2, C).$$

We first claim that the image of the map

$$i_0 = \text{Hom}_C(P_2, i) : \text{Hom}_C(P_2, \text{rad } C) \rightarrow \text{Hom}_C(P_2, C)$$

is equal to $(\text{rad } C) \text{Hom}_C(P_2, C)$. Indeed, the product rf with $r \in \text{rad } C$ and $f \in \text{Hom}_C(P_2, C)$ is easily seen to factor through $\text{rad } C$. Therefore, we have $(\text{rad } C) \text{Hom}_C(P_2, C) \subset \text{Im } i_0$. On the other hand, there is an isomorphism of k -vector spaces

$$(\text{rad } C) \text{Hom}_C(P_2, C) \cong \text{Hom}_C(P_2, \text{rad } C).$$

Since i_0 is injective, this establishes our claim.

Now, the image of i_* is generated by the residual classes (modulo the image of $\text{Hom}_C(d_2, C)$) of the products ig , with $g \in \text{Hom}_C(P_2, \text{rad } C)$. These are the residual classes of the elements in $\text{Im } i_0$ thus, by our claim above, the residual classes of the elements of the form rf with $r \in \text{rad } C$ and $f \in \text{Hom}_C(P_2, C)$. We deduce that $\text{Im } i_* = (\text{rad } C) \text{Ext}_C^2(DC, C)$, as required.

Remark 2.5 *The proof of this lemma can easily be generalised to show that, for an algebra C of global dimension at most m , the top of the bimodule $\text{Ext}_C^m(DC, C)$ is equal to $\text{Ext}_C^m(\text{soc } DC, \text{top } C)$.*

2.5 The quiver of a relation-extension

The following theorem states that the quiver of the relation-extension algebra is obtained from the quiver of the original algebra by adding, for each pair of points x, y , one arrow from x to y for each relation from y to x . This justifies the name “relation-extension”.

Theorem 2.6 *Let $C \cong kQ_C/I$ be an algebra of global dimension at most two, such that Q_C has no oriented cycles, and let R be a system of relations for C . The quiver of the relation-extension algebra $C \rtimes \text{Ext}_C^2(DC, C)$ is constructed as follows:*

- (a) $(Q_{C \rtimes \text{Ext}_C^2(DC, C)})_0 = (Q_C)_0$
- (b) *For $x, y \in (Q_C)_0$, the set of arrows in $Q_{C \rtimes \text{Ext}_C^2(DC, C)}$ from x to y equals the set of arrows in Q_C from x to y plus $\text{Card}(R \cap (e_y I e_x))$ additional arrows.*

Proof. Let S_1, S_2, \dots, S_n denote a complete set of representatives of the isomorphism classes of simple C -modules, and set $S = \bigoplus_{i=1}^n S_i$. Since C is basic, the module S is isomorphic to the top of C and to the socle of DC . By Lemma 2.2, the relations of R correspond to a k -basis of $\text{Ext}_C^2(S, S)$. By Lemma 2.4, the C - C -bimodule $\text{Ext}_C^2(S, S)$ is isomorphic to the top of $\text{Ext}_C^2(DC, C)$. Lemma 2.3 then implies that the number of additional arrows from x to y equals the k -dimension of the vector space $e_x \text{Ext}_C^2(S, S) e_y = \text{Ext}_C^2(S_y, S_x)$, and the result follows.

In particular, the quiver of a non-hereditary relation-extension algebra always contains oriented cycles.

2.6 The indecomposable projectives

It would be useful to know a system of relations for the relation-extension algebra $C \rtimes \text{Ext}_C^2(DC, C)$ starting from one for C . In actual examples, such a system is easily obtained once we know the indecomposable projective modules. In order to state the next lemma, we need a notation: for each $x \in (Q_C)_0$, we denote by \tilde{P}_x the corresponding indecomposable projective $C \rtimes \text{Ext}_C^2(DC, C)$ -module. Also, we note that C -modules can always be considered as $C \rtimes \text{Ext}_C^2(DC, C)$ -modules under the standard embedding.

Lemma 2.7 *Let C be an algebra of global dimension at most two. Then, for each $x \in (Q_C)_0$, we have a short exact sequence in $\text{mod}(C \rtimes \text{Ext}_C^2(DC, C))$*

$$0 \rightarrow \text{Ext}_C^2(DC, P_x) \rightarrow \tilde{P}_x \xrightarrow{p_x} P_x \rightarrow 0$$

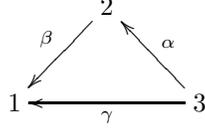
where p_x is a projective cover.

Proof. Since both P_x and \tilde{P}_x admit S_x as a simple top, there indeed exists a projective cover morphism $p_x : \tilde{P}_x \rightarrow P_x$. On the other hand, $\text{Ext}_C^2(DC, P_x) \cong e_x \text{Ext}_C^2(DC, C)$ is clearly a submodule of the $C \times \text{Ext}_C^2(DC, C)$ -module \tilde{P}_x . The result then follows from the isomorphism of k -vector spaces

$$\tilde{P}_x = e_x (C \times \text{Ext}_C^2(DC, C)) \cong P_x \oplus \text{Ext}_C^2(DC, P_x).$$

2.7 An example

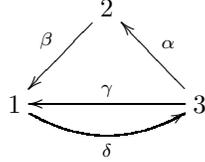
Example 2.8 Let C be given by the quiver



bound by the relation $\alpha\beta = 0$. Thus

$$C_C = 1 \oplus \begin{matrix} 2 \\ 1 \end{matrix} \oplus \begin{matrix} 3 \\ 1 \\ 2 \end{matrix} \quad \text{and} \quad (DC)_C = \begin{matrix} 2 \\ 1 \end{matrix} \oplus \begin{matrix} 3 \\ 2 \end{matrix} \oplus 3$$

where the indecomposable projectives and injectives are represented by their Loewy series. It is easily seen that the global dimension of C is two. By Theorem 2.6, the quiver of $C \times \text{Ext}_C^2(DC, C)$ is obtained by adding to Q_C a single arrow $\delta : 1 \rightarrow 3$.



We now compute the new indecomposable projective modules. A simple calculation yields

$$\text{Ext}_C^2(I_3, P_1) \cong k \quad , \quad \text{Ext}_C^2(I_3, P_3) \cong k$$

$$\text{Ext}_C^2(I_1, P_1) \cong k \quad , \quad \text{Ext}_C^2(I_1, P_3) \cong k.$$

Since the projective dimension of I_2 is one and the injective dimension of P_2 is also one, this yields $\dim_k \text{Ext}_C^2(DC, C) = 4$. Using Lemma 2.7, we get the new indecomposable projectives

$$\begin{matrix} 1 \\ 3 \\ 1 \end{matrix} \quad , \quad \begin{matrix} 3 \\ 1 \\ 3 \\ 1 \end{matrix} \quad 2 \quad , \quad \begin{matrix} 2 \\ 1 \end{matrix} .$$

Thus, a system of relations for the relation-extension algebra is $\alpha\beta = 0$, $\delta\alpha = 0$, $\beta\delta = 0$ and $\delta\gamma\delta = 0$.

3 Cluster-tilted algebras

3.1 Preliminaries

Let A be a hereditary algebra. The cluster category \mathcal{C}_A of A is defined as follows. Let F denote the automorphism of $\mathcal{D}^b(\text{mod } A)$ defined as the composition $\tau_{\mathcal{D}^b(\text{mod } A)}^{-1}[1]$, where $\tau_{\mathcal{D}^b(\text{mod } A)}^{-1}$ is the Auslander-Reiten translation in $\mathcal{D}^b(\text{mod } A)$, and $[1]$ is the shift functor. Then \mathcal{C}_A is the quotient category $\mathcal{D}^b(\text{mod } A)/F$. Its objects are the F -orbits $\tilde{X} = (F^i X)_{i \in \mathbb{Z}}$, where X is an object in $\mathcal{D}^b(\text{mod } A)$. The set of morphisms from $\tilde{X} = (F^i X)_{i \in \mathbb{Z}}$ to $\tilde{Y} = (F^i Y)_{i \in \mathbb{Z}}$ in \mathcal{C}_A is given by

$$\text{Hom}_{\mathcal{C}_A}(\tilde{X}, \tilde{Y}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(X, F^i Y).$$

It is shown in [20], that \mathcal{C}_A is a triangulated category. Furthermore, the canonical functor $\mathcal{D}^b(\text{mod } A) \rightarrow \mathcal{C}_A$ is a functor of triangulated categories. We refer to [6] for facts about the cluster category.

An object \tilde{T} in \mathcal{C}_A is called a *tilting object* provided $\text{Ext}_{\mathcal{C}_A}^1(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of \tilde{T} equals the number of isomorphism classes of simple A -modules (that is, the number of points in the quiver of A). The algebra of endomorphisms $\tilde{C} = \text{End}_{\mathcal{C}_A}(\tilde{T})$ is then called a *cluster-tilted algebra* [7].

Cluster-tilted algebras may also be expressed in terms of modules. We recall that an A -module T is called a *tilting module* provided $\text{Ext}_A^1(T, T) = 0$ and the number of isomorphism classes of indecomposable summands of T equals the number of isomorphism classes of simple A -modules. Denoting by \tilde{T} the F -orbit of T , we have the following theorem.

Theorem 3.1 ([6, 3.3]) *Let \tilde{C} be a cluster-tilted algebra, then there exist a hereditary algebra A and a tilting A -module T such that $\tilde{C} \cong \text{End}_{\mathcal{C}_A}(\tilde{T})$.*

We further recall that the endomorphism algebra of a tilting module over a hereditary algebra is called a *tilted algebra*, see, for instance, [22]. We need the following result.

Theorem 3.2 ([17]) *Let A be a hereditary algebra, T be a tilting A -module and $C = \text{End}_A(T)$ be the corresponding tilted algebra. Then*

- (a) *The derived functor $\text{RHom}_A(T, -) : \mathcal{D}^b(\text{mod } A) \rightarrow \mathcal{D}^b(\text{mod } C)$ is an equivalence of categories which maps the A -module T to the C -module C .*
- (b) *$\text{RHom}_A(T, -)$ commutes with the Auslander-Reiten translations and the shifts in the respective categories.*

3.2 Cluster-tilted algebras are trivial extensions

For any object X in $\mathcal{D}^b(\text{mod } A)$, the k -vector space $\text{Hom}_{\mathcal{D}^b(\text{mod } A)}(X, FX)$ has a natural structure of $\text{End}_{\mathcal{D}^b(\text{mod } A)}(X)$ - $\text{End}_{\mathcal{D}^b(\text{mod } A)}(X)$ -bimodule under the

action

$$\begin{aligned} \text{End}(X) \times \text{Hom}(X, FX) \times \text{End}(X) &\rightarrow \text{Hom}(X, FX) \\ (u, f, v) &\mapsto Fu \circ f \circ v \end{aligned}$$

The following lemma is proved in [3, 3.1]. We include a simple proof for the convenience of the reader.

Lemma 3.3 *Let \tilde{C} be a cluster tilted algebra. Then, for each hereditary algebra A and tilting A -module T such that $\tilde{C} = \text{End}_{\mathcal{C}_A}(\tilde{T})$, we have*

$$\tilde{C} \cong \text{End}_A(T) \ltimes \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(T, FT).$$

Proof. By definition of \mathcal{C}_A , we have

$$\tilde{C} = \text{End}_{\mathcal{C}_A}(\tilde{T}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(T, F^i T)$$

as k -vector spaces, and the multiplication is given by

$$(g_i)_{i \in \mathbb{Z}} (f_j)_{j \in \mathbb{Z}} = \left(\sum_{i+j=l} F^j g_i \circ f_j \right)_{l \in \mathbb{Z}}.$$

Since A is hereditary, then, for any two A -modules M and N , we have that $\text{Hom}_{\mathcal{D}^b(\text{mod } A)}(M, N[i]) = 0$ for all $i \geq 2$. Therefore, as a k -vector space

$$\tilde{C} = \text{End}_{\mathcal{C}_A}(\tilde{T}) = \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(T, T) \oplus \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(T, FT).$$

The multiplication of two elements $f, g \in \text{End}_{\mathcal{C}_A}(\tilde{T})$ is given as follows. Assume $f = (f_0, f_1)$ and $g = (g_0, g_1)$, with $f_0, g_0 \in \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(T, T)$ and $f_1, g_1 \in \text{Hom}_{\mathcal{D}^b(\text{mod } A)}(T, FT)$, then, since $Fg_1 \circ f_1 = 0$,

$$gf = (g_0 \circ f_0, Fg_0 \circ f_1 + f_0 \circ g_1).$$

In view of the bimodule structure of $\text{Hom}_{\mathcal{D}^b(\text{mod } A)}(T, FT)$ defined above, this shows indeed that $\tilde{C} = \text{End}_{\mathcal{C}_A}(\tilde{T})$ is the trivial extension of $\text{End}_{\mathcal{D}^b(\text{mod } A)}(T) = \text{End}_A(T)$ by the bimodule $\text{Hom}_{\mathcal{D}^b(\text{mod } A)}(T, FT)$.

Since the algebra $\text{End}_A(T)$ of the lemma is tilted, any cluster-tilted algebra is a trivial extension of a tilted algebra. However, the hereditary algebra A and the A -module T above are not unique. Therefore, one cannot apply directly the lemma to construct a map from cluster tilted algebras to tilted algebras.

3.3 The main result

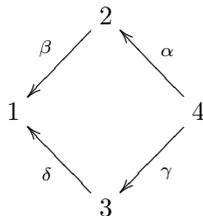
We are now able to prove the main theorem of this section.

Theorem 3.4 *An algebra \tilde{C} is cluster-tilted if and only if there exists a tilted algebra C such that \tilde{C} is the relation-extension of C .*

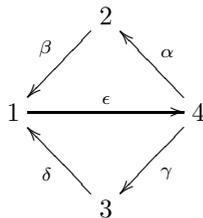
where matrices are assumed to have only finitely many non-zero coefficients, $C_i = C$ and $M_i = \text{Ext}_C^2(DC, C)$ for all $i \in \mathbb{Z}$, all the remaining coefficients are zero. The addition is the usual addition of matrices while the multiplication is induced from the bimodule structure of $\text{Ext}_C^2(DC, C)$ and the zero map $\text{Ext}_C^2(DC, C) \otimes_C \text{Ext}_C^2(DC, C) \rightarrow 0$. Clearly, \hat{C} is a Galois covering of $C \ltimes \text{Ext}_C^2(DC, C)$ with group \mathbb{Z} : the identity maps $C_i \rightarrow C_{i+1}$, $M_i \rightarrow M_{i+1}$ induce an automorphism η of \hat{C} and $\hat{C}/\eta \cong C \ltimes \text{Ext}_C^2(DC, C)$.

- (c) As observed before, different tilted algebras C may correspond to the same cluster-tilted algebra \tilde{C} (thus, the surjective map $C \mapsto \tilde{C}$ is not injective). We give an example of such an occurrence.

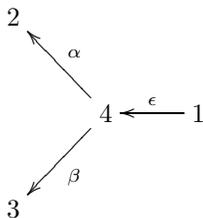
Example 3.5 Let C_1 be given by the quiver



bound by $\alpha\beta = \gamma\delta$. This is a tilted algebra of Dynkin type D_4 , and the corresponding cluster-tilted (relation-extension) algebra \tilde{C}_1 is given by the quiver



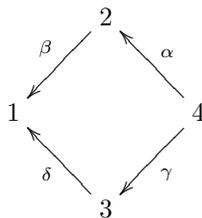
bound by $\alpha\beta = \gamma\delta$, $\beta\epsilon = 0$, $\delta\epsilon = 0$, $\epsilon\alpha = 0$, $\epsilon\gamma = 0$. Let now C_2 be the tilted algebra given by the quiver



bound by $\epsilon\alpha = 0$, $\epsilon\beta = 0$. Then it is easily seen that $\tilde{C}_1 = \tilde{C}_2$.

- (d) Not surprisingly, it is possible that C is representation-finite whereas \tilde{C} is representation-infinite: it suffices to have two points $x, y \in (Q_C)_0$ such that $\dim_k \text{Ext}_C^2(I_y, P_x) > 1$. We give an example of such a situation.

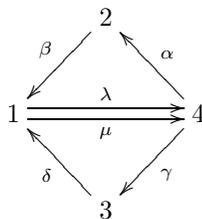
Example 3.6 Let C be given by the quiver



bound by $\alpha\beta = 0, \gamma\delta = 0$. This is a representation-finite tilted algebra of euclidean type \tilde{A}_3 . The injective resolution

$$0 \rightarrow P_1 \rightarrow I_1 \rightarrow I_2 \oplus I_3 \rightarrow I_4 \oplus I_4 \rightarrow 0$$

shows that $\dim_k \text{Ext}_C^2(I_4, P_1) = 2$. The corresponding cluster-tilted algebra \tilde{C} is given by the quiver



bound by $\alpha\beta = 0, \gamma\delta = 0, \delta\lambda = 0, \lambda\gamma = 0, \beta\mu = 0, \mu\alpha = 0$. The indecomposable projective \tilde{C} -modules are given by

$$\begin{array}{cc} & 1 \\ 4 & 4 \\ 2 & 3 \end{array}, \quad \begin{array}{cc} & 2 \\ 4 & \\ 2 & \end{array}, \quad \begin{array}{cc} & 3 \\ 1 & \\ 4 & 3 \end{array}, \quad \begin{array}{ccc} & & 4 \\ 2 & & 3 \end{array}$$

Clearly, \tilde{C} is representation-infinite.

- (e) The relation-extension algebra in Example 2.8 is not a cluster-tilted algebra. This follows from the fact that cluster-tilted algebras contain no oriented cycles of length two.

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