Cohomologies of pullbacks

Juan Carlos Bustamante, Julie Dionne, and David Smith

Abstract. We study the Hochschild and simplicial (co)homology groups or a certain family of pullbacks of algebras, called *oriented pullback*. We prove that for such a pullback, there exists Mayer-Vietoris long exact sequences of Hochschild and simplicial (co)homology groups. Finally, we study the fundamental group, the simple connectedness and the strong simple connectedness of a particular family of pullbacks, the so-called articulated algebras.

Introduction

Let k be a field, and A be a finite dimensional $k-$algebra (with unit). Given an $A-A-$bimodule ${}_AM_A$, the Hochschild homology and cohomology groups of A with coefficients in ${}_AM_A$ are the groups $\mathrm{H}_i(A,M) = \mathrm{Tor}_i^{A\otimes_k A^{op}}(M,A)$ and $\mathrm{H}^i(A,M) = \mathrm{Ext}^i_{A\otimes_k A^{op}}(A,M)$, respectively. In case ${}_AM_A = {}_A A_A$, we simply write $\mathrm{H}_i(A)$, or $\mathrm{H}^i(A)$. These groups are closely related: for every $i \geq 0$ there exists an isomorphism $\phi_i : \mathrm{DH}^i(A, \mathrm{D}X) \to \mathrm{H}_i(A,X)$, where $\mathrm{D} = \mathrm{Hom}_k(-,k)$ is the usual duality.

The cohomology groups corresponding to low degrees ($i \leq 2$) have clear interpretations in terms of classical algebraic structures. Moreover, it was observed by Gerstenhaber in [**19**] that $\mathrm{H}^2(A)$ and $\mathrm{H}^3(A)$ are also related to the rigidity properties of A. From the representation theoretical point of view, several recent results show links between the vanishing of $\mathrm{H}^1(A)$ or $\mathrm{H}^2(A)$ and the simple connectedness, or representation directness of A (see [**11, 4, 3**]).

Beside this, the sum $\mathrm{H}^*(A) = \coprod_{i\geq 0} \mathrm{H}^i(A)$ is a graded commutative ring under the Yoneda product, which coincides with a cup-product $\cup$ defined at the cochain level. In addition, there is another product on $\mathrm{H}^*(A)$, namely the bracket $[-,-]$ which makes of $\mathrm{H}^*(A)$ a graded Lie algebra (see [**20**]). Moreover, these two structures are compatible, but very little is known about them.

In general, the Hochschild cohomology groups are difficult to compute. Direct formulae exist in some particular cases (see [**24, 16, 28**]). Moreover, some long exact sequences of cohomology groups have been obtained in case A admits an homological ideal (see [**33**]), or is a triangular matrix algebra (see [**24, 17, 32, 22, 23, 8**]). In case A is an incidence algebra, these sequences are the Mayer-Vietoris sequences associated to some simplical complex (see [**21**]). However, it is known that Hochschild cohomology does not satisfy the excision axiom, so Mayer-Vietoris sequences cannot be obtained in general contexts.

2000 Mathematics Subject Classification: 16E40

Keywords: Pullbacks, Hochschild cohomology, Simplicial cohomology, Fundamental group

The main purpose of this paper is to study the Hochschild cohomology groups of a particular family of pullbacks of algebras, which we call *oriented pullbacks*. It turns out that for this class of algebras there is a Mayer-Vietoris sequence of Hochschild cohomology groups. More precisely, we prove:

Theorem 1. *Let R be the oriented pullback of the k-algebras projections $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$, and let X be a R-R-bimodule. Moreover, let e_1, e_2 and e be the units of A_1, A_2 and C respectively. Then there exists a long exact sequence of Hochschild cohomology groups :*

$$0 \to \mathrm{H}^0(R,X) \xrightarrow{(\alpha_1^0,\alpha_2^0)^t} \mathrm{H}^0(A_1,e_1Xe_1)\oplus \mathrm{H}^0(A_2,e_2Xe_2) \xrightarrow{(\beta_1^0,\beta_2^0)} \mathrm{H}^0(C,eXe) \to$$
$$\xrightarrow{\delta^0} \mathrm{H}^1(R,X) \xrightarrow{(\alpha_1^1,\alpha_2^1)^t} \mathrm{H}^1(A_1,e_1Xe_1)\oplus \mathrm{H}^1(A_2,e_2Xe_2) \xrightarrow{(\beta_1^1,\beta_2^1)} \mathrm{H}^1(C,eXe) \to \cdots$$

Moreover, if $X = R$, then $eXe = C$, $e_iXe_i = A_i$ $(i = 1, 2)$ and

(a) *the maps $\alpha_i^n : \mathrm{H}^n(R) \to \mathrm{H}^n(A_i)$ and $\beta_i^n : \mathrm{H}^n(A_i) \to \mathrm{H}^n(C)$ induce ring homomorphisms $\alpha_i^* : \mathrm{H}^*(R) \to \mathrm{H}^*(A_i)$ and $\beta_i^* : \mathrm{H}^*(A_i) \to \mathrm{H}^*(C)$.*
(b) *if R is a triangular algebra, the maps $\alpha_i^* : \mathrm{H}^*(R) \to \mathrm{H}^*(A_i)$ and $\beta_i^* : \mathrm{H}^*(A_i) \to \mathrm{H}^*(R)$ are also morphisms of Lie algebras.*

Theorem 2. *Let R be the oriented pullback of the k-algebras projections $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$, and let X be a R-R-bimodule. Moreover, let e_1, e_2 and e be the units of A_1, A_2 and C respectively. Then there exists a long exact sequence of Hochschild homology groups :*

$$\cdots \to \mathrm{H}_1(C,eXe) \to \mathrm{H}_1(A_1,e_1Xe_1)\oplus \mathrm{H}_1(A_2,e_2Xe_2) \to \mathrm{H}_1(R,X) \to$$
$$\to \mathrm{H}_0(C,eXe) \to \mathrm{H}_0(A_1,e_1Xe_1)\oplus \mathrm{H}_0(A_2,e_2Xe_2) \to \mathrm{H}_0(R,X) \to 0.$$

Moreover, in case where $X = R$, we have $eXe = C$ and $e_iXe_i = A_i$ $(i = 1, 2)$.

On the other hand, if an algebra A admits a semi-normed basis one can define its simplicial homology and cohomology groups ([**10, 30**]). In case k is algebraically closed, and A is basic, there exists a unique quiver Q_A and (several) surjective maps $\nu : kQ_A \to A$ (see [**9**]) so that $A \simeq kQ_A/\mathrm{Ker}\ \nu$. To the pair $(Q_A, \mathrm{Ker}\ \nu)$ one can associate its fundamental group $\pi_1(Q_A, \mathrm{Ker}\ \nu)$ (see [**31**]). The strong simple connectedness of an algebra $A = kQ/I$ can be defined in terms of some fundamental groups. There exist several links between the Hochschild (co)homology groups, the simplicial (co)homology groups, and the fundamental groups associated to an algebra A. In the same vein of Theorems 1 and 2, we establish analogous results concerning the simplicial (co)homology groups arising from oriented pullbacks.

The paper is organized as follows. In Section 1, we fix notation and terminology, and recall the definition of a pullback of algebras. In Section 2, we recall the relevant facts about Hochschild (co)homology, give the definition of oriented pullback of algebras and prove Theorem 1 and Theorem 2. Moreover, we give some consequences and examples, and compare the sequence of Theorem 1 with other existing long exact sequences. Finally, in Section 3, we show the existence of a long exact sequence relating the simplicial (co)homology groups of the algebras involved in a pullback. In addition, we study the fundamental group and the (strong) simple connectedness of a particular case of pullback, namely the articulated algebras.

1. Preliminaries

1.1. Quivers, algebras and categories. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 and Q_1 are two finite sets, respectively called the *set of vertices* and the *set of arrows*, and s and t are two maps from Q_1 to Q_0 associating to each arrow respectively, its *source* and its *target*.

A *path* is a sequence of arrows $\omega = \alpha_1\alpha_2\cdots\alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for each i, with $1 \leq i \leq n-1$. We denote by $l(\omega)$ the *length* of the path ω, that is, the number of arrows in ω. The source of such a path is $s(\alpha_1)$, and its target is $t(\alpha_n)$. Also, to each vertex x of Q, we associate the *trivial path* ε_x. Two paths ω and ω' are *parallel* if $s(\omega) = s(\omega')$ and $t(\omega) = t(\omega')$. A vertex x of Q is a *source* of Q if there is no arrow α such that $t(\alpha) = x$. The *sinks* are defined dually.

Given a commutative field k, we define the *path algebra* kQ as the k-vector space having as basis the set of all paths in Q endowed with the following multiplication: let ω_1 and ω_2 be two basis elements, then $\omega_1\omega_2$ is their composition if $t(\omega_1) = s(\omega_2)$, and 0 otherwise.

A *relation* from x to y, with $x, y \in Q_0$, is a linear combination $\rho = \sum_{i=1}^{r} \lambda_i\omega_i$ where, for each i, ω_i is a path of length at least two from x to y in Q and λ_i is a non-zero scalar in k. Let F be the two-sided ideal of kQ generated by all arrows of Q. An ideal I of kQ is *admissible* if there exists a $m \geq 2$ such that $F^m \subseteq I \subseteq F^2$. In this case the pair (Q, I) is called a *bound quiver* and the algebra kQ/I is finite dimensional and basic. In this case, by abuse of notation we simply denote the primitive pairwise orthogonal idempotents $\varepsilon_x + I$ by ε_x. If Q has no oriented cycles, kQ/I is said to be *triangular*.

Conversely, for every finite dimensional, connected and basic algebra A over an algebraically closed field k there exists a unique connected quiver Q and a surjective algebra morphism $\nu : kQ \to A$ with $I = \mathrm{Ker}\ \nu$ admissible. The map ν, or, equivalently, the pair (Q, I) is called a *presentation* of A. Any basic algebra $A = kQ/I$ can equivalently be regarded as a locally bounded k-category **[9]** having as object class the set $A_0 = Q_0$ and as morphism set from x to y the k-vector space $A(x, y) = \varepsilon_x A \varepsilon_y$. A subcategory B of the category A is called *full* if for all x, $y \in B_0$, $B(x, y) = A(x, y)$, and *convex* if any vertex z which lies on a path in Q from $x \in B_0$ to $y \in B_0$ also belongs to B_0.

In this paper, k will always denote a commutative field. Moreover, all the algebras considered will be of the form kQ/I, for a certain bound quiver (Q, I).

1.2. Pullbacks of algebras. Let $A_1 = kQ_1/I_1$, $A_2 = kQ_2/I_2$ and $C = kQ_C/I_C$ be algebras and $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$ be two morphisms of algebras. The *pullback* R of the morphisms f_1 and f_2 is the set of pairs (a_1, a_2) in $A_1 \times A_2$ such that $f_1(a_1) = f_2(a_2)$. It is easy to see that R becomes an algebra with the natural operations.

In this paper, we consider the particular case where Q_C is a full and convex subquiver of Q_1 and Q_2 and I_C is the restriction of I_1 and I_2 to Q_C. In this case, let e be the unit element of C and let $f_i : A_i \to C$ be the canonical projections of algebras defined by $f_i(a) = eae$, for $a \in A_i$. Also, for $i = 1, 2$, let $j_i : Q_C \to Q_i$ be the natural inclusions of quivers. From **[25, 26]** (see also **[29]**), we have the following result:

Proposition 1.2.1. *Let R be the pullback of f_1 and f_2. Then $R \cong kQ_R/I_R$, where $Q_1 \amalg_{Q_C} Q_2 = Q_R$ is the pushout of j_1 and j_2 and $I_R = I_1 + I_2 + <\overline{\rho}>$, with*

$\overline{\rho}$ the set of paths linking $(Q_1)_0 \setminus (Q_C)_0$ and $(Q_2)_0 \setminus (Q_C)_0$ in Q_R. Moreover Q_1 and Q_2 are convex in Q_R. $\square$

Example 1.2.2. Let R be the algebra given by the quiver

$$\begin{tikzcd} & 2 \arrow[r, "\beta_1"] & 4 \arrow[dr, "\gamma_1"] & \\ 1 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & & & 6 \\ & 3 \arrow[r, "\beta_2"'] & 5 \arrow[ur, "\gamma_2"'] & \end{tikzcd}$$

bound by F^3. Then R is the pullback of the projections of A_1 and A_2 on C, where A_1, A_2 and C are the full subcategories respectively generated by the set of objects $\{1,2,3,4,5\}$, $\{2,3,4,5,6\}$ and $\{2,3,4,5\}$.

2. Hochschild (co)homology

2.1. Definitions and notations. Let R be a k-algebra and ${}_RX_R$ be an R-R-bimodule. Following [**14**], the Hochschild cohomology groups of R with coefficients in X are the groups $\mathrm{Ext}^i_{R\otimes_k R^{op}}(R,X)$ and we denote them as $\mathrm{H}^i(R,X)$. In the case where ${}_RX_R = {}_RR_R$, we simply write $\mathrm{H}^i(R)$.

Let $J = \mathrm{rad}R$ and $E = R/J$. By $J^{\otimes n}$ we denote the n^{th} tensor power of J with itself over E. With these notations, and following [**15**, (1.2)], the Hochschild cohomology groups $\mathrm{H}^i(R,X)$ are the cohomology groups of the following sequence, denoted by $\mathcal{C}^\bullet_{R,X} = (\mathcal{C}^i_{R,X}, \delta^i_{R,X})$ in the sequel :

$$0 \longrightarrow \mathrm{Hom}_{E-E}(E,X) \xrightarrow{\delta^0_{R,X}} \mathrm{Hom}_{E-E}(J,X) \xrightarrow{\delta^1_{R,X}} \mathrm{Hom}_{E-E}(J^{\otimes n},X) \xrightarrow{\delta^2_{R,X}} \cdots$$

where $(\delta^0_{R,X}f)(\alpha) = \alpha f(\varepsilon_1) - f(\varepsilon_2)\alpha$, for $\alpha \in \varepsilon_1 J \varepsilon_2$ and

$$\begin{aligned} (\delta^n_{R,X}f)(x_1\otimes\cdots\otimes x_n) &= x_1 f(x_2\otimes\cdots\otimes x_n) + (-1)^n f(x_1\otimes\cdots\otimes x_{n-1})x_n \\ &+ \sum_{i=1}^{n-1}(-1)^i f(x_1\otimes\cdots\otimes x_i x_{i+1}\otimes\cdots\otimes x_n), \end{aligned}$$

for $n \geq 1$ and $x_1\otimes\cdots\otimes x_n \in J^{\otimes n}$.

In the case where R is the pullback of algebras as described in (1.2), we denote by J_i the Jacobson radical of A_i and by E_i the quotient A_i/J_i, for $i = 1,2$. Also, J_C denotes the Jacobson radical of C and $E_C = C/J_C$. Finally, A'_i denotes the full subcategory generated by $(A_i)_0 \setminus C_0$ and e_i, e'_i and e are the identities of the algebras A_i, A'_i and C, respectively. In this case e_i is the sum of the idempotents associated to the vertices of A_i, that is $e_i = \sum_{x\in(Q_i)_0}\varepsilon_x$. Note that with these notations we have $e'_1Re'_2 = e'_2Re'_1 = 0$. This will be important in the sequel. Also, given $\tau_1,\ \tau_2 \in J_C$, we can consider them as elements of J_1, J_2 or J. Nevertheless, we have the following equalities:

$$\tau_1 \otimes_{E_C} \tau_2 = \tau_1 \otimes_{E_1} \tau_2 = \tau_1 \otimes_{E_2} \tau_2 = \tau_1 \otimes_E \tau_2.$$

The Yoneda product in $\mathrm{H}^*(R) = \amalg_{i>0}\mathrm{H}^i(R)$ is induced by a cup-product defined at the cochain level as follows:

Given $f \in \mathcal{C}^n_{R,R}$, $g \in \mathcal{C}^m_{R,R}$ we define $f \cup g \in \mathcal{C}^{n+m}_{R,R}$ by the rule:

$$(f\cup g)(r_1\otimes\cdots\otimes r_{n+m}) = f(r_1\otimes\cdots\otimes r_n)g(r_{n+1}\otimes\cdots\otimes r_{n+m}).$$

On the other hand, if R is a triangular algebra, the elements $f \in \mathcal{C}^n_{R,R}$ take values in the radical of R, so that, given $i \in \{1, \ldots, n\}$, we can define $f \circ_i g$ by the rule :

$$(f \circ_i g)(r_1 \otimes \cdots \otimes r_{n+m-1}) = f(r_1 \otimes \cdots \otimes g(r_i \otimes \cdots \otimes r_{i+m-1}) \otimes \cdots \otimes r_{n+m-1}).$$

Then we define the composition $f \circ g = \begin{cases} 0, & \text{if } n = 0; \\ \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g, & \text{if } n > 0. \end{cases}$
and finally the bracket of f and g as $[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f$.

These two structures are related. In fact $\mathrm{H}^*(R)$ is a Gerstenhaber algebra (see [**20**]), that is a graded k-vector space endowed with a product $\cup$ which makes of $\mathrm{H}^*(R)$ a graded commutative algebra, and a Lie bracket $[-,-]$ of degree -1 that makes of $\mathrm{H}^*(R)$ a graded Lie algebra, and such that $[x, yz] = [x, y]z + (-1)^{(|x|-1)|y|} y[x, z]$.

On the other hand, following [**14**], the Hochschild homology groups of R with coefficients in X are the groups $\mathrm{Tor}_i^{R \otimes_k R^{op}}(X, R)$ and we denote them as $\mathrm{H}_i(R, X)$. Consequently, there exists an isomorphism $\phi_i : D\mathrm{H}^i(R, DX) \to \mathrm{H}_i(R, X)$, where $D = \mathrm{Hom}_k(-, k)$ is the usual duality [**14**].

2.2. Main results. In this section, we give the proof of Theorem 1 and Theorem 2. We begin with some definitions and preliminary results.

Definition 2.2.1. Let $R = kQ_R/I_R$ be the pullback of the k-algebra projections $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$ as defined in (1.2), where $A_i \cong kQ_i/I_i$ for $i \in \{1, 2\}$ and $C \cong kQ_C/I_C$. Then R is called *oriented pullback* if one of the following conditions is satisfied:

(1) $eRe'_1 = 0$ and $eRe'_2 = 0$;
(2) $e'_1Re = 0$ and $e'_2Re = 0$;
(3) $e'_1Re = 0$ and $eRe'_2 = 0$;

Using the decomposition $1_R = e'_1 + e + e'_2$, one can see that the oriented pullback algebras respectively have one of the following form:

$$1.\ R = \begin{pmatrix} A'_1 & e'_1Re & 0 \\ 0 & C & 0 \\ 0 & e'_2Re & A'_2 \end{pmatrix}; \quad 2.\ R = \begin{pmatrix} A'_1 & 0 & 0 \\ eRe'_1 & C & eRe'_2 \\ 0 & 0 & A'_2 \end{pmatrix};$$

$$3.\ R = \begin{pmatrix} A'_1 & 0 & 0 \\ eRe'_1 & C & 0 \\ 0 & e'_2Re & A'_2 \end{pmatrix}$$

Clearly, this definition generalizes the definition of "Nakayama oriented pullback" introduced in [**29**]. Moreover, in this case, one easily see that if ${}_RX_R = {}_RR_R$ then $eXe = C$ and $e_iXe_i = A_i$, for $i = 1, 2$. For the remainder of this section, we suppose that R is an oriented pullback of projections $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$, and we use the notations of (2.2.1). Also, note that oriented pullbacks are particular cases of extensions of algebras (compare with [**17, 32, 22, 8**]).

Lemma 2.2.2. *Under the above assumptions, there is no non zero path from A'_2 to A'_1 or from A'_1 to A'_2 in (Q_R, I_R). Moreover,*

(1) *If $eRe'_i = 0$, then there is no non zero path from C to A'_i;*
(2) *If $e'_iRe = 0$, then there is no non zero path from A'_i to C;*
(3) *The categories A_1 and A_2 are full and convex subcategories of R.*

Proof : (1). Let $eRe'_1 = 0$ and suppose that $\alpha_1\alpha_2 \cdots \alpha_n$ is a path from a vertex x of C to a vertex y of A'_1. Let j =min$\{i \mid s(\alpha_i) \notin (A'_1)_0$ and $t(\alpha_i) \in (A'_1)_0\}$. Then $\alpha_j \in \varepsilon_{s(\alpha_j)} R \varepsilon_{t(\alpha_j)}$, thus $eRe'_1 \neq 0$ or $e'_2Re'_1 \neq 0$, a contradiction. Statement (2) is shown in the same way. For statement (3), fullness follows from the fact that $e_jRe'_i = eRe'_i = 0$, or $e'_iRe_j = e'_iRe = 0$ when $1 \le i \ne j \le 2$. The convexity follows from (1) and (2). □

Lemma 2.2.3. *Under the above assumptions, $J_i = e_iJe_i$ for $i = 1, 2$.*

Proof : Since J is the k-vector space generated by $\{\omega + I_R \mid \omega$ is a path in Q and $l(\omega) \ge 1\}$, e_iJe_i is generated by

$$\begin{array}{rl} & \{e_i\omega e_i + I_R \mid \omega \text{ is a path in } Q \text{ and } l(\omega) \ge 1\} \\ = & \{e_i\omega e_i + I_R \mid \omega \text{ is a path in } Q \text{ and } l(e_i\omega e_i) \ge 1\} \\ = & \{\omega' + I_R \mid \omega' \text{ is a path in } Q_i \text{ and } l(\omega') \ge 1\} \\ = & \{\omega' + I_i \mid \omega' \text{ is a path in } Q_i \text{ and } l(\omega') \ge 1\} \\ = & J_i, \end{array}$$

where the second equality follows from the convexity of A_i and the third one follows from the fact that I_i is the restriction of I_R to A_i. □

Lemma 2.2.4. *Under the above assumptions, $e_iJ^{\otimes n}e_i = (e_iJe_i)^{\otimes n}$, for $i = 1, 2$.*

Proof : First, we prove that $(e_iJe_i)^{\otimes n} \subseteq e_iJ^{\otimes n}e_i$. Clearly, e_iJe_i is included in J, and hence, $(e_iJe_i)^{\otimes n} \subseteq J^{\otimes n}$. We only have to multiply by e_i on each side to obtain the result.

On the other hand, let $\tau = e_ix_1 \otimes x_2 \otimes \cdots \otimes x_ne_i$ be a non-zero element of $e_iJ^{\otimes n}e_i$. We can assume without loss of generality that x_j is a path in Q for each j. Thus $e_ix_1 \otimes x_2 \otimes \cdots \otimes x_ne_i = e_1\varepsilon_{s(x_1)}x_1\varepsilon_{t(x_1)} \otimes \cdots \otimes \varepsilon_{s(x_n)}x_n\varepsilon_{t(x_n)}e_i$ and it follows from the convexity of A_i that $x_j = \varepsilon_{s(x_j)}x_j\varepsilon_{t(x_j)} = e_ix_je_i$ for each j. Then, $\tau \in (e_iJe_i)^{\otimes n}$. □

Lemma 2.2.5. *Under the above assumptions, the following hold, for $i = 1, 2$:*

(1) *If $eRe'_i = 0$, then $J_i^{\otimes n} = e'_iJ^{\otimes n}e'_i \oplus e'_iJ^{\otimes n}e \oplus eJ^{\otimes n}e$, and;*
(2) *If $e'_iRe = 0$, then $J_i^{\otimes n} = e'_iJ^{\otimes n}e'_i \oplus eJ^{\otimes n}e'_i \oplus eJ^{\otimes n}e$;*

Proof : Keeping in mind that e'_1, e and e'_2 are pairwise orthogonal idempotents, this follows from the equality $e_i = e'_i + e$ and the fact that $J_i^{\otimes n} = (e_iJe_i)^{\otimes n} = e_iJ^{\otimes n}e_i = (e'_i + e)J^{\otimes n}(e'_i + e)$. □

This allows us to prove Theorem 1.

PROOF OF THEOREM 1 : We only prove the existence of the long exact sequence in case (1) of the definition of oriented pullbacks (2.2.1), since the proof in the other cases is similar.

We first observe that, since $1_R = e'_1 + e + e'_2$, we have

$$\begin{array}{rcl} J^{\otimes n} & = & e'_1J^{\otimes n}e'_1 \oplus e'_1J^{\otimes n}e \oplus e'_1J^{\otimes n}e'_2 \oplus eJ^{\otimes n}e'_1 \oplus eJ^{\otimes n}e \\ & & \oplus eJ^{\otimes n}e'_2 \oplus e'_2J^{\otimes n}e'_1 \oplus e'_2J^{\otimes n}e \oplus e'_2J^{\otimes n}e'_2 \\ & = & e'_1J^{\otimes n}e'_1 \oplus e'_1J^{\otimes n}e \oplus eJ^{\otimes n}e \oplus e'_2J^{\otimes n}e \oplus e'_2J^{\otimes n}e'_2, \\ & \subseteq & J_1^{\otimes n} \oplus J_2^{\otimes n}. \end{array}$$

and consequently,

$$\begin{array}{rcl} \mathrm{Hom}_{E-E}(J^{\otimes n}, X) & = & \mathrm{Hom}_{E-E}(e'_1J^{\otimes n}e'_1, X) \oplus \mathrm{Hom}_{E-E}(e'_1J^{\otimes n}e, X) \\ & & \oplus\ \mathrm{Hom}_{E-E}(eJ^{\otimes n}e, X) \\ & & \oplus\ \mathrm{Hom}_{E-E}(e'_2J^{\otimes n}e'_2, X) \oplus \mathrm{Hom}_{E-E}(e'_2J^{\otimes n}e, X) \\ & \cong & \mathrm{Hom}_{E_1-E_1}(e'_1J^{\otimes n}e'_1, e_1Xe_1) \oplus \mathrm{Hom}_{E_1-E_1}(e'_1J^{\otimes n}e, e_1Xe_1) \\ & & \oplus\ \mathrm{Hom}_{E_C-E_C}(eJ^{\otimes n}e, eXe) \\ & & \oplus\ \mathrm{Hom}_{E_2-E_2}(e'_2J^{\otimes n}e'_2, e_2Xe_2) \oplus \mathrm{Hom}_{E_2-E_2}(e'_2J^{\otimes n}e, e_2Xe_2) \end{array}$$

where the isomorphism follows from immediate identifications.

Similarly, for $i = 1, 2$, we obtain, from Lemma (2.2.5), that

$$\begin{aligned}\mathrm{Hom}_{E_i-E_i}(J_i^{\otimes n}, e_iXe_i) \cong\ & \mathrm{Hom}_{E_i-E_i}(e_i'J^{\otimes n}e_i', e_iXe_i) \oplus \mathrm{Hom}_{E_i-E_i}(e_i'J^{\otimes n}e, e_iXe_i) \\ & \oplus \mathrm{Hom}_{E_i-E_i}(eJ^{\otimes n}e, e_iXe_i)\end{aligned}$$

With these equalities in mind define, for $i = 1, 2$, the natural projection

$$\begin{array}{rccc}\alpha_i^n : & \mathrm{Hom}_{E-E}(J^{\otimes n}, X) & \longrightarrow & \mathrm{Hom}_{E_i-E_i}(J_i^{\otimes n}, e_iXe_i) \\ & f & \longmapsto & e_ife_i,\end{array}$$

that is $\alpha_i^n(f)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = f(e_ix_1 \otimes x_2 \otimes \cdots \otimes x_ne_i)$, for $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in J_i^{\otimes n}$. Clearly, $\alpha^n = (\alpha_1^n, \alpha_2^n)^t$ is injective for each n. Moreover, $\alpha^\bullet = (\alpha_1^\bullet, \alpha_2^\bullet)^t$ is a morphism of complexes, as a straightforward computation shows.

On the other hand, it follows from the above decompositions that the cokernel of $\alpha^\bullet$ is $\beta^\bullet$, where, for each n, β^n is defined by the rule: $\beta^n(f, g) = efe - ege$.

Therefore, with the notation of section 2.1, we obtain the following short exact sequence of complexes :

$$0 \longrightarrow \mathcal{C}^\bullet_{R,X} \xrightarrow{\alpha^\bullet} \mathcal{C}^\bullet_{A_1,e_1Xe_1} \oplus \mathcal{C}^\bullet_{A_2,e_2Xe_2} \xrightarrow{\beta^\bullet} \mathcal{C}^\bullet_{C,eXe} \longrightarrow 0,$$

and the result follows.

In order to prove statement a), we show that α_i^* is compatible with the cup-product. But this follows from the fact that if $x \in J_i^{\otimes(n+m)}$, then $x = e_ixe_i$. So $(\alpha_i^n(f)) \cup (\alpha_i^m(g))(x) = (f \cup g)(x) = \alpha_i^{n+m}(f \cup g)(x)$, for $f \in \mathcal{C}^n_{R,R}$ and $g \in \mathcal{C}^m_{R,R}$. A similar argument applies to β_i^*.

In order to prove statement b), we show that α_i^* is compatible with the bracket. Let $f \in \mathcal{C}^n_{R,R}$, $g \in \mathcal{C}^m_{R,R}$ and $x_1 \otimes \cdots \otimes x_{n+m-1} \in J_i^{\otimes n+m-1}$. Then

$$\begin{aligned}\alpha_i(f \circ_j g)&(x_1 \otimes \cdots \otimes x_{n+m-1}) \\ &= (f \circ_j g)(e_ix_1 \otimes \cdots \otimes x_{n+m-1}e_i) \\ &= f(e_ix_1 \otimes \cdots \otimes g(x_j \otimes \cdots \otimes x_{j+m-1}) \otimes \cdots \otimes x_{n+m-1}e_i) \\ &= f(e_ix_1 \otimes \cdots \otimes g(e_ix_j \otimes \cdots \otimes x_{j+m-1}e_i) \otimes \cdots \otimes x_{n+m-1}e_i) \\ &= (\alpha_i(f)) \circ_j (\alpha_i(g))(x_1 \otimes \cdots \otimes x_{n+m-1}).\end{aligned}$$

The result follows by linearity. Similarly, β_i^* is compatible with the bracket. $\square$

We now give an example showing that Theorem 1 does not hold true when we remove the assumption of oriented pullback.

Example 2.2.6. *Let R be the algebra given by the quiver* $\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$ *bound by the relations $\alpha\beta = 0$ and $\beta\alpha = 0$. Then R is a pullback of the projections of A_1 and A_2 on C, where A_1 and A_2 are the subcategories respectively generated by the arrows α and β and C is the semisimple algebra generated by the vertices. However, R is not an oriented pullback and $\mathrm{H}^i(R) \neq 0$ for each $i \geq 0$, but $\mathrm{H}^i(A_1) = \mathrm{H}^i(A_2) = \mathrm{H}^i(C) = 0$ for each $i > 0$.*

We now use the isomorphism $\phi_i : D\mathrm{H}^i(R, DX) \to \mathrm{H}_i(R, X)$ mentioned in section 2.1 and prove Theorem 2.

PROOF OF THEOREM 2 : The first observation to make is that, for $i = 1, 2$, we clearly have an isomorphism $e_i\mathrm{Hom}_k(X, k)e_i \cong \mathrm{Hom}_k(e_iXe_i, k)$. Similarly, $e\mathrm{Hom}_k(X, k)e \cong \mathrm{Hom}_k(eXe, k)$.

On the other hand, it follows from Theorem 1 that there exists a long exact sequence of cohomology groups

$$\cdots \to \mathrm{H}^i(R, DX) \to \mathrm{H}^i(A_1, e_1(DX)e_1) \oplus \mathrm{H}^i(A_2, e_2(DX)e_2) \to \mathrm{H}^i(C, e(DX)e) \to \cdots$$

and consequently a long exact sequence of homology groups

$$\begin{tikzcd}
\cdots & D\mathrm{H}^i(R,DX) \arrow[l] \arrow[d, "\cong"] & D\mathrm{H}^i(A_1,e_1(DX)e_1)\oplus D\mathrm{H}^i(A_2,e_2(DX)e_2) \arrow[l] \arrow[d, "\cong"] & D\mathrm{H}^i(C,e(DX)e) \arrow[l] \arrow[d, "\cong"] & \cdots \arrow[l] \\
\cdots & D\mathrm{H}^i(R,DX) \arrow[l] \arrow[d, "\phi_i"', "\cong"] & D\mathrm{H}^i(A_1,D(e_1Xe_1))\oplus D\mathrm{H}^i(A_2,D(e_2Xe_2)) \arrow[l] \arrow[d, "\phi_i"', "\cong"] & D\mathrm{H}^i(C,D(eXe)) \arrow[l] \arrow[d, "\phi_i"', "\cong"] & \cdots \arrow[l] \\
\cdots & \mathrm{H}_i(R,X) \arrow[l] & \mathrm{H}_i(A_1,e_1Xe_1)\oplus \mathrm{H}_i(A_2,e_2Xe_2) \arrow[l] & \mathrm{H}_i(C,eXe) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

which proves the statement. $\square$

2.3. The case where C is semisimple. We have a better description of the Hochschild cohomology groups in the case where R is an oriented pullback as above and C is a semisimple algebra. Observe that this particular case of oriented pullback corresponds exactly with the "unidirectional articulated algebras" of [**18**], see (3.2.1) for more details. We begin with the following result on the center of an algebra.

Lemma 2.3.1. *Let $A = kQ/I$ be a connected algebra, ε_i be the idempotent at the vertex i for each $i \in Q_0$ and ω belongs to the center $Z(A)$ of A. Then, $\omega = \sum_{i\in Q_0} \omega_i + \lambda 1_A$, where $\lambda \in k$ and $\omega_i \in \varepsilon_i \mathrm{rad} A \varepsilon_i$ for each i.*

Proof : Since classes modulo I of the paths in Q form a basis of kQ/I, ω can be written as $\omega = \sum_{i,j\in Q_0} \omega_{i,j}$, where $\omega_{i,j} \in \varepsilon_i A \varepsilon_j$ for all i, j. For each vertex l in Q_0, the centrality of ω gives

$$\sum_{j\neq l} \omega_{l,j} + \omega_{l,l} = \varepsilon_l \omega = \omega \varepsilon_l = \sum_{j\neq l} \omega_{j,l} + \omega_{l,l}.$$

Hence, $\sum_{j\neq l} \omega_{l,j} = 0 = \sum_{j\neq l} \omega_{j,l}$. Therefore, we have $\omega = \sum_{i\in Q_0} \omega_i + \sum_{i\in Q_0} \lambda_i \varepsilon_i$, where $\omega_i \in \varepsilon_i \mathrm{rad} A \varepsilon_i$ and $\lambda_i \in k$ for each $i \in Q_0$. To end the proof, it remains to show that $\lambda_i = \lambda_j$ for all $i, j \in Q_0$. Let $\alpha : l \to m$ be an arrow in Q. Then, $\alpha\omega_m + \lambda_m \alpha = \alpha\omega = \omega\alpha = \omega_l \alpha + \lambda_l \alpha$. The admissibility of the ideal I gives $\alpha\omega_m = \omega_l\alpha$ and $\lambda_l\alpha = \lambda_m\alpha$. In particular, $\lambda_l = \lambda_m$. Since Q is connected, $\lambda_i = \lambda_j$ for all $i, j \in Q_0$. $\square$

Proposition 2.3.2. *Let R be an oriented pullback of k-algebra projections $f_i : A_i \to C$, $i = 1, 2$, and assume that C is a semisimple algebra with m vertices. Then:*

(a) $\dim_k \mathrm{H}^0(R) = \dim_k \mathrm{H}^0(A_1) + \dim_k \mathrm{H}^0(A_2) - 1$;
(b) $\dim_k \mathrm{H}^1(R) = \dim_k \mathrm{H}^1(A_1) + \dim_k \mathrm{H}^1(A_2) + (m - 1)$;
(c) $\mathrm{H}^i(R) = \mathrm{H}^i(A_1) \oplus \mathrm{H}^i(A_2)$ *for* $i \geq 2$.

Proof : (a). Since the 0-Hochschild cohomology group $\mathrm{H}^0(A)$ of an algebra A is isomorphic to its center $Z(A)$, it is sufficient to show that $\dim_k Z(R) = \dim_k Z(A_1) + \dim_k Z(A_2) - 1$.

For $j = 1, 2$, let $\mathbb{B}_j$ be a k-basis of $Z(A_j)$ containing 1_{A_j}, the identity of A_j. We claim that $\mathbb{B} = (1_R \cup \mathbb{B}_1 \cup \mathbb{B}_2) \setminus \{1_{A_1}\} \cup \{1_{A_2}\}$ is a k-basis of $Z(R)$. Since this set is clearly linearly independent, it remains to show that it is a generating set in $Z(R)$. Let ω be an element of $Z(R)$. By (2.3.1), we have $\omega = \sum_i \omega_i + \lambda 1_R$, where $\lambda \in k$ and $\omega_i \in e_i \mathrm{rad} R e_i$ for each i. Moreover, since C is semisimple, we have $\omega = \sum_{i\notin Q_C} \omega_i + \lambda 1_R$. Therefore, we can write without ambiguity $\omega = \rho_1 + \rho_2 + \lambda 1_R$, where $\rho_j = \sum_{i\in Q_{A'_j}} \omega_i$, $j = 1, 2$. It then remains to show that $\rho_j \in Z(A_j)$ for $j = 1, 2$ since, by construction, $\rho_j \in Z(A_j)$ if and only if ρ_j belongs to the k-vector space generated by $\mathbb{B}_j \setminus \{1_{A_j}\}$. Let η_1 be a path in Q_1. We have :

$$\begin{aligned}
\rho_1\eta_1 + \eta_1 &= \rho_1\eta_1 + 0 + \eta_1 = \rho_1\eta_1 + \rho_2\eta_1 + 1_r\eta_1 = \omega\eta_1 \\
&= \eta_1\omega = \eta_1\rho_1 + \eta_1\rho_2 + \eta_1 1_R = \eta_1\rho_1 + 0 + \eta_1 \\
&= \eta_1\rho_1 + \eta_1
\end{aligned}$$

and hence $\rho_1\eta_1 = \eta_1\rho_1$. Therefore, $\rho_1 \in Z(A_1)$. Since, similarly, $\rho_2 \in Z(A_2)$, ω belongs to the k-vector space generated by $\mathbb{B}$ and $\mathbb{B}$ is a k-basis of $Z(R)$. In particular, $\dim_k Z(R) = \dim_k Z(A_1) + \dim_k Z(A_2) - 1$, and this proves (a).

(b). Since C is a semisimple algebra, we have $\mathrm{H}^i(C) = 0$ for each $i \geq 1$ and thus we have the following exact sequence :

$$0 \to \mathrm{H}^0(R) \to \mathrm{H}^0(A_1) \oplus \mathrm{H}^0(A_2) \to \mathrm{H}^0(C) \to \mathrm{H}^1(R) \to \mathrm{H}^1(A_1) \oplus \mathrm{H}^1(A_2) \to 0$$

and hence,

$$\begin{aligned} \dim_k \mathrm{H}^1(R) = \quad & \dim_k \mathrm{H}^1(A_1) + \dim_k \mathrm{H}^1(A_2) \\ & + \dim_k \mathrm{H}^0(C) + \dim_k \mathrm{H}^0(R) - \dim_k \mathrm{H}^0(A_1) - \dim_k \mathrm{H}^0(A_2) \end{aligned}$$

Since C is semisimple with m vertices, we have $\dim_k \mathrm{H}^0(C) = m$. Moreover $\dim_k \mathrm{H}^0(R) - \dim_k \mathrm{H}^0(A_1) - \dim_k \mathrm{H}^0(A_2) = -1$ by (a).
(c). Follows directly from Theorem 1. □

2.4. Examples and comparison. We now compare the sequence of Theorem 1 with the existing ones. First of all, let

$$R = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$$

and X is a R-R-bimodule, where A and B are arbitrary finitely generated k-algebras and M a finitely generated B-A-bimodule. Moreover, let e_A and e_B be the unit elements of A and B. By [**17, 32, 22, 23**], we have the following result.

Theorem 2.4.1. *Let R be as above and let X be a R-R-bimodule. Then, there exists a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{H}^0(R, X) \to \mathrm{H}^0(A, e_A X e_A) \oplus \mathrm{H}^0(B, e_B X e_B) \to \mathrm{Ext}^0_{B \otimes_k A^{op}}(M, e_B X e_A) \to \\ \to \mathrm{H}^1(R, X) \to \mathrm{H}^1(A, e_A X e_A) \oplus \mathrm{H}^1(B, e_B X e_B) \to \mathrm{Ext}^1_{B \otimes_k A^{op}}(M, e_B X e_A) \to \cdots \end{aligned}$$

□

If $B = k$, the algebra R is a one-point extension. The above long exact sequence has been obtained by D. Happel [**24**] in this case. The following examples show the difference between the exact sequence of Theorem 1 and the exact sequence recalled above.

Examples 2.4.2.

(1) The following example shows that the long exact sequences of Theorem 1 and (2.4.1) may differ from each other.
Let R be the radical square zero algebra given by the quiver

$$\begin{tikzcd} 5 \arrow[d] \arrow[dr] \arrow[drr] & & 6 \arrow[d] \arrow[dl] \arrow[dll] \\ 2 \arrow[dr] & 3 \arrow[d] & 4 \arrow[dl] \\ & 1 & \end{tikzcd}$$

Then R is the oriented pullback (satisfying the condition (1) of (2.2.1)) of the k-algebras projections $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$, where A_1, A_2 and C are the full subcategories generated by the sets of vertices $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 6\}$ and $\{1, 2, 3, 4\}$ respectively. By [**16**, (p. 96)],

$$\mathrm{H}^i(A_1) = \mathrm{H}^i(A_2) = \begin{cases} k, & \text{if } i = 0; \\ k^2, & \text{if } i = 1; \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad \mathrm{H}^i(R) = \begin{cases} k, & \text{if } i = 0; \\ k^4, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, since the ordinary quiver of C is a tree, $\mathrm{H}^0(C) = k$ and $\mathrm{H}^i(C) = 0$ for $i \geq 1$. By Theorem 1, there exists an exact sequence :

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathrm{H}^0(R) \arrow[r] \arrow[d, equal] & \mathrm{H}^0(A_1)\oplus\mathrm{H}^0(A_2) \arrow[r] \arrow[d, equal] & \mathrm{H}^0(C) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(R) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(A_1)\oplus\mathrm{H}^1(A_2) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(C) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & k \arrow[r] & k\oplus k \arrow[r] & k \arrow[r] & k^4 \arrow[r] & k^2\oplus k^2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

On the other hand, if we respectively denote by S_x and P_x the simple module and the projective module associated to the vertez x in the quiver, one can see R as the one-point extension

$$A_1[M] = \begin{pmatrix} A_1 & 0 \\ M & k \end{pmatrix},$$

where $M = \mathrm{rad}P_6 = S_2 \oplus S_3 \oplus S_4$ is a semisimple module without self-extension. In this case, the long exact sequence of (2.4.1) corresponds with the Happel's sequence [**24**], that is :

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathrm{H}^0(R) \arrow[r] \arrow[d, equal] & \mathrm{H}^0(A_1) \arrow[r] \arrow[d, equal] & (\mathrm{Hom}_{A_1}(M,M))/k \arrow[r] \arrow[d, equal] & \mathrm{H}^1(R) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(A_1) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{A_1}(M,M) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & k \arrow[r] & k \arrow[r] & k^2 \arrow[r] & k^4 \arrow[r] & k^2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

We observe that the two long exact sequences are different.

(2) The following example shows that the long exact sequences of Theorem 1 can be more appropriate than the sequence of (2.4.1) in some circumstances.
Let R be the algebra given by the quiver

$$\begin{tikzcd}
1 \arrow[loop above, "\varepsilon"] & 2 \arrow[l, "\delta"'] & 3 \arrow[l, "\gamma"'] & 4 \arrow[l, "\beta"'] \arrow[loop below, "\alpha"]
\end{tikzcd}$$

bound by the relations $\alpha^2 = \alpha\beta = \beta\gamma = \delta\varepsilon = \varepsilon^2 = 0$. Then R is the oriented pullback (satisfying the condition (3) of (2.2.1)) of the k-algebras projections $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$, where A_1, A_2 and C are the full subcategories generated by the sets of vertices $\{1, 2, 3\}$, $\{3, 4\}$ and $\{3\}$ respectively. A quick calculation gives:

$$\mathrm{H}^i(A_1) = \mathrm{H}^i(A_2) = \begin{cases} k^2, & \text{if } i = 0; \\ k^2, & \text{if } i = 1; \\ k, & \text{otherwise.} \end{cases}$$

Moreover, since the C is semisimple, $\mathrm{H}^0(C) = k$ and $\mathrm{H}^i(C) = 0$ for $i \geq 1$. By Theorem 1, there exists an exact sequence

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathrm{H}^0(R) \arrow[r] \arrow[d, equal] & \mathrm{H}^0(A_1)\oplus\mathrm{H}^0(A_2) \arrow[r] \arrow[d, equal] & \mathrm{H}^0(C) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(R) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(A_1)\oplus\mathrm{H}^1(A_2) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(C) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & k^3 \arrow[r] & k^2\oplus k^2 \arrow[r] & k \arrow[r] & k^4 \arrow[r] & k^2\oplus k^2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

from which we obtain that $\mathrm{H}^1(R) \cong k^4$ and $\mathrm{H}^i(R) \cong \mathrm{H}^i(A_1) \oplus \mathrm{H}^i(A_2) \cong k \oplus k$ for $i \geq 2$. On the other hand, one can see R as given by the matrix algebra

$$R = \begin{pmatrix} A_1 & 0 \\ M & A_2' \end{pmatrix},$$

where A_2' is the full subcategory generated by the vertex 4 and M is the A_2'-A_1-bimodule given by the arrow β. In order to apply the sequence of (2.4.1), one needs to compute $\mathrm{Ext}^i_{A_2' \otimes A_1{}^{op}}(M, M)$ for $i \geq 0$. A direct calculation leads to the fact that $A_2' \otimes A_1{}^{op}$ is given by the quiver

$$\begin{tikzcd}
4\otimes 1 \arrow[loop above, "4\otimes\varepsilon"] \arrow[loop below, "\alpha\otimes 1"] \arrow[r, "4\otimes\delta"] & 4\otimes 2 \arrow[loop below, "\alpha\otimes 2"] \arrow[r, "4\otimes\gamma"] & 4\otimes 3 \arrow[loop below, "\alpha\otimes 3"]
\end{tikzcd}$$

bound by the relations induced by the ordinary quiver of R, that is $(\alpha\otimes 1)^2 = (\alpha\otimes 2)^2 = (\alpha\otimes 3)^2 = (4\otimes\varepsilon)^2 = (4\otimes\varepsilon)(4\otimes\delta) = 0$, $(\alpha\otimes 1)(4\otimes\delta) = (4\otimes\delta)(\alpha\otimes 2)$ and $(\alpha\otimes 2)(4\otimes\gamma) = (4\otimes\gamma)(\alpha\otimes 3)$. Moreover, the $A_2'\otimes A_1{}^{op}$-module M is given by the simple module $S_{4\otimes 3}$. However, since $(\alpha\otimes 3)^2 = 0$, it follows from **[27**, (1.4)] that $\mathrm{Ext}^i_{A_2'\otimes A_1{}^{op}}(M,M) \neq 0$ for $i \geq 1$. Hence, the long exact sequence of (2.4.1) cannot efficiently be used to compute the Hochschild cohomology groups of R.

3. Simplicial (co)homology, fundamental group and simple connectedness

As mentionned in the introduction, close links exist between the (simplicial and Hochschild) (co)homology groups, the fundamental group and the simple connectedness of an algebra. In this section, we study the simplicial (co)homology, the fundamental group and the simple connectedness of a pullback of algebras. In particular, we prove the existence of a long exact sequence of simplicial (co)homology groups relating the algebras involved in a pullback of algebras.

3.1. Simplicial (co)homology. Given an algebra $R = kQ/I$ and a pair $(x,\ y)$ of vertices of Q, let ${}_y\mathbb{B}_x$ be a basis of the k-space $R(x,y)$ and $\mathbb{B} = \bigcup_{(x,y)\in Q_0\times Q_0} {}_y\mathbb{B}_x$. Then we say that $\mathbb{B}$ is a *semi-normed basis* **[30]** if:

(1) $\varepsilon_x \in {}_x\mathbb{B}_x$ for every vertex $x\in Q_0$;
(2) $\alpha + I \in {}_y\mathbb{B}_x$ for every arrow $\alpha : x \to y$;
(3) For σ and σ' in $\mathbb{B}$, either $\sigma\sigma' = 0$ or there exist $\lambda_{\sigma,\sigma'} \in k$, and $b(\sigma,\sigma') \in \mathbb{B}$ such that $\sigma\sigma' = \lambda_{\sigma,\sigma'} b(\sigma,\sigma')$.

To an algebra $R = kQ/I$ which admits a semi-normed basis, we can associate the *chain complex* $(\mathrm{SC}_\bullet(R), d)$ in the following way: $\mathrm{SC}_0(R)$ and $\mathrm{SC}_1(R)$ are the free abelian groups with basis Q_0 and $\mathbb{B}$, respectively. For $n \geq 2$, let $\mathrm{SC}_n(R)$ be the free abelian group with basis the set of n-tuples $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $\mathbb{B}^n$ such that $\sigma_1\sigma_2\cdots\sigma_n \neq 0$, and $\sigma_i \neq e_j$ for all $i, j \in Q_0$. The differential d_n is induced by the rules $d_1(\sigma) = y - x$ for $\sigma \in {}_y\mathbb{B}_x$, and, for $n \geq 1$:

$$\begin{aligned} d_n(\sigma_1,\sigma_2,\ldots,\sigma_n) \quad &= \quad (\sigma_2,\ldots,\sigma_n) + \sum_{j=1}^{n-1} (-1)^j (\sigma_1,\ldots,b(\sigma_j,\sigma_{j+1}),\ldots,\sigma_n) \\ &\qquad + (-1)^n (\sigma_1,\ldots,\sigma_{n-1}) \end{aligned}$$

We define the *i^{th}-simplicial homology group of R (with respect to the basis $\mathbb{B}$)* as the i^{th}-homology group of the chain complex $(\mathrm{SC}_\bullet(R), d)$, and we denote it by $\mathrm{SH}_i(R)$. Naturally, given an abelian group G, the *ith-simplicial cohomology group* $\mathrm{SH}^i(R,G)$ *of R with coefficients in G* is defined by applying $\mathrm{Hom}_{\mathbb{Z}}(-,G)$ on the chain complex and by taking the ith-cohomology group of the complex obtained.

The reader can easily see that if R is a pullback of the k-algebras projections $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$, as defined is (1.2), with $\mathbb{B}_1$ and $\mathbb{B}_2$ the semi-normed basis of A_1 and A_2, respectively, then $\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2$ and $\mathbb{B}_C = \mathbb{B}_1 \cap \mathbb{B}_2$ are semi-normed basis of R and C respectively. This leads to the following results.

Proposition 3.1.1. *Let R be the pullback of the k-algebra projections $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$ as defined in (1.2). Moreover, assume that $\mathbb{B}_1$ and $\mathbb{B}_2$ are semi-normed basis of A_1 and A_2. Then there exists a long exact sequence of simplicial homology groups:*

$$\cdots \to \mathrm{SH}_n(C) \to \mathrm{SH}_n(A_1) \oplus \mathrm{SH}_n(A_2) \to \mathrm{SH}_n(R) \to \mathrm{SH}_{n-1}(C) \to \cdots$$

Proof : We define

$$\begin{array}{llll} p_i : & \mathrm{SC}_i(A_1) \oplus \mathrm{SC}_i(A_2) & \longrightarrow & \mathrm{SC}_i(R) \\ & ((\beta_1, \ldots, \beta_i), (\gamma_1, \ldots, \gamma_i)) & \longmapsto & (\beta_1, \ldots, \beta_i) - (\gamma_1, \ldots, \gamma_i) \end{array}$$

The map $p.$ is a morphism of complexes. Moreover, let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a generator of $\mathrm{SC}_n(R)$. Since $e_1' R e_2' = 0 = e_2' R e_1'$ by asumption, it is easy to see that, for each j, $\alpha_j \in \mathbb{B}_1$ or $\alpha_j \in \mathbb{B}_2$. Consequently, $\alpha \in \mathrm{SC}_i(A_1)$ or $\alpha \in \mathrm{SC}_i(A_2)$, and hence p_i is surjective for each i.

Moreover, the kernel of p_i is exactly $\mathrm{SC}_i(C)$. We obtain a short exact sequence of complexes $0 \to \mathrm{SC}_\bullet(C) \to \mathrm{SC}_\bullet(A_1) \oplus \mathrm{SC}_\bullet(A_2) \to \mathrm{SC}_\bullet(R) \to 0$, from which we deduce the result. □

Proposition 3.1.2. *Let R be the pullback of the k-algebra projections $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$ as defined in (1.2). Moreover, assume that $\mathbb{B}_1$ and $\mathbb{B}_2$ are semi-normed basis of A_1 and A_2. If G is a divisible abelian group, then there exists a long exact sequence of simplicial cohomology groups :*

$$\cdots \to \mathrm{SH}^n(R, G) \to \mathrm{SH}^n(A_1, G) \oplus \mathrm{SH}^n(A_2, G) \to \mathrm{SH}^n(C, G) \to \mathrm{SH}^{n+1}(R, G) \to \cdots$$

Proof : By the proof of (3.1.1), we have an exact sequence of complexes $(*)$: $0 \to \mathrm{SC}_\bullet(C) \to \mathrm{SC}_\bullet(A_1) \oplus \mathrm{SC}_\bullet(A_2) \to \mathrm{SC}_\bullet(R) \to 0$. Since G is divisible, the functor $\mathrm{Hom}_{\mathbb{Z}}(-, G)$ is exact. Therefore the short sequence of complexes obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(-, G)$ to $(*)$ is exact and the result follows at once. □

3.2. Fundamental group. In this section, we study the fundamental group of a particular class of pullback of algebras, that of articulated algebras, see below for definition.

Let (Q, I) be a bound quiver. Recall from **[31]** that given two vertices $x, y \in Q_0$, a relation $\rho = \sum_{i=1}^n \lambda_i \omega_i \in I \cap kQ(x, y)$ is called *minimal* provided $m \geq 2$ and $\sum_{i \in J} \lambda_i \omega_i \notin I \cap kQ(x, y)$ for every proper subset J of $\{1, 2, \ldots, n\}$. We define the *homotopy relation* in the set of walks on Q as the smallest equivalence relation satisfying:

(1) For each arrow $\alpha \in Q_1$, $\alpha\alpha^{-1} \sim \epsilon_x$ and $\alpha^{-1}\alpha \sim \epsilon_y$;
(2) For each minimal relation $\sum_{i=1}^n \lambda_i \omega_i$, $\omega_i \sim \omega_j$.
(3) If u, v, w, w' are walks such that $w \sim w'$, then $uwv \sim uw'v$, whenever these compositions are defined.

Let $\pi_1(Q, x_0)$ be the fundamental group of the underlying graph of Q at the vertex x_0, that is the free group on $\chi(Q) = |Q_1| - |Q_0| + 1$ generators. Moreover, let $N(Q, I, x_0)$ be the normal subgroup of $\pi_1(Q, x_0)$ generated by the elements of the form $wuv^{-1}w^{-1}$ where w is a path beginning at x_0 and u and v are two homotopic paths. The *fundamental group* of the bound quiver (Q, I) is defined to be $\pi_1(Q, I, x) = \pi_1(Q, x)/N(Q, I, x)$.

Since we assume Q to be connected, this definition does not depend on the choice of x_0. Accordingly, we write $\pi_1(Q, I)$.

On other hand, following **[12]**, to a bound quiver (Q, I), we associate a C.W. complex $\mathcal{B}(Q, I)$ (called its *classifying space*) in the following way: the 0-cells are given by Q_0 and the n-cells are given by n-tuples $(\sigma_1, \ldots, \sigma_n)$ of homotopy classes of non trivial paths of (Q, I) such that the composition $\sigma_1\sigma_2 \cdots \sigma_n$ is a path of (Q, I) not in I. Recall (see **[12]**) that the groups $\pi_1(Q, I)$ and $\pi_1(\mathcal{B}(Q, I))$ are isomorphic. We shall use this fact, together with Van Kampen's theorem, in the proof of Proposition 3.2.3 below.

However, we first recall the following definition from **[18]**.

Definition 3.2.1. Let (Q_1, I_1) and (Q_2, I_2) be two connected bound quivers containing at least one arrow. A bound quiver (Q, I) is called *articulated along* (Q_1, I_1) *and* (Q_2, I_2), or simply *articulated*, if

(1) Q_1 and Q_2 are subquivers of Q_R;
(2) $(Q_R)_0 = (Q_1)_0 \cup (Q_2)_0$ and $(Q_R)_1 = (Q_1)_1 \cup (Q_2)_1$;
(3) $(Q_1)_0 \cap (Q_2)_0 \neq \emptyset$ and for each $x \in (Q_1)_0 \cap (Q_2)_0$, x is either:
  (a) a source of Q_1 and a sink of Q_2 or;
  (b) a source of Q_2 and a sink Q_1;
(4) $I_R = I_1 + I_2 + < \overline{\rho} >$, with $\overline{\rho}$ the set of paths linking $(Q_1)_0 \setminus (Q_2)_0$ and $(Q_2)_0 \setminus (Q_1)_0$ in Q_R.

Under these conditions, the algebra $R = kQ/I$ is called *articulated along* $A_1 = kQ_1/I_1$ *and* $A_2 = kQ_2/I_2$.

In this case R is the pullback of the projections of A_1 and A_2 to the non-empty semisimple intersection $C = A_1 \cap A_2$. Moreover, any minimal relation in I_R is either a minimal relation in I_1 or in I_2 and conversely.

Example 3.2.2.

(1) Let R, A_1, A_2 and C be the algebras of Example 2.2.6. Then R is articulated along A_1 and A_2, with $C = A_1 \cap A_2$.
(2) Let R be the algebra of Example 2.4.2(1). Then R is an articulated algebra along the algebras B_1 and B_2, where B_1 and B_2 are given by the full subcategories generated by the set of vertices $\{1, 2, 3, 4\}$ and $\{2, 3, 4, 5, 6\}$ respectively.
(3) Let R, A_1, A_2 and C be the algebras of Example 2.4.2(2). Then R is articulated along A_1 and A_2, with $C = A_1 \cap A_2$.

Proposition 3.2.3. *Let $R = kQ_R/I_R$ be an articulated algebra along $A_1 = kQ_1/I_1$ and $A_2 = kQ_2/I_2$ such that $|(Q_1)_0 \cap (Q_2)_0| = m$. Then $\pi_1(Q_R, I_R) \cong \pi_1(Q_1, I_1) * \pi_1(Q_2, I_2) * L_{m-1}$, where L_{m-1} is the free group generated in $m-1$ generators.*

Proof : Let $Q_C = Q_1 \cap Q_2$, and n_1, n_2, n_R and n denote the number of vertices of Q_1, Q_2, Q_R and Q_C, respectively. In particular, we have $n_R = n_1 + n_2 - n$. For $i \in \{1, 2\}$, let T_i be a maximal tree in Q_i and set

$$\hat{Q}_i = Q_i \coprod_{Q_C} T_i \text{ and } \hat{I}_i = I_R \cap k\hat{Q}_i.$$

Note that $\hat{I}_i = I_i + X_i$, where X_i are monomial ideals. It is easily seen that $|(\hat{Q}_i)_0| = n_R$, for $i \in \{1, 2\}$. In addition we have:

$$|(\hat{Q}_1)_1| = |(Q_1)_1| + |(T_2)_1| = |(Q_1)_1| + n_2 - 1.$$

and a similar expression for $|(\hat{Q}_2)_1|$.

But $\pi_1(\hat{Q}_1)$ is the free group in $\chi(\hat{Q}_1)$ generators, and $\chi(\hat{Q}_1) = \chi(Q_1) + (n-1)$. Thus, $\pi_1(\hat{Q}_1) \cong \pi_1(Q_1) * L_{n-1}$. It follows from the construction of $\hat{I}_i$ that the homotopy relation on $(\hat{Q}_i, \hat{I}_i)$ coincides with that of (Q_i, I_i) so that :

$$\begin{aligned} \pi_1(\hat{Q}_i, \hat{I}_i) &\cong \pi_1(\hat{Q}_i)/N(\hat{Q}_i, \hat{I}_i) \\ &\cong (\pi_1(Q_i) * L_{n-1})/N(\hat{Q}_i, \hat{I}_i) \\ &\cong \pi_1(Q_i)/N(Q_i, I_i) * L_{n-1} \\ &\cong \pi_1(Q_i, I_i) * L_{n-1}. \end{aligned}$$

Now set $Q = \hat{Q}_1 \cap \hat{Q}_2$, or, equivalently $Q = T_1 \amalg_{Q_C} T_2$ and $I = kQ \cap I_R$, which is a monomial ideal, so that there are no minimal relations in I. Consequently, the fundamental group $\pi_1(Q, I)$ is the free group on $\chi(Q) = n - 1$ generators, and therefore $\pi_1(Q, I) \cong L_{n-1}$.

Keeping in mind the expressions of $\pi_1(\hat{Q}_i, \hat{I}_i)$ and $\pi_1(Q, I)$ given above, one can see that the morphisms $q_i : \pi_1(Q, I) \to \pi_1(\hat{Q}_i, \hat{I}_i)$ given by the rules $q_i(x) = 1 * x$ are injective, having L_{n-1} as image.

The next step is to use Van Kampen's theorem. For this sake, let $\hat{\mathcal{B}_1}$, $\hat{\mathcal{B}_2}$, $\hat{\mathcal{B}_R}$ and $\hat{\mathcal{B}}$ be the classifying spaces of $(\hat{Q}_1, \hat{I}_1)$, $(\hat{Q}_2, \hat{I}_2)$, $(\hat{Q}_R, \hat{I}_R)$, $(\hat{Q}_C, \hat{I}_C)$, respectively.

By construction, $\mathcal{B}_R = \hat{\mathcal{B}_1} \cup \hat{\mathcal{B}_2}$ and the intersection $\hat{\mathcal{B}_1} \cap \hat{\mathcal{B}_2} = \mathcal{B}$ is connected. The result then follows from Van Kampen's theorem and the isomorphism $\pi_1(\mathcal{B}_R) \cong \pi_1(Q_R, I_R)$. □

3.3. Simple connectedness. The result of the preceding section give rise to immediate consequences on simple connectedness and strong simple connectedness of articulated algebras. For further details about (strong) simply connected algebras, we refer the reader to **[7, 5, 2, 1]**. We first recall the relevant definitions.

Definition 3.3.1. A connected and triangular algebra A is called *simply connected* if, for any presentation (Q, I) of A, the group $\pi_1(Q, I)$ is trivial. Moreover, it is called *strongly simply connected* if every full convex subcategory of A is simply connected.

In general, the above condition may be complicated to verify since, even if an algebra A determines a unique ordinary quiver Q, the fundamental groups of two presentations (Q, I_1) and (Q, I_2) of an algebra A can be completely different (see **[13, 6]**, for instance).

Lemma 3.3.2. *Let $R = kQ/I$ be an articulated algebra along $A_1 = kQ_1/I_1$ and $A_2 = kQ_2/I_2$ such that $|(Q_1)_0 \cap (Q_2)_0| = 1$. Then A_1 and A_2 are full and convex subcategories of R.*

Proof : We only proof the statement for A_1, the proof for A_2 being similar. Since I_1 is the restriction of I to Q_1, if A_1 is not a full and convex subcategory of R, then there exists a path $x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_n$ in Q with $x_1, x_n \in (Q_1)_0$ but $f_i \notin (Q_1)_1$ for a certain $i \in \{1, \dots, n\}$. Consequently, $f_i \in (Q_2)_1$ and since R is articulated there exist $x_j, x_k \in (Q_1)_0 \cap (Q_2)_0$, a contradiction. □

Corollary 3.3.3. *Let R be a connected articulated algebra along A_1 and A_2. Let $C = A_1 \cap A_2$. Then*

(1) *R is simply connected if and only if A_1 and A_2 are simply connected and $|C_0| = 1$;*
(2) *R is strongly simply connected if and only if A_1 and A_2 are strongly simply connected and $|C_0| = 1$.*

Proof : 1. **Necessity.** First, if $|C_0| > 1$, then it follows from (3.2.3) that $\pi_1(Q_R, I_R)$ is not trivial, a contradiction, and hence $|C_0| = 1$ since R is connected. On the other hand, let (Q_1, I_1) and (Q_2, I_2) be presentations of A_1 and A_2 respectively. By **[18]**, there exists a presentation (Q, I) of R such that (Q, I) is articulated along (Q_1, I_1) and (Q_2, I_2). It then follows from (3.2.3) that the groups $\pi(Q_1, I_1)$ and $\pi(Q_2, I_2)$ are trivial, and consequently A_1 and A_2 are simply connected.
Sufficiency. Let (Q, I) be a presentation of R. By **[18]**, (Q, I) is articulated along presentations (Q_1, I_1) and (Q_2, I_2) of A_1 and A_2 respectively. Since A_1 and A_2 are simply connected, $\pi_1(Q, I)$ is trivial by (3.2.3) and R is simply connected.

2. **Necessity.** Assume that R is strongly simply connected. In particular, R is simply connected and it follows from the first part that so are A_1 and A_2. Let $i \in \{1, 2\}$ and B be a full convex subcategory of A_i. By (3.3.2), A_i is a full convex subcategory of R, and hence so is B. Since R is strongly simply connected, B is simply connected. Therefore, A_i is strongly simply connected.
Sufficiency. Let R' be a full convex subcategory of R. If R' is subcategory of A_1 or A_2, then R' is simply connected and this finish the proof. Otherwise, since R' is convex,

one has $R'_0 \cap C_0 \neq \emptyset$, and hence $|R'_0 \cap C_0| = 1$. For $i = 1, 2$, let A'_i be the restriction of R' to A_i. In particular, A'_i is a full convex subcategory of A_i and, by construction, R' is articulated along A'_1 and A'_2. Since A'_1 and A'_2 are both simply connected, and $|A'_1 \cap A'_2| = 1$, it follows from the first part that R' is simply connected. Consequently, R is strongly simply connected. $\square$

ACKNOWLEDGEMENTS. This paper was written while the first author visited the Universities of Sherbrooke and Bishop's under a local post-doctoral fellowship, and while the second and the third author respectively had an ISM and NSERC graduate studies fellowship. The authors warmly thank Ibrahim Assem for his precious help, and Guillermo Cortiñas for pertinent questions during the XVI^{to} Coloquio Latinoamericano de Álgebra in Uruguay.

References

[1] I. Assem. Simply connected algebras. *Resenhas IME-USP*, 4(2):93–125, 1999.
[2] I. Assem and J. A. de la Peña. The fundamental groups of a triangular algebra. *Comm. Algebra*, 24(1):187–208, 1996.
[3] I. Assem and M. Lanzilotta. The simple connectedness of a tame weakly shod algebra. *Comm. Algebra*, 32(9):3685–3701, 2004.
[4] I. Assem, E. N. Marcos, and J. A. de la Peña. The Simple Connectedness or a Tame Tilted Algebra. *J. Algebra*, 237(2):647–656, 2001.
[5] I. Assem and A. Skowroński. On some classes of simply connected algebras. *Proc. London Math. Soc.*, 56(3):417–450, 1988.
[6] M. J. Bardzell and E. N. Marcos. $\mathrm{H}^1(\Lambda)$ and presentations of finite dimensional algebras. In *Proc. CRASP*, number 224 in Lecture Notes in Pure and Applied Mathematics, pages 31–38. Marcel Dekker, 2001.
[7] R. Bautista, F. Larrión, and L. Salmerón. On Simply Connected Algebras. *J. London Math. Soc. (2)*, 27(2):212–220, 1983.
[8] B. Bendiffalah and D. Guin. Cohomologie de l'algèbre triangulaire et applications. *J. Algebra*, 282(2):513–537, 2004.
[9] K. Bongartz and P. Gabriel. Covering spaces in representation theory. *Invent. Math*, 65(3):331–378, 1981-1982.
[10] O. Bretscher and P. Gabriel. The standard form of a representation-finite algebra. *Bull. Soc. Math. France*, 111:21–40, 1983.
[11] R. O. Buchweitz and S. Liu. Hochschild cohomology and representation-finite algebras. *Proc. London Math. Soc.(3)*, 88(2):355–380, 2004.
[12] J. C. Bustamante. The classifying space of a bound quiver. *J. Algebra*, 277(2):431–455, 2004.
[13] J. C. Bustamante and D. Castonguay. Fundamental groups and presentations of algebras. *arXiv:math/0405127*, preprint.
[14] H. Cartan and S. Eilenberg. *Homological Algebra.* Number 19 in Princeton mathematical series. Princeton University Press, Princeton, 1956.
[15] C. Cibils. Cohomology of incidence algebras and simplicial complexes. *J. Pure Appl. Algebra*, 56:221–232, 1989.
[16] C. Cibils. Hochschild cohomology algebra of radical square zero algebras. In *Algebras and Modules II*, number 24 in CMS Conf. Proc., pages 93–101, Providence, RI, 1998. Amer. Math. Soc.
[17] C. Cibils. Tensor Hochschild homology and cohomology. In *Interactions between ring theory and representations of algebras (Murcia)*, volume 210 of *Lecture Notes in Pure and Appl. Math.*, pages 35–51, New-York, 2000. Dekker.
[18] J. Dionne and D. Smith. Articulations of algebras and their homological properties. J. Algebra Appl., to appear, 2005.
[19] M. Gerstenhaber. On the deformations of rings and algebras. *Ann. Math. Stud.*, 79(2):59–103, 1964.
[20] M. Gerstenhaber. The Cohomology structure on an associative ring. *Ann. of Math.*, 78(2):267–288, 1978.
[21] M. Gerstenhaber and S. D. Schack. Simplicial cohomology is Hochschild cohomology. *J. Pure Appl. Algebra*, 30:143–156, 1983.

[22] E. Green and Ø. Solberg. Hochschild cohomology rings and triangular rings, Representations of algebra. *Beijing Norm. Univ. Press*, I:192–200, 2002.

[23] E. L. Green, E. N. Marcos, and N. Snashall. The Hochschild cohomology ring of a one point extension. *Comm. Algebra*, 31(1):357–379, 2003.

[24] D. Happel. *Hochschild cohomology of finite-dimensional algebras*, pages 108–126. Number 1404 in Lecture Notes in Mathematics. Springer-Verlag, Berlin Heidelberg New York Tokyo, 1989.

[25] K. Igusa, M. I. Platzeck, G. Todorov, and D. Zacharia. Auslander algebras of finite representation type. *Comm. Algebra*, 13(1-2):377–424, 1987.

[26] S. Jagadeeshan and M. Kleiner. Projectively stable Artin algebras : representation theory and quivers with relations. *Comm. Algebra*, 27(5):2277–2316, 1999.

[27] S. Liu and J. P. Morin. The strong no loop conjecture for special biserial algebras. *Proc. Amer. Math. Soc.*, 132(12):3513–3523, 2004.

[28] A. C. Locateli. Hochschild cohomology of truncated quiver algebras. *Comm. Algebra*, 27(2):645–664, 1999.

[29] J. Lévesque. Nakayama oriented pullbacks and stably hereditary algebras. *Preprint*, 2004.

[30] Ma. I. R. Martins and J. A. de la Peña. Comparing the simplicial and the Hochschild cohomologies of a finite dimensional algebra. *J. Pure Appl. Algebra*, 138(1):45–58, 1999.

[31] R. Martínez-Villa and J. A. de la Peña. The universal cover of a quiver with relations. *J. Pure Appl. Algebra*, 30(3):873–887, 1983.

[32] S. Michelena and M. I. Platzeck. Hochschild cohomology of triangular matrix algebras. *J. Algebra*, 233(2):502–525, 2000.

[33] J. A. de la Peña and C. C. Xi. Hochschild cohomology of algebras with homological ideals. Tsukuba J. Math., to appear.

J. C. Bustamante; Departamento de Matemáticas, Universidad San Francisco de Quito, Quito, Ecuador.
E-mail address: *juan.bustamante@usfq.edu.ec*

J. Dionne; Département de Mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada.
E-mail address: *julie.dionne2@usherbrooke.ca*

D. Smith; Département de Mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada.
E-mail address: *david.smith@usherbrooke.ca*