Supplementary Material for the Paper "Nonparametric Recursive Estimation of the Copula"

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This technical report (a) presents in Section 1 a pseudo algorithm summarizing the recursive copula estimation procedure and (b) establishes the proof of Theorem 2 of Camirand Lemyre and Decrouez (2020). The reader is referred to this paper for the definition of all mathematical quantities and conditions. The theorem is recalled in Section 2, and its detailed proof is given in Section 3. In Section 4, we present a result of Fabian (1968) which is adapted to our context. The latter result is key in our derivations of the asymptotic normality of the recursive estimators.

1 Pseudo Algorithm

For convenience, Algorithm 1 summarises the procedure for recursive empirical copula estimation, corresponding mainly to equations (2.1) and (2.2) of Camirand Lemyre and Decrouez (2020). As a rule of thumb, we may take $\kappa = 0.1$, $\nu = 1$ and at step n, $\mu_j = 0.1 \times n^{-1/4}$ for all $j = 1, \ldots, d$. Choices for the bandwidth $b_{n,j}$ and $\tilde{h}_{n,j}$ are discussed in Section 4 of the main paper.

Algorithm 1 is easily modified for the recursive kernel version, where the indicator functions of the form $\mathbf{1}(\mathbf{X}_\cdot \leqslant \boldsymbol{\mathcal{Q}}_*^{\mathbf{u}})$ appearing in the expressions of the copula at line 6 and line 11 of the algorithm below are replaced by $W_{h_n}(\boldsymbol{\mathcal{Q}}_*^{\mathbf{u}} - \mathbf{X}_\cdot)$. As for the choice of h_n, we may take $h_n = b_n$ as discussed in Section 4 of the main paper. For the recursive empirical copula and for the recursive kernel

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version, the Epanechnikov kernel $K(x) = 3/4(1-x^2)\mathbf{1}(|x| \leqslant 1)$ can be used, since it satisfies the assumptions of Theorem 1 and Theorem 2.
Recall that we use the notation $K_a(u) \equiv K(u/a)/a$ and $H_a(u) \equiv H(u/a)/a$ for $a > 0$.

Algorithm 1 Recursive Empirical Copula Algorithm.

Inputs: $\mathfrak{D} = \{(\mathbf{X}_i)\}_{i=1,2,...}$ sequentially observed;
n_0: number of observations for initialization;
Parameters κ, $\nu, \mu_j > 0$;
Sequences of bandwidths $\mathfrak{B} = \{b_i, \tilde{h}_i\}_{i=1,2,...}$
procedure RECURSIVE EMPIRICAL COPULA$(\mathfrak{D}, n_0, \kappa, \nu, \{\mu_j\}_{j=1}^d, \mathfrak{B})$
Initialisation
$\mathcal{Q}_{n_0,j}^{u_j} \leftarrow \inf_v \{{n_0}^{-1} \sum_{i=1}^{n_0} \mathbf{1}(X_{i,j} \leqslant v) \geqslant u_j\}$
$f_{n_0,j}^{u_j} \leftarrow {n_0}^{-1} \sum_{i=1}^{n_0} K_{\tilde{h}_{n_0,j}}(\mathcal{Q}_{n_0}^{u_j} - X_{i,j})$
$a_{n_0,j}^{u_j} \leftarrow f_{n_0,j}^{u_j}$
$C_{n_0}(\mathbf{u}) \leftarrow {n_0}^{-1} \sum_{i=1}^{n_0} \mathbf{1}(\mathbf{X}_i \leqslant \boldsymbol{\mathcal{Q}}_{n_0}^{\mathbf{u}})$,
for $n \geqslant n_0 + 1$ **do**
$\mathcal{Q}_{n,j}^{u_j} \leftarrow \mathcal{Q}_{n-1,j}^{u_j} + (n\kappa a_{n-1,j}^{u_j})^{-1}\big[u_j - H\big\{b_{n,j}^{-1}(\mathcal{Q}_{n-1,j}^{u_j} - X_{n,j})\big\}\big]$;
$a_{n,j}^{u_j} \leftarrow \max[\mu_j, \min\{f_{n,j}^{u_j}, \nu(\log n + 1)\}]$
$f_{n,j}^{u_j} \leftarrow (1-n^{-1}) f_{n-1,j}^{u_j} + n^{-1} K_{\tilde{h}_{n,j}}(\mathcal{Q}_{n-1,j}^{u_j} - X_{n,j})$
$C_n(\mathbf{u}) \leftarrow \big(1 - n^{-1}\big) C_{n-1}(\mathbf{u}) + n^{-1} \mathbf{1}(\mathbf{X}_n \leqslant \boldsymbol{\mathcal{Q}}_{n-1}^{\mathbf{u}})$
end for
end procedure

2 Statement of Theorem 2

Theorem 2 *Suppose that K is compactly supported and vanishes outside of $[-1,1]$. Under the Assumptions of Theorem 1 in Camirand Lemyre and Decrouez (2020) and of $(\mathcal{B}^\star)$, the random vector $n^{1/2}\boldsymbol{\mathcal{C}}_n \equiv n^{1/2}(\mathcal{C}_n(\mathbf{u}_1) - C(\mathbf{u}_1), \dots, \mathcal{C}_n(\mathbf{u}_k) - C(\mathbf{u}_k))^\top$ with $\mathbf{u}_1, \dots, \mathbf{u}_k \in [a,b]^d \subset (0,1)^d$, $k \geqslant 1$, converges in distribution to a multivariate normal distribution with covariance matrix Σ^C and mean $(B(q^{\mathbf{u}_1}), \dots, B(q^{\mathbf{u}_k}))^\top$, where $B(\mathbf{x}) \equiv (\nu/2) \sum_{j=1}^d z_j \, \partial^2 F(\mathbf{x})/(\partial x_j^2)$.*

3 Proof of Theorem 2

3.1 Auxiliary lemma

Before starting the proof of Theorem 2, we consider the following lemma, which will be helpful to characterise the first two moments of the recursive kernel copula estimator.

Lemma 1 *Assume that K is compactly supported and vanishes outside of $[-1,1]$. Let $[a,b] \subset (0,1)$, and put $W_{h_n}^{\mathbf{u}} \equiv W_{h_n}(\boldsymbol{\mathcal{Q}}_{n-1}^{\mathbf{u}} - \mathbf{X}_n)$. Then, under Assumptions $(\mathcal{K})$ and $(\mathcal{D})$,* *(a)* *For* $\mathbf{u} \in [a,b]^d$,

$$\mathbf{E}\left(W_{h_n}^{\mathbf{u}} \mid \mathcal{F}_{n-1}\right) = F(\mathcal{Q}_{n-1}^{\mathbf{u}}) + B_{h_n}(q^{\mathbf{u}}) + o_{a.s}\left(\max_{1 \leqslant j \leqslant d} h_{n,j}^2\right),$$

where

$$B_{h_n}(\mathbf{x}) \equiv \frac{\upsilon}{2} \sum_{j=1}^{d} h_{n,j}^2 \frac{\partial^2 F(\mathbf{x})}{\partial x_j^2}.$$

(b) *For* $\mathbf{u}, \mathbf{v} \in [a,b]^d$,

$$cov\left(W_{h_n}^{\mathbf{u}}, W_{h_n}^{\mathbf{v}} \mid \mathcal{F}_{n-1}\right) = F(\mathcal{Q}_{n-1}^{\mathbf{u}} \wedge \mathcal{Q}_{n-1}^{\mathbf{v}}) - F(\mathcal{Q}_{n-1}^{\mathbf{u}})F(\mathcal{Q}_{n-1}^{\mathbf{v}}) + O_{a.s}\left(\max_{1 \leqslant j \leqslant d} h_{n,j}\right).$$

(c) *For* $\mathbf{u}, \mathbf{v} \in [a,b]^d$, *and* $\ell \in \{1, \ldots, d\}$,

$$cov\left\{W_{h_n}^{\mathbf{v}}, \mathbf{1}\left(X_{n,\ell} \leqslant \mathcal{Q}_{n-1,\ell}^{u_\ell}\right) \mid \mathcal{F}_{n-1}\right\} = C\{\mathbf{v}^{(\ell)}(u_\ell)\} - u_\ell C(\mathbf{v}) + o_{a.s}(1).$$

Proof. First, Assumptions $(\mathcal{K})$ and $(\mathcal{D})$ allow us to deduce from the proof of Lemma A3 of Liu and Yang (2008) that uniformly in $\mathbf{x} \in \mathcal{T}_{a,b}$,

$$\mathbf{E}\left\{W_{h_n}(\mathbf{x} - \mathbf{X}_n)\right\} = F(\mathbf{x}) + B_{h_n}(\mathbf{x}) + o\left(\max_{1 \leqslant j \leqslant d} h_{n,j}^2\right).$$

As Theorem 1 in Amiri and Thiam (2014) ensures that under Assumptions $(\mathcal{B})$, $(\mathcal{K})$ and $(\mathcal{D})$ $\mathcal{Q}_{n-1,j}^{u_j} = q_j^{u_j} + o_{\text{a.s}}(1)$, the result in (a) follows. To show part (b), we first establish that

$$\mathbf{E}\left(W_{h_n}^{\mathbf{u}} W_{h_n}^{\mathbf{v}} \mid \mathcal{F}_{n-1}\right) = F\left(\mathcal{Q}_{n-1}^{\mathbf{u}} \wedge \mathcal{Q}_{n-1}^{\mathbf{v}}\right) + O_{\text{a.s}}\left(\max_{1 \leqslant j \leqslant d} h_{n,j}\right). \quad (3.1)$$

To prove the above equality, notice first that since $K(u) = 0$ for any $u \notin [-1,1]$, and because $\int K(u)\mathrm{d}u = 1$, it follows that for any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{w} \in \mathbb{R}^d$, $W_{h_n}(\mathbf{x}_1 - \mathbf{w})W_{h_n}(\mathbf{x}_2 - \mathbf{w}) = 0$ when $\mathbf{w} \ngeqslant \mathbf{x}_1 \wedge \mathbf{x}_2 + h_n$ and $W_{h_n}(\mathbf{x}_1 - \mathbf{w})W_{h_n}(\mathbf{x}_2 - \mathbf{w}) = 1$ when $\mathbf{w} \leqslant \mathbf{x}_1 \wedge \mathbf{x}_2 - h_n$. Therefore,

$$\begin{aligned}
&\mathbf{E}\left\{W_{h_n}(\mathbf{x}_1 - \mathbf{X}_n)W_{h_n}(\mathbf{x}_2 - \mathbf{X}_n)\right\} \\
&= \int_{-\infty}^{\mathbf{x}_1 \wedge \mathbf{x}_2 - h_n} \mathrm{d}F(\mathbf{w}) + \int_{\mathbf{x}_1 \wedge \mathbf{x}_2 - h_n}^{\mathbf{x}_1 \wedge \mathbf{x}_2 + h_n} W_{h_n}(\mathbf{x}_1 - \mathbf{w})W_{h_n}(\mathbf{x}_2 - \mathbf{w})\mathrm{d}F(\mathbf{w}) \\
&= F(\mathbf{x}_1 \wedge \mathbf{x}_2 - h_n) + O\left(\sum_{j=1}^{d} h_{n,j}\right) = F(\mathbf{x}_1 \wedge \mathbf{x}_2) + O\left(\max_{1 \leqslant j \leqslant d} h_{n,j}\right),
\end{aligned}$$

where the second-to-last equality follows from the fact that W and the partial derivatives of F are bounded. This proves (3.1), and part (b) follows. Part (c) is derived similarly. □

3.2 Main part of the proof

For any $\mathbf{u} \in [a,b]^d$, put

$$\mathcal{U}_n^{\mathbf{u}} \equiv \mathcal{C}_n(\mathbf{u}) - C(\mathbf{u}) , \qquad \boldsymbol{\mathcal{U}}_n^{\mathbf{u}} \equiv \left(\mathcal{U}_n^{\mathbf{u}} + \overline{U}_n^{\mathbf{u}} \quad \overline{U}_n^{\mathbf{u}} \right)^\top .$$

We first show that, for any $\mathbf{u}_1, \ldots, \mathbf{u}_k$, where, for $1 \leqslant i \leqslant k$, $\mathbf{u}_i = (u_{i,1}, \ldots, u_{i,d}) \in [a,b]^d$, the sequence of random vectors

$$n^{1/2}\boldsymbol{\mathcal{U}}_n \equiv n^{1/2}\big((\boldsymbol{\mathcal{U}}_n^{\mathbf{u}_1})^\top, \ldots, (\boldsymbol{\mathcal{U}}_n^{\mathbf{u}_k})^\top\big)^\top$$

converges in distribution to a multivariate normal distribution with $1 \times 2k$ mean vector $2\mathbf{T} = 2(\mathbf{T}^{\mathbf{u}_i})$, where

$$\mathbf{T}^{\mathbf{u}_i} = \big(B(q^{\mathbf{u}}), 0\big)^\top ,$$

with

$$B(\mathbf{x}) = \frac{\nu}{2} \sum_{j=1}^{d} z_j \frac{\partial^2 F(\mathbf{x})}{\partial x_j^2} ,$$

and covariance matrices (Σ_{ij}^U), where Σ_{ij}^U is given at the beginning of the proof of Theorem 1 in Camirand Lemyre and Decrouez (2020).

Step I: Recurrence relation for $\boldsymbol{\mathcal{U}}_n^{\mathbf{u}}$. We next establish a recurrence relation for the terms $\overline{U}_n^{\mathbf{u}}$ and $\mathcal{U}_n^{\mathbf{u}}$ appearing in the definition of $\boldsymbol{\mathcal{U}}_n^{\mathbf{u}}$. We have from Step 1 in Camirand Lemyre and Decrouez (2020) that

$$\overline{U}_n^{\mathbf{u}} = \big\{1 - n^{-1}(\kappa^{-1} + o_{\text{a.s}}(1))\big\}\, \overline{U}_{n-1}^{\mathbf{u}} - n^{-1}\{1 + o_{\text{a.s}}(1)\}\overline{V}_{n-1}^{\mathbf{u}}. \tag{3.2}$$

To obtain a similar relation for $\mathcal{U}_n^{\mathbf{u}}$, we first recall the kernel copula recursion

$$\mathcal{C}_n(\mathbf{u}) = \left(1 - n^{-1}\right) \mathcal{C}_{n-1}(\mathbf{u}) + n^{-1} W_{h_n}(\boldsymbol{\mathcal{Q}}_{n-1}^{\mathbf{u}} - \mathbf{X}_n) . \tag{3.3}$$

From there, we get the decomposition

$$\begin{aligned} \mathcal{U}_n^{\mathbf{u}} &= \mathcal{C}_n(\mathbf{u}) - C(\mathbf{u}) \\ &= \left(1 - n^{-1}\right) \mathcal{U}_{n-1}^{\mathbf{u}} + n^{-1} \left(\mathcal{V}_{n-1}^{\mathbf{u}} + G_{n-1}^{\mathbf{u}}\right) + n^{-3/2} T_{n-1}^{\mathbf{u}} , \end{aligned} \tag{3.4}$$

where we have introduced

$$\mathcal{V}_n^{\mathbf{u}} \equiv W_{h_{n+1}}^{\mathbf{u}} - \mathbf{E}\left\{W_{h_{n+1}}^{\mathbf{u}} \mid \mathcal{F}_n\right\} . \tag{3.5}$$

and

$$T_n^{\mathbf{u}} \equiv n^{1/2} \left(\mathbf{E}\left\{W_{h_{n+1}}^{\mathbf{u}} \mid \mathcal{F}_n\right\} - F\left(\boldsymbol{\mathcal{Q}}_n^{\mathbf{u}}\right)\right) .$$

Under Assumption $(\mathcal{D})$, we have from a Taylor expansion of order 1 that

$$G_{n-1}^{\mathbf{u}} = F(\boldsymbol{\mathcal{Q}}_{n-1}^{\mathbf{u}}) - C(\mathbf{u}) = \kappa^{-1}(1-\kappa)\{1 + o_{\text{a.s}}(1)\}\overline{U}_{n-1}^{\mathbf{u}} . \tag{3.6}$$

It immediately follows from Equation (3.2), (3.4) and (3.6) that

$$\mathcal{U}_n^{\mathbf{u}} + \overline{U}_n^{\mathbf{u}} = \left(1 - n^{-1}\{1 + o_{\rm a.s}(1)\}\right)\left(\mathcal{U}_{n-1}^{\mathbf{u}} + \overline{U}_{n-1}^{\mathbf{u}}\right) + n^{-1}\{1 + o_{\rm a.s}(1)\}\left(\mathcal{V}_{n-1}^{\mathbf{u}} - \overline{V}_{n-1}^{\mathbf{u}}\right) + n^{-3/2} T_{n-1}^{\mathbf{u}}. \quad (3.7)$$

Introducing $\overline{\Gamma} = \text{diag}(1\,,\,\kappa^{-1})$, $\mathbf{T}_n^{\mathbf{u}} = (T_n^{\mathbf{u}}\,,\,0)^\top$ and $\boldsymbol{\mathcal{V}}_n^{\mathbf{u}} = ((\mathcal{V}_n^{\mathbf{u}} - \overline{V}_n^{\mathbf{u}})\ (\text{-}\overline{V}_n^{\mathbf{u}}))^\top$, we obtain

$$\begin{aligned}\boldsymbol{\mathcal{U}}_n^{\mathbf{u}} &= (\ \mathcal{U}_n^{\mathbf{u}} + \overline{U}_n^{\mathbf{u}}, \overline{U}_n^{\mathbf{u}}\ )^\top \\ &= \left\{\mathcal{I}_2 - n^{-1}(\overline{\Gamma} + o_{a.s}(1))\right\}\boldsymbol{\mathcal{U}}_{n-1}^{\mathbf{u}} + n^{-1}\left\{\mathcal{I}_2 + o_{a.s}(1)\right\}\boldsymbol{\mathcal{V}}_{n-1}^{\mathbf{u}} \\ &\qquad + n^{-3/2}\mathbf{T}_{n-1}^{\mathbf{u}}. \end{aligned} \quad (3.8)$$

Since (3.8) holds for any $\mathbf{u} \in \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, it can be generalised by introducing

$$\boldsymbol{\mathcal{V}}_n \equiv (\boldsymbol{\mathcal{V}}_n^{\mathbf{u}_1}, \ldots, \boldsymbol{\mathcal{V}}_n^{\mathbf{u}_k})^\top\,, \qquad \mathbf{T}_n \equiv (\mathbf{T}_{n-1}^{\mathbf{u}_1}, \ldots, \mathbf{T}_n^{\mathbf{u}_k})^\top\,.$$

We finally obtain

$$\boldsymbol{\mathcal{U}}_n = \left\{\mathcal{I}_{2k} - n^{-1}\left(\Gamma + o_{\rm a.s}(1)\right)\right\}\boldsymbol{\mathcal{U}}_{n-1} + n^{-1}\left\{\mathcal{I}_{2k} + o_{a.s}(1)\right\}\boldsymbol{\mathcal{V}}_{n-1} + n^{-3/2}\mathbf{T}_{n-1}, \quad (3.9)$$

where we recall from Camirand Lemyre and Decrouez (2020) that

$$\Gamma = \text{diag}(1\,,\,\kappa^{-1}\,,\,\ldots\,1\,,\,\kappa^{-1}\,,\,\ldots\,1\,,\,\kappa^{-1})\,,$$

Step II. Expression (3.9) is in the form required for an application of Lemma 2. Since under Condition $(\mathcal{K})$ W is bounded, which implies that each entry of $\boldsymbol{\mathcal{V}}_n$ is bounded, it follows that to apply the aforementioned Lemma to $\boldsymbol{\mathcal{U}}_n$, it remains to (i) find an almost sure representation for $\mathbf{T}_{n-1}$ and to (ii) derive the almost sure limit of the covariance matrix

$$\mathbf{E}\left\{\boldsymbol{\mathcal{V}}_n^{\mathbf{u}_j}\,(\boldsymbol{\mathcal{V}}_n^{\mathbf{u}_j})^\top \mid \mathcal{F}_{n-1}\right\}\,.$$

We start with (ii). From Lemma 1 we have that

$$\begin{aligned}\mathbf{E}(\mathcal{V}_n^{\mathbf{u}}\mathcal{V}_n^{\mathbf{v}} \mid \mathcal{F}_n) &= \sigma^{(1)}(\mathbf{u},\mathbf{v}) + o_{\rm a.s}(1) \\ \mathbf{E}(\overline{V}_n^{\mathbf{u}}\mathcal{V}_n^{\mathbf{v}} \mid \mathcal{F}_n) &= \sigma^{(2)}(\mathbf{u},\mathbf{v}) + o_{\rm a.s}(1). \end{aligned} \quad (3.10)$$

Also,

$$\begin{aligned}\mathbf{E}\left(\overline{V}_n^{\mathbf{u}}\ \overline{V}_n^{\mathbf{v}} \mid \mathcal{F}_n\right) &= (1-\kappa)^{-2}\sum_{\ell,m=1}^{d} C^{(\ell)}(\mathbf{u})C^{(m)}(\mathbf{v})\mathbf{E}\left(V_{n,\ell}^{u_\ell}, V_{n,m}^{v_m} \mid \mathcal{F}_n\right) \\ &= (1-\kappa)^{-2}\sum_{\ell,m=1}^{d} C^{(\ell)}(\mathbf{u})C^{(m)}(\mathbf{v})\left[C_{\ell,m}(u_\ell, v_m) - u_\ell v_m\right] + o_{\rm a.s}(1) \\ &= (1-\kappa)^{-2}\sigma^{(3)}(\mathbf{u},\mathbf{v}) + o_{\rm a.s}(1)\,. \end{aligned}$$

Therefore,

$$\mathbf{E}\{\boldsymbol{\mathcal{V}}_n^{\mathbf{u}_j}(\boldsymbol{\mathcal{V}}_n^{\mathbf{u}_j})^\top \mid \mathcal{F}_{n-1}\} = \Sigma_{ij} + o_{\mathrm{a.s}}(1)\,,$$

where Σ_{ij} is defined in the proof of Theorem 1 in Camirand Lemyre and Decrouez (2020) (see *Step II* therein), so that (ii) follows. Finally, to get (i), we use Lemma 1 to obtain, as $n^{1/2}(h_{n,1}^2,\ldots,h_{n,d}^2) \to (z_1,\ldots,z_d)$, that

$$\begin{aligned} T_n^{\mathbf{u}} &= n^{1/2}\left(\mathbf{E}\left\{W_{h_n}^{\mathbf{u}} \mid \mathcal{F}_n\right\} - F\left(\boldsymbol{\mathfrak{Q}}_n^{\mathbf{u}}\right)\right) \\ &= \frac{\nu}{2}\sum_{j=1}^{d}\left(n^{1/2}h_{n,j}^2\right)\frac{\partial^2 F(q^{\mathbf{u}})}{\partial x_j^2} + o_{\mathrm{a.s}}(1) \\ &= B(q^{\mathbf{u}}) + o_{\mathrm{a.s}}(1)\,. \end{aligned}$$

This argument holds for any $\mathbf{u} \in [a,b]^d$, and we obtain $\mathbf{T}_n = \mathbf{T} + o_{\mathrm{a.s}}(1)$. This concludes the proof of the lemma. □

The asymptotic normality of $n^{1/2}(\mathcal{U}_n^{\mathbf{u}_1},\ldots,\mathcal{U}_n^{\mathbf{u}_k})^\top$ follows from from the previous lemma with identical arguments as the ones exposed fin Step III of the proof of Theorem 1 in Camirand Lemyre and Decrouez (2020).

4 Adaptation of Fabian's result

In the followings, let $\mathbb{R}^{k\times k}$ denote the space of all $k \times k$ matrices

Lemma 2 *[Fabian (1968)] Suppose k is a positive integer and $\mathcal{F}_n$ is a non-decreasing sequence of $\sigma-$fields. Also, let $U_n, V_n, T_n, T \in \mathbb{R}^k$ and $\Gamma_n, \Phi_n, \Gamma, \Sigma \in \mathbb{R}^{k\times k}$ such that Γ_n and Γ are diagonal, $\min \Gamma_{ii} > 1/2$ and Σ is positive definite. Suppose also that $\Gamma_{n-1}, \Phi_{n-1}, V_{n-1}$ are $\mathcal{F}_{n-1}$-measurables, that $\Gamma_n = \Gamma + o_{a.s}(1)$, that V_n satisfies $\mathbf{E}\{V_n \mid \mathcal{F}_n\} = 0$ and $\|V_n\|_\infty < \Upsilon$ for some positive $\Upsilon < \infty$, that $\Phi_n = \mathcal{I}_k + o_{a.s}(1)$ and that U_n satisfies $U_n = (\mathcal{I}_k - n^{-1}\Gamma_{n-1})U_{n-1} + n^{-1}\Phi_{n-1}V_{n-1} + n^{-3/2}T_{n-1}$. Provided that $\mathbf{E}\{V_nV_n^\top \mid \mathcal{F}_n\} = \Sigma + o_{a.s}(1)$ and that $T_n = T + o_{a.s}(1)$, then, $n^{1/2}U_n$ converges as $n \to \infty$ to a centered multivariate normal distribution with mean $(\Gamma - (1/2)\,\mathcal{I}_k)^{-1}T$ and covariance matrix with entries $\Sigma_{ij} \times (\Gamma_{ii} + \Gamma_{jj} - 1)^{-1}$.*

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