A note on the non-stochastic ordering of some quadratic forms [1]

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Summary

For $Y = \|aZ + \theta\|^2$, $a > 0$, $Z \sim N_p(\theta, I_p)$, $\theta \neq \{0\}$, we show that the distribution of Y is not stochastically ordered in $a > 0$. We provide extensions to spherically symmetric, elliptically symmetric, and skew-normal distributions, as well as to other quadratic forms.

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1. Introduction

This note studies stochastic ordering of $Y = \|aZ + \theta\|^2$, in $a > 0$ for fixed $\theta \in \mathbb{R}^p$, for $Z = (Z_1, \ldots, Z_p)^\top \sim N_p(0, I_p)$ and more generally when Z has a spherically symmetric distribution with density $g(\|z\|^2)$. We show that for all $p \geq 1$, $\theta \neq 0$, the distribution of Y is **not** stochastically ordered in $a > 0$ for the normal case, as well as in general as long as $t^{p/2}\, g(t)$ is decreasing in $t > t_0$ for some t_0. Of course for $\theta = 0$, Y is distributed as a scale family on $\mathbb{R}_+$ with scale parameter a^2 and thus has strict monotone increasing likelihood ratio in a. The interesting issue arose in recent work of ours (i.e., Marchand & Strawderman, 2020) pertaining to a minimax shrinkage estimation problem under balanced loss and a multivariate normal distributed Z. The negative result necessitated alternative approaches to show minimaxity, but its implications go well beyond the specifics of the above-mentioned paper.

The negative result was somewhat surprising to us in that $\mathbb{E}[Y] = a^2\, trCov(Z) + \|\theta\|^2$, whenever it exists, and is thus clearly increasing in $a \in (0, \infty)$. We are unaware of a demonstration of the non-stochastic monotonicity (and hence non-monotone likelihood ratio) of the family of distributions. It thus seems worthwhile to record the result, as it relates to the ubiquitous non-central chi-square distribution, as well as so-called generalized noncentral chi-square distributions which arise as quadratic forms for spherically symmetric observables (e.g., Cacoullos & Koutras, 1984; Fan, 1990; Hsu, 1990). Moreover, we also expand on an analogous result for elliptical symmetric densities $Z \sim g(z^\top \Sigma^{-1} z)$, as well as for quadratic forms $(aZ + \theta)^\top Q (aZ + \theta)$ with positive definite Q. Finally, we extend the non-stochastic property to multivariate skew-normal distributions (e.g.,

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Azzalini & Dalla-Valle, 1996). The ensemble capitalizes on various properties of such multivariate distributions and represents different interesting approaches for establishing a non-stochastic ordering.

2. The main result

The notion of stochastic ordering, as well as related orderings between probability distributions, has a rich history (e.g., Marshall & Olkin, 2007) and may be defined as follows.

Definition 2.1. *A family of cumulative distributions functions F_γ on the real line, $\gamma \in \Gamma \subset \mathbb{R}^1$, is said to be stochastically ordered if $\gamma < \gamma'$ implies either: (i) $F_\gamma(z) \geq F_{\gamma'}(z)$ for all z (increasing), or (ii) $F_\gamma(z) \leq F_{\gamma'}(z)$ for all z (decreasing).*

We first observe that the family of distributions of $Y = \|aZ+\theta\|^2$ cannot be stochastically decreasing, and that it will thus be only necessary to focus on the (non) stochastically increasing property. Indeed, if $Cov(Z)$ exists, as already pointed out, the expectation $E_a(Y)$ is increasing in $a > 0$, which is inconsistent with a stochastically decreasing property. More generally when $Cov(Z)$ does not exist, take $a_1 = \|\theta\|$, $a_2 = k\|\theta\|$, $Y_1 = \|a_1 Z + \theta\|^2$, and $Y_2 = \|a_2 Z + \theta\|^2 = k^2 \|\theta\|^2 \left(\|Z + \frac{\theta}{k\|\theta\|}\|^2\right)$. Then for $z > 0$, observe that, while $\mathbb{P}(Y_1 \leq z)$ does not depend on k, $\mathbb{P}(Y_2 \leq z)$ does depend on k and can be seen to converge to 0 as $k \to \infty$. By choosing z such that $\mathbb{P}(Y_1 \leq z) > 0$, and then large enough k, we obtain $\mathbb{P}(Y_1 \leq z) > \mathbb{P}(Y_2 \leq z) > 0$ so that the ordering here is indeed inconsistent with a stochastically decreasing property.

We establish a non stochastically ordered finding first in the multivariate normal case. Since $Y = a^2\|Z+\frac{\theta}{a}\|^2$ has the distribution of a scaled (by a^2) chi-square distribution with p degrees of freedom, and non-centrality parameter $\frac{\|\theta\|^2}{a^2}$, the usual Poisson and mixture representation of a non-central chi-square gives the density of Y as

$$f_a(y) = \frac{1}{2a^2} e^{-\|\theta\|^2/2a^2} \sum_{k=0}^{\infty} \frac{(\|\theta\|^2/2a^2)^k \, (y/2a^2)^{\frac{p+2k}{2}-1}}{k! \, \Gamma(\frac{p+2k}{2})} \, e^{-y/2a^2} \,. \tag{2.1}$$

Here is the main result.

Theorem 2.1. *For any $p \geq 1$ and fixed $\theta \neq 0$, the family of distributions of Y with densities in (2.1) is not stochastically ordered in $a > 0$. (Hence, there exist monotone non-decreasing functions $g(Y)$ such that $\mathbb{E}_a g(Y)$ is not monotone.)*

Proof. Fix p and $\theta \neq 0$. It suffices to show that there exists $a_2 > a_1$ such that $f_{a_2}(y) > f_{a_1}(y)$ for $y \in (0, y_0)$ since this implies that $F_{a_2}(y) > F_{a_1}(y)$ in the same range. To this end note that, for fixed $a_2 > a_1 > 0$,

$$\lim_{y \to 0^+} \frac{f_{a_2}(y)}{f_{a_1}(y)} = (\frac{a_1^2}{a_2^2})^{p/2} \, e^{-\frac{\|\theta\|^2}{2}(\frac{1}{a_2^2} - \frac{1}{a_1^2})} \,. \tag{2.2}$$

For $a_1^2 = A$ and $a_2^2 = A + \epsilon$ with $\epsilon > 0$, the r.h.s. of side of (2.2) is given by

$$H(A, \epsilon, \|\theta\|^2) = (\frac{A}{A+\epsilon})^{p/2} \, e^{\frac{\|\theta\|^2}{2}(\frac{\epsilon}{A(A+\epsilon)})} \,. \tag{2.3}$$

Then, with the inequality $\log(1+u) < u$ for $u > 0$, it follows that

$$\begin{aligned}
\log H(A, \epsilon, \|\theta\|^2) &= -\frac{p}{2}\log(1+\frac{\epsilon}{A}) + \frac{\|\theta\|^2}{2}(\frac{\epsilon}{A(A+\epsilon)}) \\
&> -\frac{p\,\epsilon}{2A} + \frac{\|\theta\|^2}{2}(\frac{\epsilon}{A(A+\epsilon)}) \\
&= -\frac{p\,\epsilon}{2A}\left(1 - \frac{\|\theta\|^2}{p(A+\epsilon)}\right) \\
&> 0\,,
\end{aligned}$$

$$\text{provided} \quad A + \epsilon < \frac{\|\theta\|^2}{p}\,. \tag{2.4}$$

Hence, we have that for any $\|\theta\|^2 > 0$, there are values of $A > 0$ and $\epsilon > 0$ such that (2.3) takes positive values. This implies that (2.2) takes on values greater than 1 and hence there exists $a_2 > a_1 > 0$ and $y_0 > 0$ such that $f_{a_2}(y) > f_{a_1}(y)$ for $y \in (0, y_0)$. This completes the proof. □

As noted previously, when $\theta = 0$, the family of distributions of Y has increasing monotone likelihood ratio in Y with parameter a, and is thus stochastically ordered. It is interesting to note that for $\theta \neq 0$ there are opposing forces influencing the ordering. In fact, $\|Z + \frac{\theta}{a}\|^2$ is stochastically decreasing in a, while for any positive random variable X, the distribution of aX is stochastically increasing in a, $a > 0$. Thus $Y = a\||Z + \frac{\theta}{a}\|^2$ is being drawn in each direction.

In the application of interest to us, we wished to show that $\mathbb{E}_a(\frac{1}{Y}) = \mathbb{E}(\frac{1}{\|aZ+\theta\|^2})$ is non-increasing in $a \in (0, \infty)$ (whenever it exists). Clearly by Jensen's inequality,

$$\mathbb{E}(\frac{1}{\|aZ+\theta\|^2}) \geq \frac{1}{\mathbb{E}(\,\|aZ+\theta\|^2\,)} = \frac{1}{a^2 trCov(Z) + \|\theta\|^2}\,,$$

which is decreasing in a and suggests the non-increasing property. Theorem 2.1 tells us that we cannot rely on a stochastic ordering for the distribution of Y. However, using the Poisson representation of the non-central χ^2 distribution of $\|Z + \frac{\theta}{a}\|^2$, we have for $p \geq 3$ and $\theta \neq 0$

$$\begin{aligned}
\mathbb{E}_a(\frac{1}{Y}) &= \frac{1}{a^2}\,\mathbb{E}\frac{1}{\|Z+\frac{\theta}{a}\|^2} \\
&= \frac{1}{a^2}\,e^{-\|\theta\|^2/2a^2}\sum_{k\geq 0}\frac{(\|\theta\|^2/2a^2)^k}{k!}\;\mathbb{E}[\frac{1}{\chi^2_{p+2k}}] \\
&= \frac{1}{a^2}\,e^{-\|\theta\|^2/2a^2}\sum_{k\geq 0}\frac{(\|\theta\|^2/2a^2)^k}{k!}\;\frac{1}{p+2k-2} \\
&= \frac{2}{\|\theta\|^2}\,e^{-\|\theta\|^2/2a^2}\sum_{k\geq 0}\frac{(\|\theta\|^2/2a^2)^{k+1}}{(k+1)!}\;\frac{k+1}{p+2k-2} \\
&= \frac{2}{\|\theta\|^2}\,e^{-\|\theta\|^2/2a^2}\sum_{k\geq 0}\frac{(\|\theta\|^2/2a^2)^k}{k!}\,U(k) \\
&= \frac{2}{\|\theta\|^2}\,\mathbb{E}_{\frac{\|\theta\|^2}{2a^2}}\,U(K)\,,
\end{aligned} \tag{2.5}$$

where $K \sim$ Poisson$(\lambda = \|\theta\|^2/2a^2)$ and $U(K) = \frac{K}{p+2K-4}\,\mathbb{I}_{\mathbb{N}_+}(K)$. Since $U(k)$ is increasing in $k \in \mathbb{N}$ for $p \geq 3$, since λ is decreasing in a, and since the Poisson(λ) distribution has increasing monotone likelihood ratio in K with parameter λ, it follows from the above that $\mathbb{E}_a(\frac{1}{Y})$ is decreasing indeed in a for $p \geq 3$.

Remark 2.1. *Such inverse moments have arisen on several occasions, in particular in the frequentist risk evaluation of shrinkage estimators of a multivariate normal mean. For $p = 4$, expression (2.5) simplifies with $U(K) = \frac{1}{2}\,\mathbb{I}_{\mathbb{N}_+}(K)$ and $\mathbb{E}(Y^{-1}) = \frac{1}{\|\theta\|^2}\mathbb{P}(K \geq 1) = \frac{1}{\|\theta\|^2}\left(1 - e^{-\|\theta\|^2/2a^2}\right)$, a result given for instance by Egerton and Laycock (1982).*

We conclude this section with an illustration.

Example 2.1. *We illustrate Theorem 2.1's negative result for $Z \sim N_4(0, I_4)$, $Y = \|aZ + \theta\|^2$ with $\|\theta\|^2 = 5$. As mentioned in the Introduction, we have here $\mathbb{E}_a(Y) = 4a^2 + 5$ which clearly yields an expected value ordering. However, it follows from Theorem 2.1 that the cumulative distribution function $F_a(y)$ is for some small enough positive y not monotone in $a > 0$ indicating the lack of a stochastic ordering. Moreover, it must be the case that $F_{a_1}(y)$ and $F_{a_2}(y)$ cross for some $y = y_0 \in \mathbb{R}_+$ as long as $a_2^2 \leq 5/4$ which comes from (2.4). The c.d.f.'s F_{a_1} and F_{a_2} for $a_1 = 1/2$ and $a_2 = 1$ are graphed in Figure 1, illustrating indeed that $F_1(y) \geq F_{1/2}(y)$ for $y \leq y_0 \approx 3.384436$.*

3. An extension to the spherically symmetric case

We investigate a possible extension to the spherically symmetric case. We assume that Z has Lebesgue density $g(\|z\|^2)$ with continuous g and establish that the distribution of $Y = \|aZ + \theta\|^2$ is not stochastically ordered in $a \in (0, \infty)$, for $\theta \neq 0$, under the weak condition that $t^{p/2}\,g(t)$ is decreasing in $t > t_0$, for some $t_0 > 0$. We make use of a known expression for the distribution $\|Z + \mu\|^2$, often referred to as a generalized chi-square distribution.

Lemma 3.1. *(Hsu, 1990) Let X have Lebesgue density $g(\|x - \mu\|^2)$, $x, \mu \in \mathbb{R}^p$, and $p > 1$. Let $h(v) = \frac{\Gamma(p/2)2^{p/2}}{v^{p/2-1}e^{-v/2}}\,g(v)$, $d = t + \|\mu\|^2$, $c = 2\sqrt{t}\|\mu\|$ for $t > 0$. Then, $\|X\|^2$ has density*

$$f_\mu(t) \;=\; \frac{t^{p/2-1}\,e^{-d/2}}{\Gamma(\frac{p-1}{2})\,\Gamma(\frac{1}{2})\,2^{p/2}}\;\; I^*_\mu(t)\,, \tag{3.6}$$

$$\text{with} \;\; I^*_\mu(t) \;=\; \int_0^1 (1-w^2)^{(p-3)/2}\left(e^{-\frac{cw}{2}}\,h(cw+d) \;+\; (e^{\frac{cw}{2}}\,h(-cw+d)\right)\,dw\,.$$

We now have the following extension of Theorem 2.1.

Theorem 3.2. *Assume $Z \in \mathbb{R}^p$, $p \geq 1$, has Lebesgue density $g(\|z\|^2)$ with continuous g such that*

$$t^{p/2}\,g(t) \text{ is decreasing in } t > t_0,\; \text{for some } t_0 > 0. \tag{3.7}$$

Then, for fixed $\theta \neq 0$, the family of distributions of $Y = \|aZ + \theta\|^2$ is not stochastically ordered in $a > 0$.

Proof. Proceed as in Theorem 2.1, focussing on the densities $f_{Y,a}$ of Y, with the aim of showing that there exists $a_2 > a_1 > 0$, $y_0 > 0$, such that $f_{Y,a_2}(y) > f_{Y,a_1}(y)$ for $y \in (0, y_0)$. To this end, it will suffice to establish that

$$L \;=\; \lim_{y \to 0^+} \frac{f_{Y,a_2}(y)}{f_{Y,a_1}(y)} \;>\; 1 \;\text{ for some } a_2 > a_1 > 0\,. \tag{3.8}$$

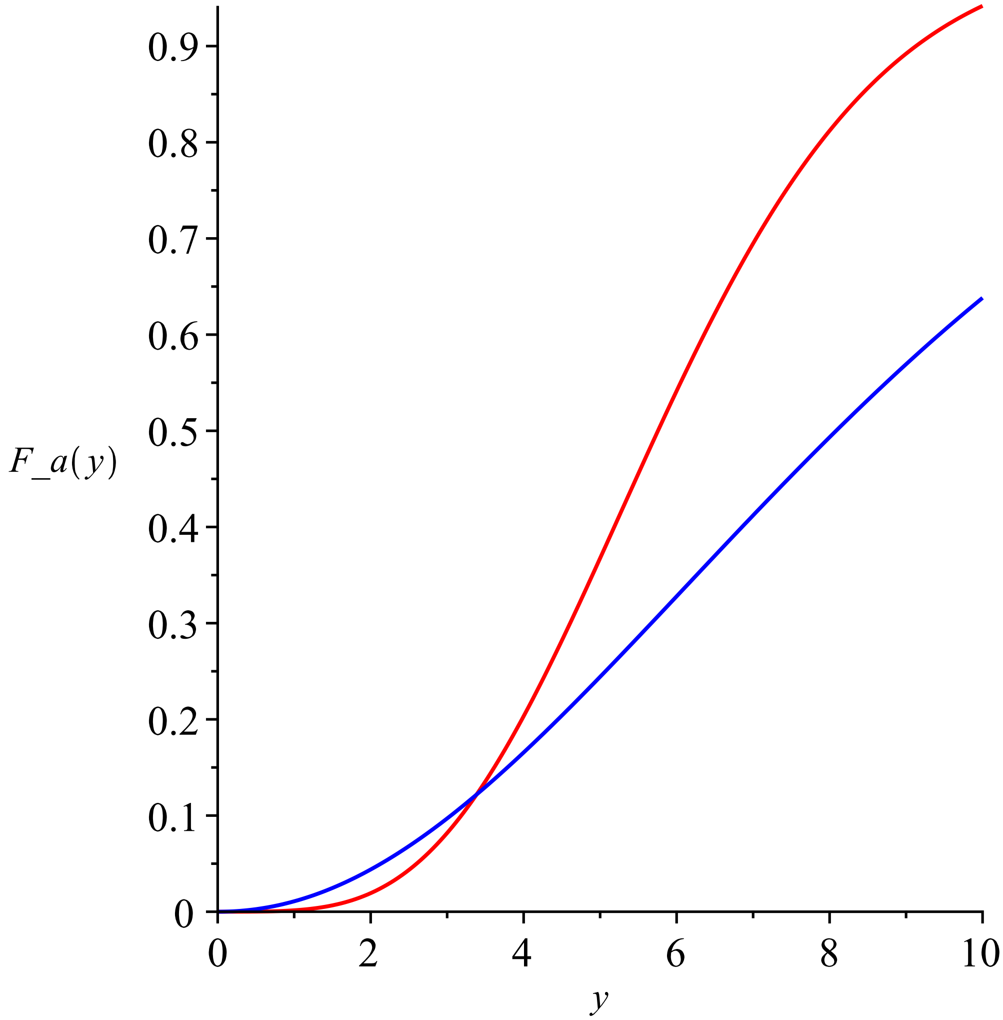

Figure 1: Cumulative distributions functions F_a of $Y = \|aZ + \theta\|^2$ for $p = 4, \|\theta\|^2 = 5$, $a_1 = 1/2$ (red), $a_2 = 1$ (blue)

Since $Y = a^2 \, \|Z + \frac{\theta}{a}\|^2$, it follows that

$$f_{Y,a}(y) \;=\; \frac{1}{a^2}\, f_{\frac{\theta}{a}}(\frac{y}{a^2})\,,$$

with f_μ the density of $\|Z + \mu\|^2 =^d \|X\|^2$ given in Lemma 3.1. From (3.6) for $p > 1$, we obtain for $y > 0$

$$\frac{f_{Y,a_2}(y)}{f_{Y,a_1}(y)} \;=\; \left(\frac{a_1^2}{a_2^2}\right) \frac{f_{\frac{\theta}{a_2}}(\frac{y}{a_2^2})}{f_{\frac{\theta}{a_1}}(\frac{y}{a_1^2})} \;=\; \left(\frac{a_1^2}{a_2^2}\right)^{p/2} \, e^{-\frac{\|\theta\|^2}{2}(\frac{1}{a_2^2}-\frac{1}{a_1^2})} \, \frac{I^*_{\frac{\theta}{a_2}}(\frac{y}{a_2^2})}{I^*_{\frac{\theta}{a_2}}(\frac{y}{a_1^2})}\,.$$

From this, with $I^*_\mu(0) \;= 2\, h(\|\mu\|^2)\,)\int_0^1 (1-w^2)^{p-3/2}\, dw$, we obtain

$$L \;=\; \left(\frac{a_1^2}{a_2^2}\right)^{p/2} \, \frac{g(\frac{\|\theta\|^2}{a_2^2})}{g(\frac{\|\theta\|^2}{a_1^2})}\,. \tag{3.9}$$

Setting $t_i = \|\theta\|^2/a_i\,, i = 1,2$ and choosing a_1 and a_2 so that $t_0 < t_2 < t_1 < \infty$, we have by assumption

$$1 < \frac{t_2^{p/2}\, g(t_2)}{t_1^{p/2}\, g(t_1)} = L\,.$$

Hence (3.8) holds and the result is established for $p > 1$. Finally, for $p = 1$, we can evaluate directly the density of Y. This yields the ratio (3.9) as well and completes the proof. □

We now provide several examples where Theorem 3.2's condition on g, which is weak, is satisfied and thus applies.

Example 3.2. *(Generalized multivariate Student distribution)*

In the context of Theorem 3.2, consider

$$Z \sim g(\|z\|^2) \propto (A + \|z\|^2)^{\frac{p+m}{2}}\,, z \in \mathbb{R}^p\,,$$

with $A, m > 0$. Such densities are multivariate Student with m degrees of freedom and scale parameter $\sqrt{A/m}$. From this, we obtain

$$\frac{d}{dt}\{\log(t^{p/2}\, g(t))\} = \frac{p}{2t} - \frac{p+m}{2(A+t)}.$$

Hence $t^{p/2}\, g(t)$ is decreasing in t for $t > t_0 = pA/m$, and the non-stochastic ordering of $Y = \|aZ + \theta\|^2$ in a, for $\theta \neq 0$, follows from Theorem 3.2.

Example 3.3. *In the context of Theorem 3.2, consider*

$$Z \sim g(\|z\|^2) \propto \|z\|^{2A}\, e^{-B(\|z\|^{2\alpha})}\,, z \in \mathbb{R}^p\,, \tag{3.10}$$

with $A > -p/2, \alpha > 0, B > 0$. These densities form a large subclass of spherically symmetric densities, with the particular case $A = 0$ reducing to exponential power distributions (e.g., West, 1987), and the particular case $\alpha = 1, A \neq 0$ corresponding to the Kotz distribution. As in the previous example, it is easy to verify that condition (3.7) holds for $t_0 = (\frac{A+p/2}{B\alpha})^{1/\alpha}$. We thus infer, for the densities in (3.10), the non-stochastic ordering of $Y = \|aZ + \theta\|^2$ in a, for $\theta \neq 0$.

The last examples concern scale mixtures of normal distributions. These include Student, Logistic, Laplace densities, as well as some of the densities in Example 3.3, but also many more such as finite mixtures.

Example 3.4. *In the context of Theorem 3.2, i.e., $Z \sim g(\|z\|^2)$, consider:* **(I)** *finite scale mixtures of normal densities with*

$$g(t) = \sum_{i=1}^{n} \alpha_i (v_i/2\pi)^{p/2}\, e^{-v_i t/2}\,, \tag{3.11}$$

such that $\alpha_i > 0$ for $i = 1, \ldots, n$, and $\sum_i \alpha_i = 1$; and **(II)** *continuous mixing with*

$$g(t) \propto \int_0^\infty v^{p/2}\, e^{-vt}\, h(v)\, dv, \tag{3.12}$$

for $h(\cdot)$ a probability density on $\mathbb{R}_+$ such that $\lim_{v\to 0^+} h(v)/v^\alpha = c > 0$ for some $\alpha > -1$. In both cases, we show that condition (3.7) holds.

(I) *Focussing on the elements of the sum in (3.11), observe that* $t^{p/2}\, e^{-v_i t/2}$ *is decreasing in* t *for* $t > p/(2v_i)$. *Therefore, condition (3.7) holds for* $t_0 = \frac{p}{2} \max\{\frac{1}{v_1}, \cdots, \frac{1}{v_n}\}$, *and Theorem 3.2 tells us that the family of distributions of* $Y = \|aZ + \theta\|^2$ *is not stochastically ordered for* $\theta \neq 0$.

(II) *Here, we have*

$$\frac{d}{dt}\{t^{p/2}\, g(t)\} \;=\; \int_0^\infty (\frac{p}{2t} - v)\,(vt)^{p/2}\, e^{-vt}\, h(v)\, dv\,. \tag{3.13}$$

By an Abelian theorem (e.g., Theorem 3.1.4. of Fourdrinier et al., 2018), we have for $A > 0$:

$$\lim_{t\to\infty} \frac{\int_0^\infty v^A\, e^{-vt}\, h(v)\, dv}{t^{-\,(A+\alpha+1)}\,\Gamma(A+\alpha+1)} \;=\; 1\,. \tag{3.14}$$

Now, from (3.13) and (3.14), we have

$$\begin{aligned}\lim_{t\to\infty} t^{\alpha+2}\,\frac{d}{dt}\{t^{p/2}\, g(t)\} \;&=\; \lim_{t\to\infty} \frac{\int_0^\infty (\frac{p}{2t} - v)\,(v)^{p/2}\, e^{-vt}\, h(v)\, dv}{t^{-\,(p/2+\alpha+2)}} \\ &=\; \frac{p}{2}\,\Gamma(\frac{p}{2}+\alpha+1) \;-\; \Gamma(\frac{p}{2}+\alpha+2) \\ &=\; -(\alpha+1)\,\Gamma(\frac{p}{2}+\alpha+1) \;<\; 0\,.\end{aligned}$$

Thus, there exists t_0 *such that* $\frac{d}{dt}\{t^{p/2}\, g(t)\}$ *is negative for* $t > t_0$ *and Theorem 3.2 applies as in the examples above.*

4. An extension to the elliptically symmetric case

We extend here the results of the previous section to the elliptically symmetric case, with

$$Z \sim g(z^\top \Sigma^{-1} z) \tag{4.15}$$

where g is continuous, and to more general quadratic forms,

$$Y = (aZ+\theta)^\top Q\,(aZ+\theta)\,, \tag{4.16}$$

where Q $(p\times p)$ is symmetric and positive definite. Under the same condition (3.7), i.e., $t^{p/2}g(t)$ is decreasing in $t > t_0$, for some $t_0 > 0$, we show that the family of distributions for Y is not stochastically ordered in a whenever $\theta \neq 0$.

Theorem 4.3. *For* Z *and* Y *as in (4.15) and (4.16), under condition (3.7), the family of distributions for* Y *is not stochastically ordered in* a *whenever* $\theta \neq 0$.

Proof. It suffices to consider the case $Q = I_p$ as, otherwise, the standard decomposition $Q = C^\top C$, with C $(p\times p)$ symmetric and positive definite, permits us to write $Y = (aZ_* + \theta_*)^\top (aZ_* + \theta_*)$ with $Z_* = CZ$, $\theta_* = C\theta$, and with Z_* having a density proportional to (4.15) with shape matrix given by $\Sigma_* = C\Sigma C^\top$.

First, note that for $y > 0$,

$$\mathbb{P}(Y \le y) = \mathbb{P}(\|Z + \frac{\theta}{a}\|^2 \le \frac{y}{a^2}) = \mathbb{P}(Z \in B_{-\frac{\theta}{a},\sqrt{\frac{y}{a^2}}}), \quad (4.17)$$

where $B_{\gamma,r}$ is the ball in $\mathbb{R}^p$ centered at γ and of radius r. Since $g(\cdot)$ is continuous, we have denoting $\mu(B_{\gamma,r})$ as the ball's volume

$$\lim_{r\to 0^+} \frac{\mathbb{P}(Z \in B_{\gamma,r})}{g(\gamma^\top \Sigma^{-1}\gamma)\, \mu(B_{\gamma,r})} = 1. \quad (4.18)$$

With $\mu(B_{\gamma,r}) = r^p \, \mu(B_{0,1})$, it follows from (4.17) and (4.18) that for fixed $0 < a_1 < a_2$ and $\theta \neq 0$:

$$\lim_{y\to 0^+} \frac{\mathbb{P}(\|a_2 Z + \theta\|^2 \le y)}{\mathbb{P}(\|a_1 Z + \theta\|^2 \le y)} = \frac{g(\theta^\top \Sigma^{-1}\theta/a_2^2)}{g(\theta^\top \Sigma^{-1}\theta/a_1^2)} \, (\frac{a_1^2}{a_2^2})^{p/2}. \quad (4.19)$$

Now, set M as the value in (4.19) and let $t_i = \frac{\theta^\top\Sigma^{-1}\theta}{a_i^2}$, $i = 1, 2$. To complete the proof, it suffices to specify a_1 and a_2 such that M is greater than 1. But, given condition (3.7), we have

$$M = \frac{t_2^{p/2}\, g(t_2)}{t_1^{p/2}\, g(t_1)} > 1,$$

provided $t_0 < t_2 < t_1$. Finally, small enough choices of a_1 and a_2 such that $\frac{\theta^\top\Sigma^{-1}\theta}{t_0} > a_2^2 > a_1^2 > 0$ implies precisely the desired ordering $t_0 < t_2 < t_1$. The proof is thus complete.

□

5. An extension to skew-symmetric distributions

As we have seen above, the non-stochasticity ordering property holds for a wide range of multivariate densities which exhibit either spherical or elliptical symmetry. It is naturally of interest to study asymmetric models, and multivariate skew-normal densities (e.g., Azzalini & Dalla-Valle, 1996), as well as skew-symmetric and skew-elliptical extensions (e.g., Azzalini, 2014) come to mind. In this regard, the non-stochastic ordering properties above carry-over and we provide in this section an analogous result for multivariate skew-normal models

$$Z \sim g(z) = 2\phi(z)\, \Phi(\alpha^\top z), \; z \in \mathbb{R}^p, \quad (5.20)$$

and the quadratic form

$$Y = \|aZ + \theta\|^2. \quad (5.21)$$

Here $\alpha \in \mathbb{R}^p$ is a shape parameter while the quantities $a \in \mathbb{R}_+$ and $\theta \in \mathbb{R}^p$ play the same role as above. The distribution of such quadratic forms have been studied by several researchers (see for instance Wang et al., 2009, and references therein). For $a = 1$, the distribution of Y is referred to as a noncentral skew chi-square distribution and $Y \sim \chi_p^2(0)$ independently of α whenever $a = 1$ and $\theta = 0$. As mentioned in the Introduction, the family of distributions of Y is stochastically increasing in $a > 0$. For $\theta \neq 0$, this property does not hold as addressed by the next result.

Theorem 5.4. *For Z and Y as in (5.20) and (5.21), under condition (3.7), the family of distributions for Y is not stochastically ordered in a whenever $\theta \neq 0$.*

Proof. As in the proof of Theorem 4.3, we have for $0 < a_1 < a_2$:

$$M \;=\; \lim_{y \to 0^+} \frac{\mathbb{P}(\|a_2 Z + \theta\|^2 \leq y)}{\mathbb{P}(\|a_1 Z + \theta\|^2 \leq y)} \;=\; \frac{\phi(-\frac{\theta}{a_2})\,\Phi(-\frac{\alpha^\top \theta}{a_2})}{\phi(-\frac{\theta}{a_1})\,\Phi(-\frac{\alpha^\top \theta}{a_1})} \;(\frac{a_1^2}{a_2^2})^{p/2}\,,$$

and it will suffice to show that $M > 1$ for some $0 < a_1 < a_2$. As in the proof of Theorem 2.1, set $a_1^2 = A, a_2^2 = A + \epsilon$ with $\epsilon > 0$, and $M = H(A, \epsilon, \|\theta\|^2)$. We have from above:

$$\log H(A, \epsilon, \|\theta\|^2) \;>\; -\frac{p\,\epsilon}{2A}\left(1 \,-\, \frac{\|\theta\|^2}{p(A+\epsilon)}\right) \,+\, \log \frac{\Phi(-\frac{\alpha^\top \theta}{\sqrt{A+\epsilon}})}{\Phi(-\frac{\alpha^\top \theta}{\sqrt{A}})}\,. \tag{5.22}$$

To pursue, consider the separate cases: (i) $\alpha^\top \theta \geq 0$ and (ii) $\alpha^\top \theta < 0$. In (i), we have indeed $\log H(A, \epsilon, \|\theta\|^2) \;>\; 0$ for $A + \epsilon \leq \frac{\|\theta\|^2}{p}$. Finally in (ii), take $A = \epsilon$ and observe that

$$\lim_{\epsilon \to 0} \log \frac{\Phi(-\frac{\alpha^\top \theta}{\sqrt{2\epsilon}})}{\Phi(-\frac{\alpha^\top \theta}{\sqrt{\epsilon}})} \;=\; 0\,.$$

Combined with (5.22), we obtain that $\log H(\epsilon, \epsilon, \|\theta\|^2) > 0$ for small enough $\epsilon > 0$, thus completing the proof. □

Concluding remarks

For $Z \sim N_p(0, I_p)$ and the quadratic form $Y = \|aZ + \theta\|^2$ with $\theta \neq 0$ and $a > 0$, we have shown that the distribution of Y is not stochastically ordered in $a > 0$ in contrast with the monotonone increasing property of $\mathbb{E}_a(Y)$, and of $\mathbb{E}_a(\frac{1}{Y})$ for $p \geq 3$. Moreover, the lack of a stochastic ordering persists for more general quadratic forms $(aZ + \theta)^\top Q(aZ + \theta)$, with Q positive definite, and for a wide range of other distributions for Z including spherically symmetric, elliptically symmetric, and multivariate skew-normal distributions.

With the importance of quadratic forms in statistical distributional theory and inference, we feel that the results and techniques presented here are of further potential use and may well lead to further implications. For instance, it would be interesting to assess whether the absence of a stochastic ordering persists or vanishes for subsets of values $a \geq a_0 > 0$ given the proofs of a non-stochastic ordering came about by considering small enough values of a.

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