THE LEFT PART AND THE AUSLANDER-REITEN COMPONENTS OF AN ARTIN ALGEBRA

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ABSTRACT. The left part $\mathcal{L}_A$ of the module category of an artin algebra A consists of all indecomposables whose predecessors have projective dimension at most one. In this paper, we study the Auslander-Reiten components of A (and of its left support A_λ) which intersect $\mathcal{L}_A$ and also the class $\mathcal{E}$ of the indecomposable Ext-injectives in the addditive subcategory add$\mathcal{L}_A$ generated by $\mathcal{L}_A$.

INTRODUCTION

Let A be an artin algebra and mod$A$ denote the category of finitely generated right $A-$modules. The class $\mathcal{L}_A$, called the *left part* of mod$A$, is the full subcategory of mod$A$ having as objects all indecomposable modules whose predecessors have projective dimension at most one. This class, introduced in [15], was heavily investigated and applied (see, for instance, the survey [4]).

Our objective in this paper is to study the Auslander-Reiten components of an artin algebra which intersect the left part. Some information on these components was already obtained in [2, 3]. Here we are interested in the components which intersect the class $\mathcal{E}$ of the indecomposable Ext-injectives in the full additive subcategory add $\mathcal{L}_A$ having as objects the direct sums of modules in $\mathcal{L}_A$. We start by proving the following theorem.

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Theorem (A). *Let A be an artin algebra, and Γ be a component of the Auslander-Reiten quiver of A. If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

(a) Each τ_A-orbit of $\Gamma \cap \mathcal{L}_A$ intersects $\mathcal{E}$ exactly once.

(b) The number of τ_A-orbits of $\Gamma \cap \mathcal{L}_A$ equals the number of modules in $\Gamma \cap \mathcal{E}$.

(c) $\Gamma \cap \mathcal{L}_A$ contains no module lying on a cycle between modules in Γ.

If, on the other hand, $\Gamma \cap \mathcal{E} = \emptyset$, then either $\Gamma \in \mathcal{L}_A$ or else $\Gamma \cap \mathcal{L}_A = \emptyset$.

We recall that, by [3] (3.3), the class $\mathcal{E}$ contains only finitely many non-isomorphic modules (hence only finitely many Auslander-Reiten components intersect $\mathcal{E}$).

As a consequence, we give a complete description of the Auslander-Reiten components lying entirely inside the left part.

We then try to describe the intersection of $\mathcal{E}$ with a component Γ of the Auslander-Reiten quiver $\Gamma(\mathrm{mod}A)$. We find that, in general, $\Gamma \cap \mathcal{E}$ is not a section in Γ (in the sense of [20, 23]) but is very nearly one. This leads us to our second theorem, for which we recall that a component Γ of $\Gamma(\mathrm{mod}A)$ is called *generalised standard* if $\mathrm{rad}_A^\infty(X,Y) = 0$ for all $X, Y \in \Gamma$, see [23].

Theorem (B). *Let A be an artin algebra and Γ be a component of $\Gamma(\mathrm{mod}A)$ such that all projectives in Γ belong to $\mathcal{L}_A$. If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

(a) $\Gamma \cap \mathcal{E}$ is a section in Γ.

(b) Γ is generalised standard.

(c) $A/\mathrm{Ann}(\Gamma \cap \mathcal{E})$ is a tilted algebra having Γ as a connecting component and $\Gamma \cap \mathcal{E}$ as a complete slice.

In particular, such a component Γ has only finitely many τ_A-orbits.

The situation is better if we look instead at the intersection of $\mathcal{E}$ with the Auslander-Reiten components of the left support A_λ of A. We recall from [3, 24] that the *left support* A_λ of A is the endomorphism algebra of the direct sum of the indecomposable projective A-modules lying in $\mathcal{L}_A$. It is shown in [3, 24] that every connected component of A_λ is a quasi-tilted algebra (in the sense of [15]). We prove the following theorem.

theorem (C). *Let A be an artin algebra and Γ be a component of the Auslander-Reiten quiver of the left support A_λ of A. If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

(a) $\Gamma \cap \mathcal{E}$ is a section in Γ.

(b) Γ *is directed, and generalised standard.*

(c) $A_\lambda/\mathrm{Ann}(\Gamma \cap \mathcal{E})$ *is a tilted algebra having* Γ *as a connecting component and* $\Gamma \cap \mathcal{E}$ *as a complete slice.*

We then apply our results to the study of left supported algebras. We recall from [3] that an artin algebra A is *left supported* provided $\mathrm{add}\mathcal{L}_A$ is contravariantly finite in $\mathrm{mod}A$. Several classes of algebras are left supported, such as all representation-finite algebras, and all laura algebras which are not quasi-tilted (see [3, 4]). It is shown in [1] that an artin algebra A is left supported if and only if $\mathcal{L}_A$ consists of all the predecessors of the modules in $\mathcal{E}$. We give here a proof of this fact which, in contrast to the homological nature of the proof in [1], uses our theorem and the full power of the Auslander-Reiten theory of quasi-tilted algebras. Our proof also yields a new characterisation: an algebra A is left supported if and only if every projective A-module which belongs to $\mathcal{L}_A$ is a predecessor of $\mathcal{E}$. We end the paper with a short proof of the theorem of D. Smith [25] (3.8) which characterises the left supported quasi-tilted algebras.

Clearly, the dual statements about the right part of the module category, also hold true. Here, we only concern ourselves with the left part, leaving the primal-dual translation to the reader.

We now describe the contents of the paper. After a brief preliminary section 1, the sections $2, 3$ and 4 are respectively devoted to the proofs of our theorems (A), (B) and (C). In our final section 5, we consider the applications to left supported algebras.

1. Preliminaries.

1.1. **Notation.** For a basic and connected artin algebra A, let $\mathrm{mod}A$ denote its category of finitely generated right modules and $\mathrm{ind}A$ a full subcategory consisting of exactly one representative from each isomorphism class of indecomposable modules. We sometimes consider A as a category, with objects a complete set $\{e_1, \cdots, e_n\}$ of primitive orthogonal idempotents, and where e_iAe_j is the set of morphisms from e_i to e_j. An algebra B is a *full subcategory* of A if there is an idempotent $e \in A$, which is a sum of some of the distinguished idempotents e_i, such that $B = eAe$. It is *convex* in A if, for any sequence $e_i = e_{i_0}, e_{i_1}, \cdots, e_{i_t} = e_j$ of objects of A such that $e_{i_{l+1}}A\, e_{i_l} \neq 0$ (with $0 \leq l < t$) and e_i, e_j objects of B, all e_{i_l} are in B.

Given a full subcategory $\mathcal{C}$ of $\mathrm{mod}A$, we write $M \in \mathcal{C}$ to indicate that M is an object in $\mathcal{C}$, and we denote by $\mathrm{add}\mathcal{C}$ the full subcategory with objects the direct sums of summands of modules in $\mathcal{C}$. Given a module M, let $\mathrm{pd}M$ stand for its projective dimension. We also denote by $\Gamma(\mathrm{mod}A)$ the Auslander-Reiten quiver of A and by $\tau_A = \mathrm{DTr}$, $\tau_A^{-1} = \mathrm{TrD}$ the Auslander-Reiten translations. For further notions or facts needed on mod$A$ we refer to [7, 22].

1.2. **Paths.** Let A be an artin algebra and $M, N \in \mathrm{ind}A$. A *path* $M \rightsquigarrow N$ is a sequence

$$(*) \qquad M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \longrightarrow M_{t-1} \xrightarrow{f_t} M_t = N$$

where the f_i are non-zero morphisms and the M_i lie in ind$A$. We call M a *predecessor* of N and N a *successor* of M. A path from M to M involving at least one non-isomorphism is a *cycle*. An indecomposable module M lying on no cycle is called *directed* . A path $(*)$ is called *sectional* if each f_i is irreducible and $\tau_A M_{i+1} \neq M_{i-1}$ for all i. A *refinement* of $(*)$ is a path

$$M = M'_0 \xrightarrow{f'_1} M'_1 \xrightarrow{f'_2} \cdots \longrightarrow M'_{t-1} \xrightarrow{f'_t} M'_t = N$$

such that there exists an order-preserving injection $\sigma : \{1, \cdots, t-1\} \longrightarrow \{1, \cdots, s-1\}$ with $M_i = M'_{\sigma(i)}$ for all i. A full subcategory $\mathcal{C}$ of ind$A$ is *convex* if, for any path $(*)$ with $M,\ N \in \mathcal{C}$, all the M_i lie in $\mathcal{C}$.

2. Ext-injectives in the left part.

2.1. Let A be an artin algebra. The *left part* $\mathcal{L}_A$ *of* mod$A$ is the full subcategory of ind$A$ defined by

$$\mathcal{L}_A = \{M \in \mathrm{ind}A \mid \mathrm{pd}\, L \leq 1 \text{ for any predecessor } L \text{ of } M\}.$$

An indecomposable module $M \in \mathcal{L}_A$ is called *Ext-projective* (or *Ext-injective*) in add$\mathcal{L}_A$ if $\mathrm{Ext}^1_A(M, -)|_{\mathcal{L}_A} = 0$ (or $\mathrm{Ext}^1_A(-, M)|_{\mathcal{L}_A} = 0$, respectively), see [9]. While the Ext-projectives in add$\mathcal{L}_A$ are the projective modules lying in $\mathcal{L}_A$ (see [3] (3.1)), the Ext-injectives are more interesting. Before stating their characterisations we recall that, by [9] (3.7), $M \in \mathcal{L}_A$ is Ext-injective in add$\mathcal{L}_A$ if and only if $\tau_A^{-1} M \notin \mathcal{L}_A$.

Lemma [5] (3.2), [3] (3.1). *Let $M \in \mathcal{L}_A$.*

(a) The following are equivalent :

(i) *There exists an indecomposable injective module I such that $\mathrm{Hom}_A(I, M) \neq 0$.*

(ii) *There exist an indecomposable injective module I and a path $I \rightsquigarrow M$.*

(iii) *There exist an indecomposable injective module I and a sectional path $I \rightsquigarrow M$.*

(b) The following conditions are equivalent for $M \in \mathcal{L}_A$ which does not satisfy conditions (a):

(i) *There exists an indecomposable projective module $P \notin \mathcal{L}_A$ such that* $\mathrm{Hom}_A(P, \tau_A^{-1}M) \neq 0$.
(ii) *There exist an indecomposable projective module $P \notin \mathcal{L}_A$ and a path $P \rightsquigarrow \tau_A^{-1}M$.*
(iii) *There exist an indecomposable projective module $P \notin \mathcal{L}_A$ and a sectional path $P \rightsquigarrow \tau_A^{-1}M$.*

Letting $\mathcal{E}_1$ (or $\mathcal{E}_2$) denote the set of all $M \in \mathcal{L}_A$ verifying (a) (or (b), respectively), and setting $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, then M is Ext-injective in $\mathrm{add}\mathcal{L}_A$ *if and only if $M \in \mathcal{E}$.* $\square$

2.2. The following lemma will also be useful.

Lemma [3] (3.2) (3.4). *(a) Any path of irreducible morphisms in $\mathcal{E}$ is sectional.*

(b) Let $M \in \mathcal{E}$ and $M \rightsquigarrow N$ with $N \in \mathcal{L}_A$. Then this path can be refined to a sectional path and $N \in \mathcal{E}$. In particular, $\mathcal{E}$ is convex in $\mathrm{ind}A$. $\square$

2.3. The following immediate corollary will be useful in the proof of our theorem (A).

Corollary *All modules in $\mathcal{E}$ are directed.*

Proof. Assume $M = M_0 \to M_1 \to \cdots \to M_s = M$ is a cycle in $\mathrm{ind}A$, with $M \in \mathcal{E}$. By (2.2) above, such a cycle can be refined to a sectional cycle with all indecomposables lying in $\mathcal{E}$. Now compose two copies of this cycle to form a larger cycle in $\mathcal{E}$ of irreducible morphisms. By (2.2), this cycle is also sectional, in contradiction to [11, 12] . $\square$

2.4. Theorem (A). *Let A be an artin algebra, and Γ be a component of the Auslander-Reiten quiver of A. If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

(a) Each τ_A-orbit of $\Gamma \cap \mathcal{L}_A$ intersects $\mathcal{E}$ exactly once.

(b) The number of τ_A-orbits of $\Gamma \cap \mathcal{L}_A$ equals the number of modules in $\Gamma \cap \mathcal{E}$.

(c) $\Gamma \cap \mathcal{L}_A$ contains no module lying on a cycle between modules in Γ.

If, on the other hand, $\Gamma \cap \mathcal{E} = \emptyset$, then either $\Gamma \in \mathcal{L}_A$ or else $\Gamma \cap \mathcal{L}_A = \emptyset$.

Proof . Assume first that $\Gamma \cap \mathcal{E} \neq \emptyset$, that is, the component Γ contains an Ext-injective in $\mathrm{add}\mathcal{L}_A$.

(a) If Γ contains an injective module, then the statement follows from [3] (3.5). We may thus assume that Γ contains no injective. But then $\Gamma\cap\mathcal{E}_1=\emptyset$, and therefore $\Gamma\cap\mathcal{E}_2=\Gamma\cap\mathcal{E}\neq\emptyset$. Thus, by (2.1), there exist an indecomposable projective P in Γ such that $P\notin\mathcal{L}_A$, a module $M\in\Gamma\cap\mathcal{E}_2$ and a sectional path $P\rightsquigarrow\tau_A^{-1}M$. Now let $X\in\Gamma\cap\mathcal{L}_A$. Since Γ contains no injective, there exists $s>0$ such that $\tau_A^{-s}X$ is a successor of P. Hence $\tau_A^{-s}X\notin\mathcal{L}_A$. Since X itself lies in $\mathcal{L}_A$, there exists $j\geq 0$ such that $\tau_A^{-j}X\in\mathcal{L}_A$ but $\tau_A^{-j-1}X\notin\mathcal{L}_A$, so that $\tau_A^{-j}X$ is Ext-injective in add $\mathcal{L}_A$. This shows that every τ_A-orbit of $\Gamma\cap\mathcal{L}_A$ intersects $\mathcal{E}$ at least once.

Furthermore, it intersects it only once: if Y and $\tau_A^{-t}Y$ (with $t>0$) both belong to $\Gamma\cap\mathcal{E}$ then, by (2.2), all the modules on the path

$$Y\to *\to\tau_A^{-1}Y\to\cdots\to\tau_A^{-t}Y$$

belong to $\mathcal{L}_A$. In particular, $\tau_A^{-1}Y\in\mathcal{L}_A$ and this contradicts the Ext-injectivity of Y. This completes the proof of (a).

(b) It follows from (a) that the number of τ_A-orbits in $\Gamma\cap\mathcal{L}_A$ does not exceed the cardinality of $\Gamma\cap\mathcal{E}$ (note that by [3] (3.3), the cardinality of $\mathcal{E}$ is finite and does not exceed the rank of the Grothendieck group $K_0(A)$ of A). Since clearly, any element of $\Gamma\cap\mathcal{E}$ belongs to exactly one τ_A-orbit in $\mathcal{L}_A$, this establishes (b).

(c) Let $(*)\quad M_0\xrightarrow{f_1}M_1\xrightarrow{f_2}\ldots\xrightarrow{f_t}M_t=M_0$ be a cycle with $M_0\in\Gamma\cap\mathcal{L}_A$ and all M_i in Γ. Clearly, all M_i belong to $\Gamma\cap\mathcal{L}_A$. By (2.2) and (2.3), none of the M_i belongs to $\mathcal{E}$ and none of the f_i factors through an injective module. Indeed, if f_i factors through the injective I, then some indecomposable summand of I would belong to $\mathcal{L}_A$ and thus M_i would lie in $\mathcal{E}$, contradicting (2.3). Then the cycle $(*)$ induces a cycle $\tau_A^{-1}M_0\longrightarrow\tau_A^{-1}M_1\longrightarrow\cdots\longrightarrow\tau_A^{-1}M_t=\tau_A^{-1}M_0$, and every module in this cycle belongs to $\Gamma\cap\mathcal{L}_A$. We can iterate this procedure and deduce that, for any $m>0$, the module $\tau_A^{-m}M_0$ lies on a cycle in $\Gamma\cap\mathcal{L}_A$. However, as shown in (a), there exists $s>0$ such that $\tau_A^{-s}M_0$ does not belong to $\mathcal{L}_A$, and this contradiction proves (c).

Now assume that the component Γ contains no Ext-injective, that is, $\Gamma\cap\mathcal{E}=\emptyset$. If Γ contains both a module in $\mathcal{L}_A$ and a module which is not in $\mathcal{L}_A$, then there exists an irreducible morphism $X\longrightarrow Y$ with $X\in\Gamma\cap\mathcal{L}_A$ and $Y\in\Gamma\setminus\mathcal{L}_A$. Since $\Gamma\cap\mathcal{E}=\emptyset$, then $\tau_A^{-1}X\in\mathcal{L}_A$. But this is a contradiction, because $Y\notin\mathcal{L}_A$ and $\mathrm{Hom}_A(Y,\tau_A^{-1}X)\neq 0$. This shows that either $\Gamma\cap\mathcal{L}_A=\emptyset$ or $\Gamma\subseteq\mathcal{L}_A$, as required. □

We observe that part (c) of the theorem was already proven in [3] (1.5) under the additional hypothesis that Γ contains no injective module.

2.5. COROLLARY [3] (1.6). *Let A be a representation-finite artin algebra. Then $\mathcal{L}_A$ is directed.* $\square$

2.6. We have a good description of the Auslander-Reiten components which completely lie in $\mathcal{L}_A$. We need to recall a definition. The endomorphism algebra A_λ of the direct sum of all the projective modules lying in $\mathcal{L}_A$ is called the *left support* of A, see [3, 24]. Clearly, A_λ is (isomorphic to) a full convex subcategory of A, closed under successors, and any A-module lying in $\mathcal{L}_A$ has a natural A_λ-module structure. It is shown in [3] (2.3), [24] (3.1) that A_λ is a product of connected quasi-tilted algebras, and that $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$. The following corollary generalises [3] (5.5).

COROLLARY. *Let A be a representation-infinite artin algebra, and Γ be a component of $\Gamma(\text{mod}A)$ lying entirely in $\mathcal{L}_A$. Then Γ is one of the following: a postprojective component, a regular component (directed, stable tube or of type $\mathbb{Z}A_\infty$), a semiregular tube without injectives, or a ray extension of $\mathbb{Z}A_\infty$.*

Proof. Indeed, the component Γ lies entirely in $\text{mod}A_\lambda$ and thus is a component of $\Gamma(\text{mod}A_\lambda)$. Since $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$, then Γ is a component of $\Gamma(\text{mod}A_\lambda)$ lying in the left part $\mathcal{L}_{A_\lambda}$. The statement then follows from the well-known description of the Auslander-Reiten components of quasi-tilted algebras, as in [13, 18]. $\square$

3. EXT-INJECTIVES AS SECTIONS IN $\Gamma(\text{mod}A)$.

3.1. We recall the following notion from [20, 23]. Let A be an artin algebra and Γ be a component of $\Gamma(\text{mod}A)$. A full connected subquiver Σ of Γ is called a *section* if it satisfies the following conditions:

(S_1) Σ contains no oriented cycle.
(S_2) Σ intersects each τ_A-orbit of Γ exactly once.
(S_3) Σ is convex in Γ.
(S_4) If $X \to Y$ is an arrow in Γ with $X \in \Sigma$, then $Y \in \Sigma$ or $\tau_A Y \in \Sigma$.
(S_5) If $X \to Y$ is an arrow in Γ with $Y \in \Sigma$, then $X \in \Sigma$ or $\tau_A^{-1} X \in \Sigma$.

As we show next, the intersection of $\mathcal{E}$ with a component of $\Gamma(\text{mod}A)$ satisfies several of these conditions (but generally not all).

PROPOSITION. *Assume Γ is a component of $\Gamma(\text{mod}A)$ which intersects $\mathcal{E}$. Then Γ satisties (S_1), (S_3), (S_5) above, and the following conditions*

(S_2') $\Gamma \cap \mathcal{E}$ intersects each τ_A-orbit of Γ at most once.

(S_4') If $X \to Y$ is an arrow in Γ with $X \in \Gamma \cap \mathcal{E}$ and Y non-projective, then $Y \in \mathcal{E}$ or $\tau_A Y \in \mathcal{E}$.

Proof. (S_1) follows from Theorem (A) (c).

(S_2') follows from Theorem (A) (a).

(S_3) follows from (2.2).

(S_4') If $Y \in \mathcal{L}_A$, then $X \in \mathcal{E}$ and (2.2) imply $Y \in \mathcal{E}$. Otherwise, since Y is non-projective, there exists an arrow $\tau_A Y \to X$. Since $X \in \mathcal{E}$, then $\tau_A Y \in \mathcal{L}_A$. Since $Y = \tau_A^{-1}(\tau_A Y) \notin \mathcal{L}_A$, we get $\tau_A Y \in \mathcal{E}$.

(S_5) If X is injective then, since it lies in $\mathcal{L}_A$ (because it precedes Y), it belongs to $\mathcal{E}$. So assume it is not and consider the arrow $Y \to \tau_A^{-1}X$. If $\tau_A^{-1}X \notin \mathcal{L}_A$ then, again, $X \in \mathcal{E}$ while, if $\tau_A^{-1}X \in \mathcal{L}_A$, then $Y \in \mathcal{E}$ and (2.2) imply $\tau_A^{-1}X \in \mathcal{E}$. $\square$

3.2. Example. Let k be a field and A be the radical square zero k-algebra given by the quiver

$$\begin{tikzcd} & 2 \arrow[dl] & \\ 1 & & 3 \arrow[ll] \arrow[ul] \end{tikzcd}$$

Here, A is representation finite and $\mathcal{E}$ consists of the two indecomposable projectives P_1 and P_2 corresponding to the points 1 and 2, respectively. Clearly, $\mathcal{E} = \{P_1, P_2\}$ is not a section in $\Gamma(\text{mod}A)$: indeed, there is an arrow $P_1 \to P_3$ with $P_3 \notin \mathcal{E}$ and, moreover, $\mathcal{E}$ does not intersect each τ_A-orbit of $\Gamma(\text{mod}A)$.

3.3. We are now in a position to prove our second main theorem.

Theorem (B). *Let A be an artin algebra and Γ be a component of $\Gamma(\text{mod}A)$ such that all projectives in Γ belong to $\mathcal{L}_A$. If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

(a) $\Gamma \cap \mathcal{E}$ is a section in Γ.

(b) Γ is generalised standard.

(c) $A/\text{Ann}(\Gamma \cap \mathcal{E})$ is a tilted algebra having Γ as a connecting component and $\Gamma \cap \mathcal{E}$ as a complete slice.

Proof.

(a) We start by observing that, if $X \to P$ is an arrow in Γ, with $X \in \mathcal{E}$ and P projective then, by hypothesis, $P \in \mathcal{L}_A$. Thus, (2.2) implies $P \in \mathcal{E}$. This shows that (S_4) is satisfied. In view of the lemma, it suffices to show that $\Gamma \cap \mathcal{E}$ cuts each τ_A−orbit of Γ.

We claim that if $M \in \mathcal{E}$ and $N \in \Gamma$ lie in two neighbouring orbits, then $\mathcal{E}$ intersects the τ_A-orbit of N. This claim and induction imply the statement. We assume that $\mathcal{E}$ does not intersect the orbit of N and try to reach a contradiction. There exist $n \in \mathbb{Z}$ and an arrow $\tau_A^n M \to X$ or $X \to \tau_A^n M$, with X in the τ_A-orbit of N, where we may suppose, without loss of generality, that $|n|$ is minimal.

Suppose first that $n < 0$. If there exists an arrow $X \to \tau_A^n M$ then there exists an arrow $\tau_A^{n+1} M \to X$, a contradiction to the minimality of $|n|$. If, on the other hand, there exists an arrow $\tau_A^n M \to X$, then there is a path in Γ of the form $M \to * \to \tau_A^{-1} M \rightsquigarrow X$. Since $M \in \mathcal{E}$ then $\tau_A^{-1} M \notin \mathcal{L}_A$. Hence $X \notin \mathcal{L}_A$. In particular, X is not projective, so there exists an arrow $\tau_A^{n+1} M \to \tau_A X$, contrary to the minimality of $|n|$.

Suppose now that $n > 0$. If there exists an arrow $\tau_A^n M \to X$, then there exists an arrow $X \to \tau_A^{n-1} M$, a contradiction to the minimality of $|n|$. If, on the other hand, there exists an arrow $X \to \tau_A^n M$, then there is a path in Γ of the form $X \to \tau_A^n M \rightsquigarrow M$. Hence $X \in \mathcal{L}_A$. In particular, X is not injective (otherwise, $X \in \mathcal{E}$, a contradiction). Hence there exists an arrow $\tau_A^{-1} X \to \tau_A^{n-1} M$, contrary to the minimality of $|n|$.

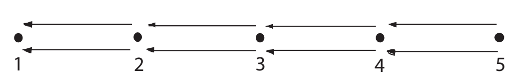
We have thus shown that necessarily $n = 0$, that is, there is an arrow $M \to X$ or $X \to M$. If $M \to X$ then, by (S_4), $X \in \mathcal{E}$ or $\tau_A X \in \mathcal{E}$, in any case a contradiction. If $X \to M$, then (3.1) yields $X \in \mathcal{E}$ or $\tau_A^{-1} X \in \mathcal{E}$, again a contradiction in any case. This completes the proof of (a).

(b) By [23], Theorem 2, it suffices to show that for any $X, Y \in \Gamma \cap \mathcal{E}$, we have $\mathrm{Hom}_A(X, \tau_A Y) = 0$. But $Y \in \mathcal{E}$ implies $\mathrm{pd} Y \leq 1$. Therefore the Ext-injectivity of X in $\mathrm{add}\mathcal{L}_A$ implies that

$$\mathrm{Hom}_A(X, \tau_A Y) \simeq \mathrm{D}\ \mathrm{Ext}_A^1(Y, X) = 0.$$

(c) This follows directly from [20] (2.2). $\square$

3.4. Example. Let k be a field and A be the radical square zero algebra given by the quiver



Let Γ be the component containing the injective I_1 corresponding to the point 1. Clearly, $I_1 \in \mathcal{E}$, so that $\Gamma \cap \mathcal{E} \neq \emptyset$. On the other hand, the only projective lying in Γ is P_3, and it belongs to $\mathcal{L}_A$. Thus, the hypotheses of the theorem apply here. Note that $A/\mathrm{Ann}(\Gamma \cap \mathcal{E})$ is equal to the left support A_λ of A, that is, the full convex subcategory with objects $\{1, 2, 3\}$.

4. Ext-injectives and the left support

4.1. In this section we study the intersection of $\mathcal{E}$ with the components of the Auslander-Reiten quiver of the left support A_λ of the artin algebra A.

We observe first that if Y is an A_λ-module and $\tau_A Y \in \mathcal{L}_A$ then $\tau_A Y = \tau_{A_\lambda} Y$. In particular, Y is not projective in $\text{mod}A_\lambda$. Indeed, since $\text{mod}A_\lambda$ is closed under extensions in $\text{mod}A$, then the inclusion $\mathcal{L}_A \subseteq \text{ind}A_\lambda$ implies that the almost split sequence in $\text{mod}A$ ending at Y is entirely contained in $\text{mod}A_\lambda$ (See also [7], p. 187). Similarly, if $\tau_A^{-1} Y \in \mathcal{L}_A$, then $\tau_A^{-1} Y = \tau_{A_\lambda}^{-1} Y$, and Y is not an injective A_λ-module.

Lemma. *If an indecomposable injective A_λ-module I is a predecessor of $\mathcal{E}$, then $I \in \mathcal{E}$.*

Proof. This is clear if I is an indecomposable injective A-module. So assume it is not. Since I precedes $\mathcal{E}$, then $I \in \mathcal{L}_A$. By the above observation we obtain that $\tau_A^{-1} I \notin \mathcal{L}_A$, because I is A_λ-injective. This proves that $I \in \mathcal{E}$, as desired. □

4.2. The following is an easy consequence of (3.1) and the results in [3].

Lemma. *Let $E = \oplus_{X \in \mathcal{E}} X$. Then E is a convex partial tilting A_λ-module . In particular, $|\mathcal{E}| \leq \text{rk } K_0(A_\lambda)$.*

Proof. Indeed, since $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$ (see [3], (2.1)), $\mathcal{E} \subseteq \mathcal{L}_A$ implies $\text{pd}_{A_\lambda} E \leq 1$. Since $\text{Ext}_A^1(E,E) = 0$ and A_λ is a full convex subcategory of A, we also have $\text{Ext}_{A_\lambda}^1(E,E) = 0$. Finally, the convexity of E in $\text{ind}A_\lambda$ follows from its convexity in $\text{ind}A$ (see (2.2)). □

4.3. theorem C. *Let A be an artin algebra and Γ be a component of the Auslander-Reiten quiver of the left support A_λ of A. If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

(a) $\Gamma \cap \mathcal{E}$ is a section in Γ.

(b) Γ is directed, and generalised standard.

(c) $A/\text{Ann}(\Gamma \cap \mathcal{E})$ is a tilted algebra having Γ as a connecting component and $\Gamma \cap \mathcal{E}$ as a complete slice.

Proof. (a) In order to show that $\Gamma \cap \mathcal{E}$ is a section in Γ, we just have to check the conditions of the definition in (3.1). Clearly, (S_1) follows from (2.3) and the observation that any cycle in $\text{ind}A_\lambda$ induces one in $\text{ind}A$. Also, (S_3) follows from (4.2). We start by proving (S_4) and (S_5).

(S_4) Assume $X \to Y$ is an arrow in Γ, with $X \in \mathcal{E}$. If $Y \in \mathcal{L}_A$, then (2.2) implies $Y \in \mathcal{E}$. Assume $Y \notin \mathcal{L}_A$. Then, in particular, Y is not a projective A_λ-module. Since Y is an A_λ-module, it is not a projective A-module either, so there is an irreducible morphism $\tau_A Y \to X$ in mod$A$. Then $\tau_A Y$ precedes $X \in \mathcal{E}$ and therefore lies in $\mathcal{L}_A$. Thus, as we observed in (4.1), $\tau_A Y = \tau_{A_\lambda} Y$. Since $\tau_A^{-1}(\tau_{A_\lambda} Y) = Y \notin \mathcal{L}_A$, we conclude that $\tau_{A_\lambda} Y \in \mathcal{E}$, as required.

(S_5) Assume $X \to Y$ is an arrow in Γ, with $Y \in \mathcal{E}$. If $X \notin \mathcal{E}$, then $\tau_A^{-1} X \in \mathcal{L}_A$ and, again by the observation in 4.1, we know that X is not an injective A_λ-module. Hence $\tau_{A_\lambda}^{-1} X = \tau_A^{-1} X \in \mathcal{L}_A$. Since there is an arrow $Y \to \tau_{A_\lambda}^{-1} X$, we conclude that $\tau_{A_\lambda}^{-1} X \in \mathcal{E}$, as required.

There remains to prove (S_2), that is, that $\mathcal{E}$ intersects each orbit of Γ exactly once. We use the same technique as in the proof of Theorem (B). Clearly, the situation is different and so the arguments vary slightly.

We start by proving that $\mathcal{E}$ intersects each orbit of Γ at least once. We claim that if $M \in \mathcal{E}$ and $N \in \Gamma$ lie in two neighbouring orbits, then $\mathcal{E}$ intersects the τ_{A_λ}-orbit of N. This claim and induction imply the statement. We assume that $\mathcal{E}$ does not intersect the orbit of N and try to reach a contradiction. There exist $n \in \mathbb{Z}$ and an arrow $\tau_{A_\lambda}^n M \to X$ or $X \to \tau_{A_\lambda}^n M$, with X in the τ_{A_λ}-orbit of N, where we may suppose, without loss of generality, that $|n|$ is minimal.

Suppose first that $n < 0$. If there exists an arrow $X \to \tau_{A_\lambda}^n M$ then there exists an arrow $\tau_{A_\lambda}^{n+1} M \to X$, a contradiction to the minimality of $|n|$. If, on the other hand, there exists an arrow $\tau_{A_\lambda}^n M \to X$, then there is a path in Γ of the form $M \to * \to \tau_{A_\lambda}^{-1} M \rightsquigarrow X$. Now, $M \in \mathcal{E}$ implies $\tau_A^{-1} M \notin \mathcal{L}_A$. By [7] p. 186, there exists an epimorphism $\tau_A^{-1} M \to \tau_{A_\lambda}^{-1} M$. Hence $\tau_{A_\lambda}^{-1} M \notin \mathcal{L}_A$ and so $X \notin \mathcal{L}_A$. In particular, X is not a projective A_λ-module, so there exists an arrow $\tau_{A_\lambda}^{n+1} M \to \tau_{A_\lambda} X$, contrary to the minimality of $|n|$.

Suppose now that $n > 0$. If there exists an arrow $\tau_{A_\lambda}^n M \to X$, then there exists an arrow $X \to \tau_{A_\lambda}^{n-1} M$, a contradiction to the minimality of $|n|$. If, on the other hand, there exists an arrow $X \to \tau_{A_\lambda}^n M$, then there is a path in Γ of the form $X \to \tau_{A_\lambda}^n M \rightsquigarrow M$, hence X is a predecessor of $\mathcal{E}$. Since $X \notin \mathcal{E}$, by hypothesis, then we know by (4.1) that X is not injective in modA_λ. Hence there exists an arrow $\tau_{A_\lambda}^{-1} X \to \tau_{A_\lambda}^{n-1} M$, contrary to the minimality of $|n|$.

This shows that necessarily $n = 0$, that is, there is an arrow $M \to X$ or $X \to M$. If $M \to X$, then (S_4) yields $X \in \mathcal{E}$ or $\tau_{A_\lambda} X \in \mathcal{E}$, in any case a contradiction. If $X \to M$, then (S_5) yields $X \in \mathcal{E}$ or $\tau_{A_\lambda}^{-1} X \in \mathcal{E}$, again a contradiction in any case.

We proved that $\mathcal{E}$ intersects each τ_{A_λ}-orbit of Γ. Suppose now that $M \in \mathcal{E}$ and $\tau_{A_\lambda}^{-t}M \in \mathcal{E}$ with $t > 0$. Then the epimorphism $\tau_A^{-1}M \to \tau_{A_\lambda}^{-1}M$ yields a path $\tau_A^{-1}M \to \tau_{A_\lambda}^{-1}M \rightsquigarrow \tau_{A_\lambda}^{-t}M$, so that $\tau_A^{-1}M \in \mathcal{L}_A$. This is a contradiction because $M \in \mathcal{E}$. Thus (b) is proven.

(b) Since, by [13], directed components of quasi-tilted algebras are postprojective, preinjective or connecting, thus always generalised standard (see [20, 23]), it suffices to show that Γ is directed. If this is not the case then, by [18] (4.3), Γ is a stable tube, of type $\mathbb{Z}A_\infty$ or obtained from one of these by finitely many ray or coray insertions.

We first notice that by (2.3), any $E_0 \in \Gamma \cap \mathcal{E}$ is directed in $\mathrm{ind}A$, hence in $\mathrm{ind}A_\lambda$. In particular, Γ is neither a stable tube, nor of type $\mathbb{Z}A_\infty$. Therefore Γ is obtained from one of these by ray or coray insertions.

Assume first that Γ is an inserted tube or component of type $\mathbb{Z}A_\infty$, and let $E_0 \in \Gamma \cap \mathcal{E}$. We claim that $E_0 \in \mathcal{E}_2$. Indeed, if this is not the case, then there exists an injective A-module I such that $\mathrm{Hom}_A(I, E_0) \neq 0$, by (2.1). However, $I \in \mathcal{L}_A$ implies that I is an A_λ-module, so that I is an injective A_λ-module. But this is impossible because no injective A_λ-module precedes an inserted tube or component of type $\mathbb{Z}A_\infty$. This establishes our claim. Thus, there exists an indecomposable projective module $P \notin \mathcal{L}_A$ such that $\mathrm{Hom}_A(P, \tau_A^{-1}E_0) \neq 0$, by (2.1). On the other hand, $\tau_{A_\lambda}^{-1}E_0 \in \Gamma$, therefore there exist a non-directed projective $P' \in \Gamma$ and a path $\tau_{A_\lambda}^{-1}E_0 \rightsquigarrow P'$ in Γ. This is clear if Γ is an inserted tube, and follows from [10, 17] if Γ is an inserted component of type $\mathbb{Z}A_\infty$. Hence there exists a path $P \to \tau_A^{-1}E_0 \to \tau_{A_\lambda}^{-1}E_0 \rightsquigarrow P'$ in $\mathrm{ind}A$. Since $P \notin \mathcal{L}_A$, then $P' \notin \mathcal{L}_A$. However, $P' \in \Gamma$, hence P' is a projective A-module lying in $\mathcal{L}_A$, a contradiction.

Assume next that Γ is a co-inserted tube or component of type $\mathbb{Z}A_\infty$, and let $E_0 \in \Gamma \cap \mathcal{E}$. Then, among the predecessors of E_0 lies a cycle $M_0 \to M_1 \to \cdots M_t = M_0$, with all $M_i \in \Gamma$. Since all M_i precede E_0 and, by hypothesis, $E_0 \in \mathcal{E} \subseteq \mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$, then this cycle lies in $\mathcal{L}_{A_\lambda}$. This contradicts Theorem (A) (c) (also [3] (1.5) (b)).

(c) This follows directly from [20] (2.2). $\square$

4.4. Example. It is important to underline that, while the components of $\Gamma(\mathrm{mod}\, A_\lambda)$ which cut $\mathcal{E}$ are directed, the same does not hold for the components of $\Gamma(\mathrm{mod}A)$. Indeed, let k be a field and A be given by the quiver

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\delta"'] & \underset{3}{\bullet} \arrow[l, shift right, "\beta"'] \arrow[l, shift left, "\gamma"] & \underset{4}{\bullet} \arrow[l, "\alpha"'] \arrow[r, "\varepsilon"] & \underset{5}{\bullet} \end{tikzcd}$$

bound by $\beta\delta = 0$, $\gamma\delta = 0$ and $\alpha\beta = 0$. The simple projective S_5 corresponding to the point 5 lies in an inserted tube Γ, it is a directed module and also an Ext-injective in add$\mathcal{L}_A$, that is, $S_5 \in \Gamma \cap \mathcal{E}$. On the other hand, Γ is clearly not directed. Observe that we have an irreducible morphism $S_5 \to P_4$ and $P_4 \notin \mathcal{L}_A$ (compare with (3.1)).

4.5. LEMMA. *Let Γ be a component of $\Gamma(\mathrm{mod}A_\lambda)$.*

(a) If Γ is a non-connecting postprojective component, then $\Gamma\cap\mathcal{E} = \emptyset$.

(b) If Γ is a non-connecting preinjective component, then $\Gamma \cap \mathcal{E} = \emptyset$.

(c) If Γ intersects $\mathcal{E}$, then Γ is connecting.

(d) If a connected component B of A_λ is not tilted, then $\mathrm{mod}B \cap \mathcal{E} = \emptyset$.

Proof. (a) Assume that Γ is a non-connecting postprojective component of $\Gamma(\mathrm{mod}A_\lambda)$ such that $\Gamma\cap\mathcal{E} \neq \emptyset$. Let B be the (unique) connected component of A_λ such that Γ consists of B-modules. We claim that Γ does not contain every indecomposable projective $B-$module. Indeed, if this is not the case, then the number of τ_B-orbits in Γ coincides with rk $K_0(B)$. By Theorem (C) (a), $\mathcal{E}$ intersects each τ_B-orbit of Γ exactly once. Hence $\Gamma \cap \mathcal{E}$ has rk $K_0(B)$ elements. From this and (4.2) we deduce that $E_0 = \bigoplus_{X\in\Gamma\cap\mathcal{E}} X$ is a convex tilting B-module. By [6], (2.1), $\Gamma \cap \mathcal{E}$ is a complete slice in mod$B$. But this is a contradiction, because Γ was assumed to be non-connecting. This establishes our claim.

Now, let $Q \notin \Gamma$ be an indecomposable projective $B-$module. Since B is a connected algebra, there exists a walk of projective $B-$modules $P = P_0 - P_1 - ... - P_s = Q$, with $P \in \Gamma$. Thus there exists i such that $P_i \in \Gamma$ and $P_{i+1} \notin \Gamma$. Since Γ does not receive morphisms from other components of $\Gamma(\mathrm{mod}B)$, then $\mathrm{Hom}_B(P_i, P_{i+1}) \neq 0$. By [21] (2.1) there exists, for each $s > 0$, a path

$$P_i = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_s} M_s = L \xrightarrow{f} P_{i+1}$$

with f_i irreducible. Since s is as large as we want, and $\mathcal{E}$ intersects each τ_B-orbit of Γ, we may choose s so that L is a proper successor of $\Gamma \cap \mathcal{E}$. On the other hand, P_{i+1} is a projective $B-$module, hence a projective $A-$module lying in $\mathcal{L}_A$. Thus $L \in \mathcal{L}_A$. Since L is a successor of $\mathcal{E}$, by (2.2), $L \in \mathcal{E}$, a contradiction which proves (a).

(b) Assume that Γ is a non-connecting preinjective component of $\Gamma(\mathrm{mod}A_\lambda)$ such that $\Gamma \cap \mathcal{E} \neq \emptyset$. Using the same reasoning as in (a), there exist $M \in \Gamma$, which is a proper predecessor of $\Gamma \cap \mathcal{E}$, and an indecomposable injective $A_\lambda-$module $I \notin \Gamma$ such that $\mathrm{Hom}_{A_\lambda}(I, M) \neq 0$. Since I precedes $\mathcal{E}$ then, by (4.1), $I \in \mathcal{E}$. The convexity of $\mathcal{E}$ yields the contradiction $M \in \mathcal{E}$. This establishes (b).

(c) It is shown in [3, 24] that every connected component of A_λ is quasi-tilted. By Theorem (C), $\mathcal{E}$ intersects only directed components of $\Gamma(\mathrm{mod}A_\lambda)$. Furthermore, directed components of quasi-tilted algebras are necessarily postprojective, preinjective or connecting. Now the result follows from (a) and (b).

(d) Is a consequence of (c). □

4.6. Proposition *Let B be a connected component of the left support A_λ, such that* $\mathrm{mod}B \cap \mathcal{E} \neq \emptyset$. *Then:*

(a) B is a tilted algebra and $\mathrm{mod}B \cap \mathcal{E}$ *is a complete slice in* $\mathrm{mod}B$.

(b) Let Γ be a component of $\Gamma(\mathrm{mod}A_\lambda)$ *such that* $\Gamma \cap \mathcal{E} \neq \emptyset$. *Then* $B = A_\lambda\ /\mathrm{Ann}(\Gamma \cap \mathcal{E})$.

Proof. (a) Let Γ be a component of $\Gamma(\mathrm{mod}A_\lambda)$ such that $\Gamma \cap \mathcal{E} \neq \emptyset$. By (c) of the previous lemma, we know that Γ is a connecting component. Since, on the other hand, $\mathcal{E}$ intersects each τ_B-orbit of Γ exactly once (by Theorem (C) (a)), we have $|\Gamma \cap \mathcal{E}| = \mathrm{rk}K_0(B)$. But by (4.2), $|\Gamma(\mathrm{mod}B) \cap \mathcal{E}| \leq \mathrm{rk}K_0(B)$. Hence $\Gamma \cap \mathcal{E} = \Gamma(\mathrm{mod}B) \cap \mathcal{E}$ and the direct sum of the modules in $\Gamma(\mathrm{mod}B) \cap \mathcal{E}$ is a convex tilting B-module. The result then follows from [6](2.1).

(b) We need to prove that $\mathrm{Ann}_{A_\lambda}(\Gamma \cap \mathcal{E})$ is the product of the connected components of A_λ different from B. Clearly $\mathrm{Ann}_{A_\lambda}(\Gamma \cap \mathcal{E})$ contains this product. To prove the other inclusion it is enough to see that $\mathrm{Ann}_B(\Gamma \cap \mathcal{E}) = 0$. But this is again a consequence of (a), since $\Gamma \cap \mathcal{E}$ is faithful in $\mathrm{mod}B$. □

5. Left supported algebras.

5.1. An artin algebra A is *left supported* if $\mathrm{add}\mathcal{L}_A$ is contravariantly finite in $\mathrm{mod}A$, in the sense of [8]. It is shown in [3] (5.1) that an artin algebra A is left supported if and only if each connected component of A_λ is tilted and the restriction of $\mathcal{E}$ to this component is a complete slice. Several other characterisations of left supported algebras are given in [1, 3]. In particular, it is shown in [1] that A is left supported if and only if $\mathcal{L}_A = \mathrm{Pred}\,\mathcal{E}$, where $\mathrm{Pred}\,\mathcal{E}$ denotes the full subcategory of $\mathrm{ind}A$ having as objects all the $M \in \mathrm{ind}A$ such that there exists $E_0 \in \mathcal{E}$ and a path $M \rightsquigarrow E_0$. Our objective in this section is to give another proof of this theorem, using the results above. Our proof also yields a new characterisation of left supported algebras.

Theorem. *Let A be an artin algebra. Then the following conditions are equivalent:*

(a) A is left supported.

(b) $\mathcal{L}_A = \text{Pred}\,\mathcal{E}$.

(c) Every projective $A-$*module which belongs to* $\mathcal{L}_A$ *is a predecessor of* $\mathcal{E}$.

Proof. (a) implies (b). Assume that A is left supported. By [3](4.2), $\mathcal{L}_A$ is cogenerated by the direct sum of the modules in $\mathcal{E}$. In particular, $\mathcal{L}_A \subseteq \text{Pred}\,\mathcal{E}$. Since the reverse inclusion is obvious, this completes the proof of (a) implies (b).

Clearly (b) implies (c). To prove that (c) implies (a) we assume that every projective $A-$module which belongs to $\mathcal{L}_A$ is a predecessor of $\mathcal{E}$. Let B be a connected component of A_λ and P be an indecomposable projective B-module. Since $P \in \mathcal{L}_A$, there exist $E_0 \in \mathcal{E}$ and a path $P \rightsquigarrow E_0$ in $\mathcal{L}_A$, hence in $\text{mod}B$. Therefore, $\text{mod}B \cap \mathcal{E} \neq \emptyset$. By (4.6), B is a tilted algebra and $\text{mod}B \cap \mathcal{E}$ is a complete slice in $\text{mod}B$. Hence A is left supported. $\square$

5.2. We end this paper with a short proof of a result by D. Smith.

Theorem.([25] (3.8)) *Let* A *be a quasi-tilted algebra. Then* A *is left supported if and only if* A *is tilted having a complete slice containing an injective module.*

Proof. Since A is quasi-tilted, then $A = A_\lambda$. Assume that A is left supported. Then $\mathcal{L}_A = \text{Pred}\,\mathcal{E}$. By (5.1), A is tilted and $\mathcal{E}$ is a complete slice in $\Gamma(\text{mod}A)$. Furthermore, since A is quasi-tilted, then all projective A-modules lie in $\mathcal{L}_A$, so that $\mathcal{E}_2 = \emptyset$ and $\mathcal{E} = \mathcal{E}_1$. Thus $\mathcal{E}$ must contain an injective module.

Conversely, if A has a complete slice containing an injective, then there exists a complete slice Σ having all its sources injective. By (2.1), $\Sigma \subseteq \mathcal{E}$. Since $|\Sigma| = \text{rk}K_0(A)$, it follows from [3] (3.3) that A is left supported. $\square$

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