Supplementary Material for the Paper "On the large-sample behavior of two estimators of the conditional copula under serially dependent data"

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This technical report establishes the proofs Lemma 1, Lemma 2, Lemma 3 and Lemma 4 of Bouezmarni et al (2019). The reader is referred to this paper for the definition of all mathematical quantities and conditions. These lemmas are first recalled in Section 1, and their detailed proofs are given in Section 2.

1 Statement of the four lemmas

Lemma 1 *Under Assumptions ($\mathcal{S}$), ($\mathcal{LL}$) and ($\mathcal{N}$), one has almost surely that as $n \to \infty$,*

$$\sup_{z \in J_x} \sup_{u : K(u) > 0} \frac{1}{K(u)} \left| \mathcal{K}_{zn}(u) - \frac{K(u)}{f_X(z)} \right| \to 0.$$

Lemma 2 *Under Assumptions ($\mathcal{S}$), ($\mathcal{H}$) and ($\mathcal{LL}$), one can find a finite constant $\omega > 0$ such that for all b satisfying $0 < b < \min\{(a-6)/a, 2/5\}$,*

$$\mathrm{E}\left\{ |\mathbb{H}_{xh}(A)|^6 \right\} \leq \omega \left\{ \frac{\nu_x(A)}{n^2 h^2} + \frac{\nu_x(A)^{2-\frac{4}{a}}}{nh} + \nu_x(A)^{3-\frac{6}{a}} + \mathcal{J}_n(h,b) \right\},$$

where $\mathcal{J}_n(h,b) = h^4 h^{2b} + h^b (nh)^{-1} + h^{5b} (nh)^{-2}$.

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Lemma 3 *Under Assumptions ($\mathcal{S}$), ($\mathcal{LL}$) and ($\mathcal{N}$), one has for n sufficiently large that for any $\epsilon > 0$ and $\delta > 2\kappa_\gamma^{-1}$,*

$$\mathbb{P}\left\{\mathfrak{W}_\delta(Z_{xn}, \mathbb{R}^2) > \epsilon\right\} \leq \mathbb{P}\left\{\mathfrak{W}_{2\delta}(Z_{xn}, T_\gamma) > \frac{\epsilon}{3}\right\}.$$

Lemma 4 *Under Assumptions ($\mathcal{S}$), ($\mathcal{H}^\star$), ($\mathcal{LL}$) and ($\mathcal{N}^\star$), the sequences $\widetilde{Z}_{1xn}$ and $\widetilde{Z}_{2xn}$ are asymptotically tight in $\ell^\infty([-1,1] \times [0,1])$.*

2 Proofs of the four lemmas

2.1 Proof of Lemma 1

Corollary 1 of Masry (1996) ensures that under Assumptions ($\mathcal{S}$), ($\mathcal{LL}$) and ($\mathcal{N}$), one has for each $j \in \{0,1,2\}$ that almost surely,

$$\sup_{z \in J_x} \left| S_{n,j}(z) - f_X(z) \int_{\mathbb{R}} u^j K(u)\,\mathrm{d}u \right| \to 0.$$

In addition, one has almost surely as $n \to \infty$ that

$$\sup_{z \in J_x} \sup_{u: K(u) > 0} \left| S_{n,2}(z) - u\, S_{n,1}(z) - f_X(z) \int u^2 K(u)\,\mathrm{d}u \right| \to 0$$

and

$$\sup_{z \in J_x} \left| S_{n,0}(z)\, S_{n,2}(z) - \{S_{n,1}(z)\}^2 - \{f_X(z)\}^2 \int u^2 K(u)\,\mathrm{d}u \right| \to 0.$$

Since Assumption ($\mathcal{LL}$) ensures that $f_X(z) > 0$ for all $z \in J_x$, an application of Slutsky's Theorem entails that almost surely as $n \to \infty$,

$$\sup_{z \in J_x} \sup_{u: K(u) > 0} \left| \frac{S_{n,2}(z) - u\, S_{n,1}(z)}{S_{n,0}(z) S_{n,2}(z) - \{S_{n,1}(z)\}^2} - \frac{1}{f_x(z)} \right| \to 0$$

The result then follows from the definition of $\mathcal{K}_{zn}$.

2.2 Proof of Lemma 2

The first step will be to show that for an arbitrary $A \subseteq \mathbb{R}^2$, there exists a finite constant $\omega' > 0$ such that

$$\mathrm{E}\left\{|\mathbb{H}_{xh}(A)|^6\right\} \leq \omega' \left\{ \frac{\nu_{xh}(A) + h^{\frac{5a-30}{a}}}{n^2 h^2} + \frac{h^{\frac{a-6}{a}} + \nu_{xh}(A)^{2-\frac{4}{a}}}{nh} + \nu_{xh}(A)^{3-\frac{6}{a}} \right\}, \quad (1)$$

where

$$\nu_{xh}(A) = \frac{1}{h}\,\mathrm{E}\left\{\nu_X(A)\, K\left(\frac{X-x}{h}\right)\right\}.$$

From the definition of $\mathcal{K}_{xn}$ and in view of Lemma 1, one can write

$$\mathbb{H}_{xh}(A) = \frac{1}{\sqrt{nh}\, f_X(x)} \sum_{i=1}^{n} \vartheta_i(A)\, K_{xh}(i)\, \{1 + o_{as}(1)\},$$

where $\vartheta_i(A) = \mathbb{I}\{(Y_{1i}, Y_{2i}) \in A\} - \nu_{X_i}(A)$ and $K_{xh}(i) = K\{(X_i - x)/h\}$. Hence, it will be sufficient to show that inequality (1) holds for

$$\widetilde{\mathbb{H}}_{xh}(A) = \frac{1}{\sqrt{nh}} \sum_{i=1}^{n} \nu_i(A)\, K_{xh}(i).$$

Letting $\mathcal{Q}_p = \{(k_1 \ldots, k_p) \in \{1, \ldots, 6\}^p : k_1 + \ldots + k_p = 6\}$, observe that since the data process is assumed to be stationary,

$$\begin{aligned} \mathrm{E}\left\{ \left| \widetilde{\mathbb{H}}_{xh}(A) \right|^6 \right\} &\leq \frac{6!}{(nh)^3} \sum_{i_1 \leq \cdots \leq i_6} \left| \mathrm{E}\left\{ \prod_{j=1}^{6} \vartheta_{i_k}(A)\, K_{xh}(i_k) \right\} \right| \\ &= \frac{6!}{(nh)^3} \sum_{p=1}^{6} \left\{ \sum_{\mathcal{Q}_p} T_p(k_1, \ldots, k_p) \right\}, \end{aligned} \tag{2}$$

where

$$T_p(k_1, \ldots, k_p) = \sum_{i_1 < \ldots < i_p} \left| \mathrm{E}\left[\prod_{j=1}^{p} \{\vartheta_{i_j}(A)\, K_{xh}(i_j)\}^{k_j} \right] \right|.$$

The remainder of the proof will essentially consist in bounding each of $T_1, \ldots, T_6$. To this end, one will make use of the inequality that states that there exists a positive constant ζ such that for any $k \in \{1, \ldots, 6\}$,

$$\left| \mathrm{E}\left[\left\{ \vartheta_{i_j}(A) K_{xh}(i_j) \right\}^k \right] \right| \leq \zeta \mathrm{E}\left\{ \nu_{X_1}(A) K_{xh}(1) \right\} \mathbb{I}(k \neq 1) = \zeta\, h\, \nu_{xh}(A) \Delta_1(k), \tag{3}$$

where $\Delta_p(k_1, \ldots, k_p) = \mathbb{I}(k_1 \neq 1, \ldots, k_p \neq 1)$ for $p \in \{1, \ldots, 6\}$. Note that (3) is a simple consequence of the fact that $\mathrm{E}\{\vartheta_{i_j}(A)\, K_{xh}(i_j)\} = 0$ and because of Assumption $(\mathcal{LL})$), K is bounded.

Now the following inequality will also be used repeatedly: for any $k \in \{2, \ldots, 6\}$, $m \in \{1, \ldots, k-1\}$, $i_1 < \cdots < i_k$ and $\ell_1, \ldots, \ell_k \in \{1, \ldots, 6\}$,

$$\left| \mathrm{Cov}\left[\prod_{j=1}^{m} \{\vartheta_{i_j}(A) K_{xh}(i_j)\}^{\ell_j}, \prod_{j=m+1}^{k} \{\vartheta_{i_j}(A) K_{xh}(i_j)\}^{\ell_j} \right] \right| \leq \zeta' \min\left\{ \Upsilon_{xh}(k), g_m^{-a} \right\}, \tag{4}$$

where $g_m = i_{m+1} - i_m$ and $\Upsilon_{xh}(k) = \min\{h^k, h\nu_{xh}(A)\}$. To establish (4), a first step is to show that

$$\left| \mathrm{Cov}\left[\prod_{j=1}^{m} \left\{ \vartheta_{i_j}(A)\, K_{xh}(i_j) \right\}^{\ell_j}, \prod_{j=m+1}^{k} \left\{ \vartheta_{i_j}(A)\, K_{xh}(i_j) \right\}^{\ell_j} \right] \right| = O\left\{ \Upsilon_{xh}(k) \right\}. \tag{5}$$

Since $i_1 < \cdots < i_k$, one has for $\mathbf{z} = (z_1, \ldots, z_k)$ and $\mathbf{u} = (u_1, \ldots, u_k)$ that

$$\begin{aligned}
\mathrm{E}\left\{\prod_{j=1}^{k} \left|\vartheta_{i_j}(A) K_{xh}(i_j)\right|^{\ell_j}\right\} &\leq \mathrm{E}\left\{\prod_{i_j=1}^{k} K_{xh}(i_j)^{\ell_j}\right\} \\
&= \int_{\mathbb{R}^k} \prod_{j=1}^{k} \left\{K\left(\frac{z_j - x}{h}\right)\right\}^{\ell_j} f_X^{(i_2-i_1,\ldots,i_k-i_1)}(\mathbf{z})\,\mathrm{d}\mathbf{z} \\
&= h^k \int_{\mathbb{R}^k} \prod_{j=1}^{k} \{K(u_j)\}^{\ell_j} f_X^{(i_2-i_1,\ldots,i_k-i_1)}(h\mathbf{u} + x)\,\mathrm{d}\mathbf{u}.
\end{aligned}$$

Since Assumption $(\mathcal{H})$ ensures that K is compactly supported and $f_X^{(i_2-i_1,\ldots,i_k-i_1)}(\mathbf{z})$ is bounded in a neighborhood of $\mathbf{z} = (x, \ldots, x)$,

$$h^k \int_{\mathbb{R}^k} \prod_{j=1}^{k} \{K(u_j)\}^{\ell_j} f_X^{(i_2-i_1,\ldots,i_k-i_1)}(h\mathbf{u} + x)\,\mathrm{d}\mathbf{u} = O(h^k).$$

Combining this with (3) yields

$$\mathrm{E}\left\{\prod_{j=1}^{k} \left|\vartheta_{i_j}(A) K_{xh}(i_j)\right|^{\ell_j}\right\} = O\left\{\min\left(h^k, h\,\nu_{xh}(A)\right)\right\} = O\left\{\Upsilon_{xh}(k)\right\}.$$

Hence, since $\Upsilon_{xh}(m)\,\Upsilon_{xh}(k-m) \leq \min\{h^k, h^2 \nu_{xh}(A)^2\} \leq \Upsilon_{xh}(k)$ when h is sufficiently small, (5) follows from the fact that

$$\begin{aligned}
&\left|\mathrm{Cov}\left[\prod_{j=1}^{m} \left\{\vartheta_{i_j}(A) K_{xh}(i_j)\right\}^{\ell_j}, \prod_{j=m+1}^{k} \left\{\vartheta_{i_j}(A) K_{xh}(i_j)\right\}^{\ell_j}\right]\right| \\
&\quad \leq \left|\mathrm{E}\left\{\prod_{j=1}^{k} \left|\vartheta_{i_j}(A) K_{xh}(i_j)\right|^{\ell_j}\right\}\right| \\
&\qquad + \left|\mathrm{E}\left[\prod_{j=1}^{m} \left\{\vartheta_{i_j}(A) K_{xh}(i_j)\right\}^{\ell_j}\right]\right| \times \left|\mathrm{E}\left[\prod_{j=m+1}^{k} \left\{\vartheta_{i_j}(A) K_{xh}(i_j)\right\}^{\ell_j}\right]\right| \\
&\quad = O\left\{\Upsilon_{xh}(k)\right\} + O\left\{\Upsilon_{xh}(m)\,\Upsilon_{xh}(k-m)\right\} \\
&\quad = O\left\{\Upsilon_{xh}(k)\right\}.
\end{aligned}$$

It then suffices to show that the left hand side of (4) is bounded above by g_m^{-a}. To this end, one can use Lemma 3.9 of Dehling and Philipp (2002) and the fact that the random variables $\vartheta_{i_1}(A) K_{xh}(i_1), \ldots, \vartheta_{i_k}(A) K_{xh}(i_k)$ are bounded to deduce that the left hand side of Equation (4) is $O(\alpha_{g_m})$. Inequality (4) then follows from the fact that $\alpha_\ell \sim \ell^{-a}$ for some $a > 6$ under Assumption $(\mathcal{S})$.

The case $p = 1$. *From* (3), *one has for* $k \in \{1, \ldots, 6\}$ *that*

$$T_1(k) = n \left| \mathrm{E}\left[\{\vartheta_1(A)\, K_{xh}(1)\}^k\right]\right| \le \zeta\, nh\, \nu_{xh}(A)\, \Delta_1(k). \qquad (6)$$

Here, $\mathcal{Q}_1 = \{6\}$ *and then*

$$\sum_{\mathcal{Q}_1} T_p(k_1, \ldots, k_p) = T_1(6) \le \zeta\, nh\, \nu_{xh}(A).$$

The case $p = 2$. *Observe that for any* $(k_1, k_2) \in \{1, \ldots, 6\}^2$,

$$\begin{aligned}
T_2(k_1, k_2) &= n \sum_{\ell=1}^{n-1} \left|\mathrm{E}\left[\{\vartheta_1(A)K_{xh}(1)\}^{k_1}\{\vartheta_{i+\ell}(A)K_{xh}(i+\ell)\}^{k_2}\right]\right| \\
&\le n \sum_{\ell=1}^{n-1} \left|\mathrm{Cov}\left[\{\vartheta_1(A)K_{xh}(1)\}^{k_1}, \{\vartheta_{i+\ell}(A)K_{xh}(i+\ell)\}^{k_2}\right]\right| \\
&\quad + n \sum_{\ell=1}^{n-1} \mathrm{E}\left[\{\vartheta_1(A)K_{xh}(1)\}^{k_1}\right] \mathrm{E}\left[\{\vartheta_{i+\ell}(A)K_{xh}(i+\ell)\}^{k_2}\right] \\
&\le n\zeta' \sum_{\ell=1}^{n-1} \min(\Upsilon_{xh}(A, 2), \ell^{-a}) + \zeta^2 n^2 h^2 \nu_{xh}(A)^2 \Delta_2(k_1, k_2),
\end{aligned}$$

where the last inequality follows from (3) *and* (4). *Since*

$$\begin{aligned}
\sum_{\ell=1}^{n-1} \min(\Upsilon_{xh}(2), \ell^{-a}) &= \sum_{\ell < \Upsilon_{xh}(2)^{-1/a}} \Upsilon_{xh}(2) + \sum_{\ell > \Upsilon_{xh}(2)^{-1/a}} \ell^{-a} \\
&\le \Upsilon_{xh}(2)^{1-\frac{1}{a}} + 2 \int_{\Upsilon_{xh}(2)^{-1/a}}^{\infty} x^{-a}\, \mathrm{d}x \\
&\le 2\, \Upsilon_{xh}(2)^{1-\frac{1}{a}},
\end{aligned}$$

one deduces that there exists a constant $\omega_2 > 0$ *such that*

$$T_2(k_1, k_2) \le \omega_2 \left\{ n\Upsilon_{xh}(2)^{1-\frac{1}{a}} + n^2 h^2 \nu_{xh}(A)^2 \Delta_2(k_1, k_2) \right\}. \qquad (7)$$

Since $\mathcal{Q}_2 = \{(k, 6-k) : 1 \le k \le 5\}$, *one finally obtains*

$$\max_{\mathcal{Q}_2} T_2(k_1, k_2) \le \omega_2 \left\{ n\Upsilon_{xh}(2)^{1-\frac{1}{a}} + n^2 h^2 \nu_{xh}(A)^2 \right\}.$$

The case $p = 3$. *Define* $\mathcal{T}_1 = \{i_1 < i_2 < i_3 : g_2 \le g_1\}$ *and* $\mathcal{T}_2 = \{i_1 < i_2 < i_3 : g_2 > g_1\}$, *where* $g_m = i_{m+1} - i_m$, *and for* $m \in \{1, 2\}$, *let*

$$W_m(k_1, k_2, k_3) = \sum_{\mathcal{T}_m} \left| \mathrm{E}\left[\prod_{j=1}^{m} \left\{\vartheta_{i_j}(A)\, K_{xh}(i_j)\right\}^{k_j}\right] \mathrm{E}\left[\prod_{j=m+1}^{3} \left\{\vartheta_{i_j}(A)\, K_{xh}(i_j)\right\}^{k_j}\right]\right|.$$

Now observe that

$$\begin{aligned}
T_3(k_1,k_2,k_3) &= \sum_{m=1}^{2}\sum_{\mathcal{T}_m}\left| \mathrm{E}\left[\prod_{j=1}^{3}\left\{\vartheta_{i_j}(A)\,K_{xh}(i_j)\right\}^{k_j}\right]\right| \\
&\le \sum_{m=1}^{2}\sum_{\mathcal{T}_m}\left| \mathrm{Cov}\left[\prod_{j=1}^{m}\left\{\vartheta_{i_j}(A)\,K_{xh}(i_j)\right\}^{k_j},\ \prod_{j=m+1}^{3}\left\{\vartheta_{i_j}(A)\,K_{xh}(i_j)\right\}^{k_j}\right]\right| \\
&\quad + \sum_{m=1}^{2} W_m(k_1,k_2,k_3).
\end{aligned}$$

Since

$$\sum_{m=1}^{2} W_m(k_1,k_2,k_3) \le T_1(k_1)\,T_2(k_2,k_3) + T_2(k_1,k_2)\,T_2(k_3),$$

one deduces from (4), (6) *and* (7) *that*

$$\begin{aligned}
T_3(k_1,k_2,k_3) &\le \zeta' n \sum_{m=1}^{2}\sum_{g_m=1}^{n-1} g_m \min\left\{\Upsilon_{xh}(3), g_m^{-a}\right\} \\
&\quad + T_1(k_1)\,T_2(k_2,k_3) + T_2(k_1,k_2)\,T_2(k_3) \\
&\le 2\,\zeta' n\,\Upsilon_{xh}(3)^{1-\frac{2}{a}} + \zeta\,\omega_2\, n^2 h \nu_{xh}(A)\,\Upsilon_{xh}(2)^{1-\frac{1}{a}}\left\{\Delta_1(k_1) + \Delta_1(k_3)\right\} \\
&\quad + n^3 h^3 \nu_{xh}(A)^3 \Delta_3(k_1,k_2,k_3). \qquad (8)
\end{aligned}$$

Hence, from the definition of $\mathcal{Q}_3$, *one deduces that for some* $\omega_3 > 0$,

$$\max_{\mathcal{Q}_3} T_3(k_1,k_2,k_3) \le \omega_3 \left\{ n\Upsilon_{xh}(3)^{1-\frac{2}{a}} + n^2 h\nu_{xh}(A)\Upsilon_{xh}(2)^{1-\frac{1}{a}} + n^3h^3\nu_{xh}(A)^3 \right\}.$$

The case $p = 4$. *Define* $\mathcal{U}_k = \{i_1 < \ldots < i_4 : g_i \le g_k\}$ *and assume without loss of generality that* $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ *are disjoint sets, which could be done by removing the redundant equalities; again, recall that* $g_m = i_{m+1} - i_m$. *Letting*

$$\begin{aligned}
U_{hx}^{(m)}(k_1,k_2,k_3,k_4) = \sum_{\mathcal{U}_m} \mathrm{E}\left[\prod_{j=1}^{m}\left\{\vartheta_{i_j}(A)\,K_{xh}(i_j)\right\}^{k_j}\right] \\
\times\, \mathrm{E}\left[\prod_{j=m+1}^{4}\left\{\vartheta_{i_j}(A)\,K_{xh}(i_j)\right\}^{k_j}\right],
\end{aligned}$$

and since by construction,

$$T_4(k_1,k_2,k_3,k_4) = \sum_{m=1}^{3}\sum_{\mathcal{U}_m}\left| \mathrm{E}\left\{\prod_{j=1}^{4}\vartheta_{i_j}(A)\,K_{xh}(i_j)\right\}\right|,$$

one has in view of (4) *that*

$$
\begin{aligned}
T_4(k_1,k_2,k_3,k_4) &\le \sum_{m=1}^{3}\sum_{\mathcal{U}_m}\left|\mathrm{Cov}\left[\prod_{j=1}^{m}\left\{\vartheta_{i_j}(A)\,K_{xh}(i_j)\right\}^{k_j},\prod_{j=m+1}^{4}\left\{\vartheta_{i_j}(A)\,K_{xh}(i_j)\right\}^{k_j}\right]\right|\\
&\quad+\sum_{m=1}^{3}U_{hx}^{(m)}(k_1,k_2,k_3,k_4)\\
&\le 3\,\zeta' n\sum_{\ell=1}^{n-1}\ell^2\min\left\{\Upsilon_{xh}(4),\ell^{-a}\right\}\\
&\quad+T_1(k_1)\,T_3(k_2,k_3,k_4)+T_2(k_1,k_2)\,T_2(k_3,k_4)+T_1(k_4)\,T_3(k_1,k_2,k_3).
\end{aligned}
$$

From (6), (7) *and* (8), *one obtains*

$$
\begin{aligned}
T_4(k_1,k_2,k_3,k_4) &\le 6\,\zeta' n\Upsilon_{xh}(4)^{1-\frac{3}{a}}+n^2\Upsilon_{xh}(2)^{2-\frac{2}{a}}\\
&\quad+n^2h\nu_{xh}(A)\Upsilon_{xh}(3)^{1-\frac{2}{a}}\left\{\Delta_1(k_1)+\Delta_1(k_4)\right\}\\
&\quad+n^3h^2\nu_{xh}(A)^2\Upsilon_{xh}(2)^{1-\frac{1}{a}}\left\{\Delta_2(k_1,k_2)+\Delta_2(k_1,k_4)+\Delta_2(k_3,k_4)\right\}\\
&\quad+\left\{n^3h^3\nu_{xh}(A)^3+n^4h^4\nu_{xh}(A)^4\right\}\Delta_4(k_1,k_2,k_3,k_4). \qquad (9)
\end{aligned}
$$

Since $(k_1,k_2,k_3,k_4)\in\mathcal{Q}_4$ *implies that at least two of the* k_i*'s are equal to* 1,

$$
\begin{aligned}
\max_{\mathcal{Q}_4} T_4(k_1,k_2,k_3,k_4) &\le n\Upsilon_{xh}(4)^{1-\frac{3}{a}}+n^2h\nu_{xh}(A)\Upsilon_{xh}(3)^{1-\frac{2}{a}}\\
&\quad+n^3h^2\nu_{xh}(A)^2\Upsilon_{xh}(2)^{1-\frac{1}{a}}+n^2\Upsilon_{xh}(2)^{2-\frac{2}{a}}.
\end{aligned}
$$

The case $p=5$. *From computations similar as those for the case* $p=4$,

$$
\begin{aligned}
T_5(k_1,k_2,k_3,k_4,k_5) &\le n\Upsilon_{xh}(5)^{1-\frac{4}{a}}\\
&\quad+T_1(k_1)\,T_4(k_2,k_3,k_4,k_5)+T_2(k_1,k_2)\,T_3(k_3,k_4,k_5)\\
&\quad+T_3(k_1,k_2,k_3)\,T_2(k_4,k_5)+T_4(k_1,k_2,k_3,k_4)\,T_1(k_5)\\
&\le n\Upsilon_{xh}(5)^{1-\frac{4}{a}}\\
&\quad+2\,T_1(2)\,T_4(1,1,1,1)\\
&\quad+T_2(1,1)\left\{T_3(1,1,2)+T_3(1,2,1)+T_3(2,1,1)\right\}\\
&\quad+T_3(1,1,1)\left\{T_2(1,2)+T_2(2,1)\right\},
\end{aligned}
$$

since $T_1(1)=0$ *and* $(k_1,k_2,k_3,k_4,k_5)\in\mathcal{Q}_5$ *implies that all but one of the* k_i*'s are equal to 1. From there, one deduces from* (6), (7), (8) *and* (9) *that*

$$
\begin{aligned}
\max_{\mathcal{Q}_5} T_5(k_1,k_2,k_3,k_4,k_5) &\le n\Upsilon_{xh}(5)^{1-\frac{4}{a}}+nh\nu_{xh}(A)\left\{n\Upsilon_{xh}(4)^{1-\frac{3}{a}}+n^2\Upsilon_{xh}(2)^{2-\frac{2}{a}}\right\}\\
&\quad+n\Upsilon_{xh}(2)^{1-\frac{1}{a}}\left\{n\Upsilon_{xh}(3)^{1-\frac{2}{a}}+n^2h\nu_{xh}(A)\Upsilon_{xh}(2)^{1-\frac{1}{a}}\right\}\\
&\quad+n^2\Upsilon_{xh}(3)^{1-\frac{2}{a}}\Upsilon_{xh}(2)^{1-\frac{1}{a}}.
\end{aligned}
$$

The case $p=6$. *Since $\mathcal{Q}_6 = \{(1,1,1,1,1,1)\}$, one can proceed as for the case $p=5$ and deduce*

$$\begin{aligned} T_6(1,1,1,1,1,1) &\le n\Upsilon_{xh}(6)^{1-\frac{5}{a}} + 2\,T_1(1)\,T_5(1,1,1,1,1) \\ &\quad +2\,T_2(1,1)\,T_4(1,1,1,1) + \{T_3(1,1,1)\}^2 \\ &\le n\Upsilon_{xh}(6)^{1-\frac{5}{a}} + n\Upsilon_{xh}(2)^{1-\frac{1}{a}} \left\{ n\Upsilon_{xh}(4)^{1-\frac{3}{a}} + n^2\Upsilon_{xh}(2)^{2-\frac{2}{a}} \right\} \\ &\quad + n^2\Upsilon_{xh}(3)^{2-\frac{4}{a}}. \end{aligned}$$

Note that the first term on the right hand side of the first inequality was obtained using the fact that $\alpha(r) \sim r^{-a}$ with $a > 6$, which ensured that $\sum_{\ell=z}^{\infty} \ell^4 \alpha(r) \le 2z^{-(a-5)}$.

Collecting the bounds for $p \in \{1,\ldots,6\}$, one has in view of Equation (2) that

$$\begin{aligned} &(nh)^3\,\mathrm{E}\left\{ \left| \widetilde{\mathbb{H}}_{xh}(A) \right|^6 \right\} \\ &\quad \le \sum_{p=1}^{3} (nh)^p \nu_{xh}(A)^p + n\sum_{p=2}^{6} \Upsilon_{xh}(p)^{1-\frac{p-1}{a}} \\ &\quad + n^2 \left\{ h\nu_{xh}(A) + \Upsilon_{xh}(2)^{1-\frac{1}{a}} \right\} \sum_{p=2}^{4} \Upsilon_{xh}(p)^{1-\frac{p-1}{a}} + n^2\Upsilon_{xh}(3)^{2-\frac{4}{a}} \\ &\quad + n^3\Upsilon_{xh}(2)^{1-\frac{1}{a}} \left\{ h^2\nu_{xh}(A)^2 + h\nu_{xh}(A)\Upsilon_{xh}(2)^{1-\frac{1}{a}} + \Upsilon_{xh}(2)^{2-\frac{2}{a}} \right\}. \end{aligned}$$

Since $\Upsilon_{xh}(p) = \min\{h^p, h\nu_{xh}(A)\}$, one has using the fact that $a > 6$ that

$$\sum_{p=2}^{6} \Upsilon_{xh}(p)^{1-\frac{p-1}{a}} \le 5hh^{\frac{5a-30}{a}} \quad \text{and} \quad \sum_{p=3}^{4} \Upsilon_{xh}(p)^{1-\frac{p-1}{a}} \le 2h^2 h^{\frac{a-6}{a}}.$$

Because $\Upsilon_{xh}(3) \le h^3$, one deduces from straightforward computations that

$$\begin{aligned} &(nh)^3\,\mathrm{E}\left\{ \left| \widetilde{\mathbb{H}}_{xh}(A) \right|^6 \right\} \\ &\quad \le \sum_{p=1}^{3} (nh)^p \nu_{xh}(A)^p + 5nhh^{\frac{5a-30}{a}} \\ &\quad + n^2 \left\{ h^2 h^{\frac{a-6}{a}} + h\nu_{xh}(A)\Upsilon_{xh}(2)^{1-\frac{1}{a}} + \Upsilon_{xh}(2)^{2-\frac{2}{a}} \right\} \\ &\quad + n^3\Upsilon_{xh}(2)^{1-\frac{1}{a}} \left\{ h^2\nu_{xh}(A)^2 + h\nu_{xh}(A)\Upsilon_{xh}(2)^{1-\frac{1}{a}} + \Upsilon_{xh}(2)^{2-\frac{2}{a}} \right\}. \end{aligned}$$

When $\nu_{xh}(A) < h$, it follows that $\Upsilon_{xh}(2) = h\nu_{xh}(A)$. Therefore, in that case,

$$\begin{aligned}(nh)^3 \,\mathrm{E}\left\{\left|\widetilde{\mathbb{H}}_{xh}(A)\right|^6\right\} &\le \sum_{p=1}^{3}(nh)^p \nu_{xh}(A)^p + 5nhh^{\frac{5a-30}{a}} \\ &\quad + n^2\left\{h^2 h^{\frac{a-6}{a}} + (h\nu_{xh}(A))^{2-\frac{2}{a}}\right\} + n^3 (h\nu_{xh}(A))^{3-\frac{3}{a}} \\ &\le nh\left\{\nu_{xh}(A) + h^{\frac{5a-30}{a}}\right\} + n^2h^2\left\{h^{\frac{a-6}{a}} + (\nu_{xh}(A))^{2-\frac{4}{a}}\right\} \\ &\quad + 2n^3h^3(\nu_{xh}(A))^{3-\frac{6}{a}}.\end{aligned}$$

When $\nu_{xh}(A) \ge h$, then $\Upsilon_{xh}(2) = h^2$, which implies that

$$\begin{aligned}(nh)^3 \,\mathrm{E}\left\{\left|\widetilde{\mathbb{H}}_{xh}(A)\right|^6\right\} &\le nh\left\{\nu_{xh}(A) + 5h^{\frac{5a-30}{a}}\right\} \\ &\quad + n^2h^2\left\{\nu_{xh}(A)^2 + h^{\frac{a-6}{a}} + \nu_{xh}(A)h^{1-\frac{2}{a}} + h^{2-\frac{4}{a}}\right\} \\ &\quad + n^3h^3\left\{\nu_{xh}(A)^3 + h^{\frac{a-2}{a}}\left(\nu_{xh}^2 + \nu_{xh}(A)h^{\frac{a-2}{a}} + h^{\frac{2a-4}{a}}\right)\right\} \\ &\le nh\left\{\nu_{xh}(A) + 5h^{\frac{5a-30}{a}}\right\} + n^2h^2\left\{h^{\frac{a-6}{a}} + 2\,\nu_{xh}(A)^{2-\frac{4}{a}}\right\} \\ &\quad + 2n^3h^3\nu_{xh}(A)^{3-\frac{6}{a}}.\end{aligned}$$

This proves the inequality in (1). From there, note that under Assumption $(\mathcal{H})$, one has for any $i \in \{j : K_{xh}(j) \neq 0\}$ that for z_i between X_i and x,

$$\nu_{X_i}(A) = \nu_x(A) + \dot{\nu}_x(A)(X_i - x) + \frac{1}{2}\,\ddot{\nu}_{z_i}(A)(X_i - x)^2.$$

From assumption $(\mathcal{LL})$, one can then deduce that

$$\begin{aligned}\mathrm{E}\left\{K_{xh}(1)\right\} &= \{1 + o(1)\} \times h\, f_X(x), \\ \mathrm{E}\left\{(X_1 - x)\, K_{xh}(1)\right\} &= \{1 + o(1)\} \times h^3 \dot{f}_X(x), \\ \mathrm{E}\left\{(X_1 - x)^2\, K_{xh}(1)\right\} &= \{1 + o(1)\} \times h^3\, f_X(x) \int_{\mathbb{R}} u^2 K(u)\,\mathrm{d}u.\end{aligned}$$

Since Assumption $(\mathcal{H})$ implies that $|\ddot{\nu}_{z_i}(A)|$ is uniformly bounded on J_x, one can find a positive constant ζ'' such that

$$\nu_{xh}(A) = \frac{1}{h}\,\mathrm{E}\left\{\nu_{X_1}(A)\, K_{xh}(1)\right\} \le \zeta''\left\{\nu_x(A) + h^2\right\}.$$

Combining this with (1) yields that for some $\omega'' > 0$,

$$\begin{aligned}\mathrm{E}\left[\{\mathbb{H}_{xh}(A)\}^6\right] \le \omega''\Bigg\{&\frac{\nu_x(A) + h^2 + h^{\frac{5a-30}{a}}}{n^2h^2} + \frac{\nu_{xh}(A)^{2-\frac{4}{a}} + h^{\frac{a-6}{a}} + h^{4-\frac{8}{a}}}{nh} \\ &+ \nu_{xh}(A)^{3-\frac{6}{a}} + h^{6-\frac{12}{a}}\Bigg\}.\end{aligned}$$

Since $a > 6$, it suffices to choose b such that $0 < b < \min\{(a-6)/a, 2/5\}$ to conclude the proof.

2.3 Proof of Lemma 3

First define for $y \in \mathbb{R}$,

$$\underline{y}_\gamma^{(1)} = \max\left\{\zeta \in T_\gamma^{(1)} : \zeta \leq y\right\}, \quad \overline{y}_\gamma^{(1)} = \min\left\{\zeta \in T_\gamma^{(1)} : \zeta > y\right\},$$
$$\underline{y}_\gamma^{(2)} = \max\left\{\zeta \in T_\gamma^{(2)} : \zeta \leq y\right\}, \quad \overline{y}_\gamma^{(2)} = \min\left\{\zeta \in T_\gamma^{(2)} : \zeta > y\right\}.$$

With this notation, one can claim that with $\kappa_\gamma = \lfloor (nh)^{1/2+\gamma} \rfloor$,

$$F_{1x}\left(\overline{y}_\gamma^{(1)}\right) - F_{1x}\left(\underline{y}_\gamma^{(1)}\right) \leq \frac{1}{\kappa_\gamma} \quad \text{and} \quad F_{2x}\left(\overline{y}_\gamma^{(2)}\right) - F_{2x}\left(\underline{y}_\gamma^{(2)}\right) \leq \frac{1}{\kappa_\gamma}.$$

Under Assumptions $(\mathcal{S})$, $(\mathcal{LL})$ and $(\mathcal{N})$, Lemma 1 entails that $\mathcal{K}_{xn} \geq 0$ almost surely for n taken sufficiently large. Hence, for any $\omega = (y, z) \in \mathbb{R}^2$, one has for $\underline{\omega}_\gamma = (\underline{y}_\gamma^{(1)}, \underline{z}_\gamma^{(2)})$ and $\overline{\omega}_\gamma = (\overline{y}_\gamma^{(1)}, \overline{z}_\gamma^{(2)})$ that almost surely,

$$Z_{xn}(\omega) - Z_{xn}(\underline{\omega}_\gamma) \leq Z_{xn}(\overline{\omega}_\gamma) - Z_{xn}(\underline{\omega}_\gamma) + \frac{1}{\sqrt{nh}} \sum_{i=1}^n \mathcal{K}_{xn}\left(\frac{X_i - x}{h}\right)\left\{H_{X_i}(\overline{\omega}_\gamma) - H_{X_i}(\underline{\omega}_\gamma)\right\}.$$

It will be shown in the sequel that the second summand on the right-hand side of the previous inequality is asymptotically negligible. Using the notation $Z_{xh}^\star = \sqrt{nh}(\bar{H}_{xh} - H_x)$ as in the proof of Proposition 1, one has under Assumptions $(\mathcal{H})$ and $(\mathcal{N})$ that

$$\begin{aligned}
\frac{1}{\sqrt{nh}} \sum_{i=1}^n \mathcal{K}_{xn}\left(\frac{X_i - x}{h}\right)&\left\{H_{X_i}(\overline{\omega}_\gamma) - H_{X_i}(\underline{\omega}_\gamma)\right\} \\
&= \sqrt{nh}\left\{H_x(\overline{\omega}_\gamma) - H_x(\underline{\omega}_\gamma)\right\} + \left\{Z_{xh}^\star(\overline{\omega}_\gamma) - Z_{xh}^\star(\underline{\omega}_\gamma)\right\} \\
&= \sqrt{nh}\left\{H_x(\overline{\omega}_\gamma) - H_x(\underline{\omega}_\gamma)\right\} \\
&\quad + \frac{1}{2}\left\{\ddot{H}_x(\overline{\omega}_\gamma) - \ddot{H}_x(\underline{\omega}_\gamma)\right\}\left(\int_{\mathbb{R}} u^2 K(u)\, \mathrm{d}u\right)\{1 + o_{as}(1)\},
\end{aligned}$$

where the last equality follows from Equation (7) in Bouezmarni et al (2019). Now since for any bivariate distribution function H with marginals F_1 and F_2, $|H(y_1, z_1) - H(y_2, z_2)| \leq |F_1(y_1) - F_1(y_2)| + |F_2(z_1) - F_2(z_2)|$, one deduces

$$\sqrt{nh}\left|H_x(\overline{\omega}_\gamma) - H_x(\underline{\omega}_\gamma)\right| \leq \sqrt{nh}\,\rho(\overline{\omega}_\gamma, \underline{\omega}_\gamma).$$

Because $\ddot{H}_x$ is uniformly continuous, one obtains that uniformly in $\omega \in \mathbb{R}^2$,

$$Z_{xn}(\omega) - Z_{xn}(\underline{\omega}_\gamma) \leq Z_{xn}(\overline{\omega}_\gamma) - Z_{xn}(\underline{\omega}_\gamma) + o(1)\{1 + o_{as}(1)\},$$
$$Z_{xn}(\underline{\omega}_\gamma) - Z_{xn}(\omega) \leq Z_{xn}(\overline{\omega}_\gamma) - Z_{xn}(\underline{\omega}_\gamma) + o(1)\{1 + o_{as}(1)\}.$$

As a consequence, for any $\omega_1, \omega_2 \in \mathbb{R}^2$,

$$
\begin{aligned}
|Z_{xn}(\omega_1) - Z_{xn}(\omega_2)| \le & \left| Z_{xn}(\overline{\omega_1}_\gamma) - Z_{xn}(\underline{\omega_1}_\gamma) \right| \\
& + \left| Z_{xn}(\overline{\omega_2}_\gamma) - Z_{xn}(\underline{\omega_2}_\gamma) \right| \\
& + \left| Z_{xn}(\underline{\omega_1}_\gamma) - Z_{xn}(\underline{\omega_2}_\gamma) \right| \\
& + o(1) \left\{ 1 + o_{as}(1) \right\}.
\end{aligned}
$$

Since it is assumed that $\delta > 2\kappa_\gamma^{-1}$, the fact that $\rho(\omega_1, \omega_2) < \delta$ implies $\rho(\underline{\omega_1}_\gamma, \underline{\omega_2}_\gamma) < 2\delta$. It follows that $\mathfrak{W}_\delta(Z_{xn}, \mathbb{R}^2) \le 3\,\mathfrak{W}_{2\delta}(Z_{xn}, T_\gamma)$. Therefore, for any $\epsilon > 0$,

$$
\mathbb{P}\left\{ \mathfrak{W}_\delta(Z_{xn}, \mathbb{R}^2) > \epsilon \right\} \le \mathbb{P}\left\{ \mathfrak{W}_{2\delta}(Z_{xn}, T_\gamma) > \frac{\epsilon}{3} \right\}.
$$

2.4 Proof of Lemma 4

In the sequel, since the cases $j = 1$ and $j = 2$ are identical, the index j is omitted; also, g will replace h_j in order to ease readability. Under Assumptions $(\mathcal{S})$, $(\mathcal{H}^\star)$, $(\mathcal{LL})$ and $(\mathcal{N}^\star)$, theorem 6 of Masry and Fan (1997) ensures that for any fixed $(t, u) \in \mathcal{I} = [-1, 1] \times [0, 1]$, the random variable $\widetilde{Z}_{xn}(t, u)$ is asymptotically normal; hence $\widetilde{Z}_{xn}(t, u)$ is asymptotically tight in $\mathbb{R}$. It remains to show the asymptotic tightness of the sequence $\widetilde{Z}_{xn}$ in $\ell^\infty(\mathcal{I})$. To this end, one uses Lemma 1 to deduce that

$$
\begin{aligned}
\widetilde{Z}_{xn}(t, u) = & \left\{ 1 + o_{as}(1) \right\} \\
& \times \frac{1}{\sqrt{ng}\, f_X(z_t)} \sum_{i=1}^{n} \left[\mathbb{I}\left\{ Y_i \le F_{z_t}^{-1}(u) \right\} - F_{X_i}\left\{ F_{z_t}^{-1}(u) \right\} \right] K\left(\frac{X_i - z_t}{g} \right).
\end{aligned}
$$

Since $f_X(z) > 0$ for $z \in J_x$, it follows that $\widetilde{Z}_{xn}$ is asymptotically tight if and only if the same holds for the sequence $\widetilde{W}_{xn}(t, u) = \sqrt{ng}\, \widetilde{\mathcal{E}}_g(u, z_t, z_t)$, where $\widetilde{\mathcal{E}}_g = \mathcal{E}_g - \overline{\mathcal{E}}_g$, with

$$
\begin{aligned}
\mathcal{E}_g(u, z, w) &= \frac{1}{ng} \sum_{i=1}^{n} \mathbb{I}(Y_i \le F_z^{-1}(u)\} \, K\left(\frac{X_i - w}{g} \right), \\
\overline{\mathcal{E}}_g(u, z, w) &= \frac{1}{ng} \sum_{i=1}^{n} F_{X_i}\{F_z^{-1}(u)\} \, K\left(\frac{X_i - w}{g} \right).
\end{aligned}
$$

For $\rho(t, u, t', u') = |t - t'| + |u - u'|$, any bounded function $f : \mathcal{I} \to \mathbb{R}$ and $T \subset \mathcal{I}$, define

$$
\mathfrak{W}_\delta(f, T) = \sup_{\substack{(t,u),(t',u') \in T \\ \rho(t,u,t',u') < \delta}} |f(t, u) - f(t', u')|.
$$

It will next be shown that $\widetilde{W}_{xn}$ is asymptotically ρ-equicontinuous, *i.e.* for any $\epsilon > 0$,

$$\lim_{\delta \downarrow 0} \lim_{n\to\infty} \mathbb{P}\left(\mathfrak{W}_\delta(\widetilde{W}_{xn}, \mathcal{I}) > \epsilon \right) = 0.$$

Letting $\kappa_\gamma = \lfloor (ng)^{1/2+\gamma} \rfloor$ for some $\gamma \in (0, 1/2)$, define $T_\gamma = J_\gamma \times I_\gamma$, where

$$I_\gamma = \left\{ 0, \frac{1}{\kappa_\gamma}, \ldots, \frac{\kappa_\gamma - 1}{\kappa_\gamma}, 1 \right\} \quad \text{and} \quad J_\gamma = \left\{ 0, \pm\frac{1}{\kappa_\gamma}, \ldots, \pm\frac{\kappa_\gamma - 1}{\kappa_\gamma}, \pm 1 \right\}.$$

For any $(t, u) \in \mathcal{I}$, one can proceed as in the proof of Lemma 3 in defining $(\underline{t}_\gamma, \underline{u}_\gamma)$ and $(\overline{t}_\gamma, \overline{u}_\gamma)$, and by writing

$$\begin{aligned} \widetilde{W}_{xn}(t,u) - \widetilde{W}_{xn}(\underline{t}_\gamma, \underline{u}_\gamma) &= \sqrt{ng}\, \widetilde{\mathcal{E}}_g(u, z_t, z_t) - \widetilde{\mathcal{E}}_g(\underline{u}_\gamma, z_{\underline{t}_\gamma}, \underline{t}_\gamma) \\ &\le \sqrt{ng}\, \widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \sqrt{ng}\, \widetilde{\mathcal{E}}_g(\underline{u}_\gamma, z_{\underline{t}_\gamma}, \underline{t}_\gamma) \\ &\quad + \sqrt{ng} \left\{ \overline{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \overline{\mathcal{E}}_g(\underline{u}_\gamma, z_t, z_t) \right\}. \end{aligned} \tag{10}$$

Starting with the second summand on the right hand side of (10), a Taylor expansion of F_{X_i} around z_t yields that for r_{ti} lying between z_t and X_i,

$$\begin{aligned} &\sqrt{ng} \left\{ \overline{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \overline{\mathcal{E}}_g(\underline{u}_\gamma, z_t, z_t) \right\} \\ &\qquad = \sqrt{ng} \left(\overline{u}_\gamma - \underline{u}_\gamma \right) S_{n,0}(z_t) \\ &\qquad\quad + \sqrt{ng} \left[\dot{F}_{z_t} \left\{ F_{z_t}^{-1}(\overline{u}_\gamma) \right\} - \dot{F}_{z_t} \left\{ F_{z_t}^{-1}(\overline{u}_\gamma) \right\} \right] g\, S_{n,1}(z_t) \\ &\qquad\quad + \frac{\sqrt{ng^5}}{2ng} \sum_{i=1}^n \left[\ddot{F}_{r_{ti}} \left\{ F_{z_t}^{-1}(\overline{u}_\gamma) \right\} - \ddot{F}_{r_{ti}} \left\{ F_{z_t}^{-1}(\underline{u}_\gamma) \right\} \right] \\ &\qquad\qquad\qquad \times \left(\frac{X_i - z_t}{g} \right)^2 K \left(\frac{X_i - z_t}{g} \right). \end{aligned}$$

Using Assumptions $(\mathcal{S})$, $(\mathcal{LL})$ and $(\mathcal{N}^\star)$, one deduces from Corollary 1 of Masry (1996) that $S_{n,0}(z) = O_{\text{a.s.}}(1)$, $S_{n,1}(z) = O_{as} \left\{ g + (ng)^{-1/2} \log(n) \right\}$ and $S_{n,2}(z) = O_{as}(1)$, uniformly in $z \in J_x$. Therefore, using Assumption $(\mathcal{H}^\star)$,

$$\begin{aligned} \sqrt{ng} \left\{ \overline{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \overline{\mathcal{E}}_g(\underline{u}_\gamma, z_t, z_t) \right\} &= \sqrt{ng}(\overline{u}_\gamma - \underline{u}_\gamma) + o(1)\, O_{as}(g \log(n)) \\ &\quad + o(1)\, O_{as}(\sqrt{ng^5}) \\ &= o_{as}(1). \end{aligned}$$

The last equality is a consequence of $\sqrt{ng^5} < \infty$ and $g \log(n) \to 0$ as $n \to \infty$, and the fact that the definition of the grids entails $\sqrt{ng}(\overline{u}_\gamma - \underline{u}_\gamma) = O\{(ng)^{-\gamma}\}$. This ensures the asymptotic negligibility of $\sqrt{ng}\{\overline{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \overline{\mathcal{E}}_g(\underline{u}_\gamma, z_t, z_t)\}$.

For $\sqrt{ng}\, \widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t)$, first note that since $z_t - z_{\underline{t}_\gamma} = Ch(t - \underline{t}_\gamma)$,

$$
\begin{aligned}
&\sqrt{ng}\left|\widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_{\underline{t}_\gamma})\right| \\
&\qquad \le \frac{1}{\sqrt{ng}}\left|\sum_{i=1}^n K\left(\frac{X_i - z_t}{g}\right) - K\left(\frac{X_i - z_{\underline{t}_\gamma}}{g}\right)\right| \\
&\qquad \le \left\{\sup_{z\in I_x}\frac{1}{ng}\sum_{i=1}^n\left|K'\left(\frac{X_i - z}{g}\right)\right|\right\} C\sqrt{ng}g^{-1}h(t-\underline{t}_\gamma).
\end{aligned}
$$

Mimicking the proof of Corollary 1 in Masry (1996) and using the fact that K has a bounded second order derivative, Assumptions $(\mathcal{S})$–$(\mathcal{LL})$ entail

$$
\sup_{z\in I_x}\frac{1}{ng}\sum_{i=1}^n\left|K'\left(\frac{X_i - z}{g}\right)\right| = O_{as}(1).
$$

Since $h/g < \infty$, it follows that

$$
\sqrt{ng}\left|\widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_{\underline{u}_\gamma})\right| = O_{\text{a.s.}}\{(ng)^{-\gamma}g^{-1}h\} = o_{\text{a.s.}}(1).
$$

Proceeding similarly, $\sqrt{ng}|\widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_{\overline{t}_\gamma})| = o_{\text{a.s.}}(1)$, so that

$$
\sqrt{ng}\max\left\{\left|\widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_{\underline{t}_\gamma})\right|, \left|\widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_{\overline{t}_\gamma})\right|\right\} \quad (11)
$$

is $o_{as}(1)$ for any $t \in [0,1]$. Now since by Assumption $(\mathcal{H}^\star)$, one has that F_z^{-1} is continuous in a neighborhood of x, one can find a constant $\eta > 0$ such that for n sufficiently large, one has uniformly in u that

$$
\begin{aligned}
F_{z_t}^{-1}(u) &\le \min\left\{F_{z_{\underline{t}_\gamma}}^{-1}(u + \eta(z_t - z_{\underline{t}_\gamma})), F_{z_{\overline{t}_\gamma}}^{-1}(u + \eta(z_{\overline{t}_\gamma} - z_t))\right\} \\
&\le \min\left\{F_{z_{\underline{t}_\gamma}}^{-1}(\overline{u}_\gamma), F_{z_{\overline{t}_\gamma}}^{-1}(\overline{u}_\gamma)\right\}.
\end{aligned}
$$

Note that the fact that $\max\{z_{\overline{t}_\gamma} - z_t, z_t - z_{\underline{t}_\gamma}\} \le C\,h\,\kappa_\gamma^{-1} \le \kappa_\gamma^{-1}$ for h sufficiently small has been used. Therefore,

$$
\left|\sqrt{ng}\,\widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) - \sqrt{ng}\,\widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_{\underline{t}_\gamma}, z_t)\right| \le \sqrt{ng}\left|\overline{\mathcal{E}}_g(\overline{u}_\gamma, z_{\overline{t}_\gamma}, z_t) - \overline{\mathcal{E}}_g(\overline{u}_\gamma, z_{\underline{t}_\gamma}, z_t)\right|.
$$

From similar computations as those yielding the negligibility of the third term of Equation (10), $\sqrt{ng}|\overline{\mathcal{E}}_g(\overline{u}_\gamma, z_{\overline{t}_\gamma}, z_t) - \overline{\mathcal{E}}_g(\overline{u}_\gamma, z_{\underline{t}_\gamma}, z_t)| = o_{\text{a.s.}}(1)$. One then deduces from (11) that

$$
\sqrt{ng}\,\widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_t, z_t) = \sqrt{ng}\,\widetilde{\mathcal{E}}_g(\overline{u}_\gamma, z_{\underline{t}_\gamma}, z_{\underline{t}_\gamma}) + o_{\text{a.s.}}(1) = \widetilde{W}_{xn}(\underline{t}_\gamma, \overline{u}_\gamma) + o_{\text{a.s.}}(1).
$$

Combining this with Equation (10) allows to deduce that for n sufficiently large, $\mathfrak{W}_\delta(\widetilde{W}_{xn}, \mathcal{I}) \le 6\mathfrak{W}_{2\delta}(\widetilde{W}_{xn}, T_\gamma)$ almost surely. In order to conclude the proof, Lemma 2 of Balacheff and Dupont (1980) will be used in a similar manner as in the proof of Proposition 1. To this end, Lemma 5 stated below and whose proof is given in Section 3 will prove useful. Before stating it, define for any rectangle $A = [u, u'] \times [t, t] \subset \mathcal{I}$,

$$\widetilde{W}_{xn}(A) = \widetilde{W}_{xn}(t', u') - \widetilde{W}_{xn}(t, u') - \widetilde{W}_{xn}(t', u) + \widetilde{W}_{xn}(t, u).$$

Let also $\mu(A) = |u' - u| \times |t - t'|$. The announced result finally follows from an application of Lemma 2 of Balacheff and Dupont (1980) combined with similar arguments as those in the proof of Proposition 1.

Lemma 5 *Under Assumptions ($\mathcal{S}$), ($\mathcal{H}^\star$), ($\mathcal{LL}$) and ($\mathcal{N}^\star$), there exists a constant $\overline{\eta} > 0$ such that for n sufficiently large and any rectangle A whose corner points are all distinct and lie in T_γ, one has for β that satisfies $0 < \beta < \min\{(a-6)/16, 1/40\}$ that*

$$\mathrm{E}\left\{\left|\widetilde{W}_{xn}(A)\right|^6\right\} \leq \overline{\eta}\mu(A)^{1+\beta}.$$

3 Proof of Lemma 5

First note that one can write

$$\widetilde{W}_{xn}(A) = \frac{1}{\sqrt{ng}} \sum_{i=1}^{n} \widetilde{\Omega}_i(A),$$

where $\widetilde{\Omega}_i = \Omega_i - \mathrm{E}(\Omega_i | X_i)$ and

$$\Omega_i(A) = \mathbb{I}\left\{Y_i \in \left]F_{z_{t'}}^{-1}(u), F_{z_{t'}}^{-1}(u')\right]\right\} K\left(\frac{X_i - z_{t'}}{g}\right) - \mathbb{I}\left\{Y_i \in \left]F_{z_t}^{-1}(u), F_{z_t}^{-1}(u')\right]\right\} K\left(\frac{X_i - z_t}{g}\right).$$

Since $|\Omega_i| \leq 2\|K\|_\infty$, one has for $\widetilde{\nu}_{xg}(A) = \mathrm{E}\{|\mathrm{E}(\Omega_i(A)|X_i)|\}/g$ and $p \in \{1, \ldots, 6\}$ that

$$\left|\mathrm{E}\left(\widetilde{\Omega}_i^p\right)\right| \leq (2\|K\|_\infty)^{p-1} h\widetilde{\nu}_{xg}(A)\Delta_1(p). \tag{12}$$

Moreover, from arguments similar as those in the proof of Lemma 2, one has for $k \in \{2, \ldots, 6\}$ and $i_1 \neq \cdots \neq i_k$ that for $\ell_1, \ldots, \ell_k \in \{1, \ldots, 6\}$,

$$\begin{aligned}
&\mathrm{E}\left\{\prod_{j=1}^{k}\left|\widetilde{\Omega}_{i_j}(A)\right|^{\ell_j}\right\} \\
&\leq \mathrm{E}\left[\prod_{i_j=1}^{k}\left\{K\left(\frac{X_i - z_t}{g}\right) + K\left(\frac{X_i - z_{t'}}{g}\right)\right\}^{\ell_j}\right] \\
&\leq (2\|K\|_\infty)^{k-1} g^k \int_{\mathbb{R}^k} \prod_{j=1}^{k} K(u_j)^{\ell_j} \times \left\{f_X^{(i_2,\ldots,i_k)}(g\mathbf{u} + z_t) + f_X^{(i_2,\ldots,i_k)}(g\mathbf{u} + z_t')\right\} \mathrm{d}\mathbf{u} \\
&= O(g^k).
\end{aligned}$$

This inequality combined with Equation (12) allow to mimic the proof of Lemma 2 (replacing $\nu_{xg}(A)$ by $\widetilde{\nu}_{xg}(A)$) and deduce that there exists a finite constant $\omega' > 0$ such that

$$\mathrm{E}\left\{\left|\widetilde{W}_{xn}(A)\right|^6\right\} \le \omega' \left\{\frac{\widetilde{\nu}_{xg}(A) + g^{\frac{5a-30}{a}}}{n^2g^2} + \frac{g^{\frac{a-6}{a}} + \widetilde{\nu}_{xg}(A)^{2-\frac{4}{a}}}{ng} + \widetilde{\nu}_{xg}(A)^{3-\frac{6}{a}}\right\}. \tag{13}$$

Under Assumption $(\mathcal{H}^\star)$, a Taylor expansion around $z_{t'}$ and z_t yields that for $z_{1i}^\star$ between $z_{t'}$, X_i and $z_{2i}^\star$ between z_t, X_i,

$$\begin{aligned}
\mathrm{E}\{\Omega_i(A)|X_i\} &= \left[F_{X_i}\left\{F_{z_{t'}}^{-1}(u')\right\} - F_{X_i}\left\{F_{z_{t'}}^{-1}(u)\right\}\right] K\left(\frac{X_i - z_{t'}}{g}\right) \\
&\quad - \left[F_{X_i}\left\{F_{z_t}^{-1}(u')\right\} - F_{X_i}\left\{F_{z_t}^{-1}(u)\right\}\right] K\left(\frac{X_i - z_t}{g}\right) \\
&= (u-u')\left\{K\left(\frac{X_i - z_{t'}}{g}\right) - K\left(\frac{X_i - z_t}{g}\right)\right] \\
&\quad + \left[\dot{F}_{z_{t'}}\left\{F_{z_{t'}}^{-1}(u')\right\} - \dot{F}_{z_{t'}}\left\{F_{z_{t'}}^{-1}(u)\right\}\right](X_i - z_{t'})\, K\left(\frac{X_i - z_{t'}}{g}\right) \\
&\quad - \left[\dot{F}_{z_t}\left\{F_{z_t}^{-1}(u')\right\} - \dot{F}_{z_t}\left\{F_{z_t}^{-1}(u)\right\}\right](X_i - z_t)\, K\left(\frac{X_i - z_t}{g}\right) \\
&\quad + \left[\ddot{F}_{z_{1i}^\star}\left\{F_{z_{t'}}^{-1}(u')\right\} - \ddot{F}_{z_{1i}^\star}\left\{F_{z_{t'}}^{-1}(u)\right\}\right](X_i - z_{t'})^2\, K\left(\frac{X_i - z_{t'}}{g}\right) \\
&\quad - \left[\ddot{F}_{z_{2i}^\star}\left\{F_{z_t}^{-1}(u')\right\} - \ddot{F}_{z_{2i}^\star}\left\{F_{z_t}^{-1}(u)\right\}\right](X_i - z_t)^2\, K\left(\frac{X_i - z_t}{g}\right).
\end{aligned}$$

Since $z_t = x + Cth$ and because Assumption $(\mathcal{LL})$ ensures that K' is bounded and compactly supported, the fact that $h/g < \infty$ as $n \to \infty$ (a consequence of Assumption $(\mathcal{N}^\star)$) entails

$$\begin{aligned}
\mathrm{E}\left\{\left|K\left(\frac{X_i - z_{t'}}{h}\right) - K\left(\frac{X_i - z_t}{h}\right)\right|\right] &= g\int\left|\int_{Cth/g}^{Ct'h/g} K'(z-u)\,\mathrm{d}u\right| f_X(x+gz)\,\mathrm{d}z \\
&\le \eta g|t-t'|
\end{aligned}$$

for some $\eta > 0$. Moreover, since $u \mapsto \dot{F}_z\{F_z^{-1}(u)\}$ is Lipchitz continuous and since by straightforward computations,

$$\mathrm{E}\left\{|X_i - z|K\left(\frac{X_i - z_t}{g}\right)\right\} = O(g^2) \quad \text{and} \quad \mathrm{E}\left\{(X_i - z)^2K\left(\frac{X_i - z_t}{g}\right)\right\} = O(g^3),$$

one obtains that there exists $\eta' > 0$ such that

$$\widetilde{\nu}_{xg}(A) \le \eta'\left\{(u-u')(t-t') + g|u-u'| + g^2\right\}.$$

Plugging it into (13) entails that for any $0 < b < \min\{(a-6)/a, 2/5\}$,

$$\mathrm{E}\left\{\left|\widetilde{W}_{xn}(A)\right|^6\right\} \leq \omega'' \left\{\frac{\mu(A)}{n^2g^2} + \frac{\mu(A)^{2-\frac{4}{a}}}{ng} + \mu(A)^{3-\frac{6}{a}} + \mathcal{J}_n(g,b) + \frac{g|u-u'|}{n^2g^2} + \frac{g^{2-\frac{4}{a}}|u-u'|^{2-\frac{4}{a}}}{ng} + g^{3-\frac{6}{a}}|u-u'|^{3-\frac{6}{a}}\right\}.$$

Note that Assumption ($\mathcal{N}^\star$) ensures that ng^5 is bounded above by some positive constant cst as $n \to \infty$. Also, since $A = [u,u'] \times [t,t']$, where $\min(u'-u, t'-t) \geq (ng)^{-1/2+\gamma}$, and because $\mu(A) \leq 2$, it follows that for $0 < \beta < \min\{(a-5)/a, 1/8\}$ and $\gamma \in (0,1/4)$,

$$\begin{aligned}\mathrm{E}\left\{\left|\widetilde{W}_{xn}(A)\right|^6\right\} &\leq \mathrm{cst}\left\{\frac{\mu(A)}{n^2g^2} + \frac{\mu(A)^{2-\frac{4}{a}}}{ng} + \mu(A)^{3-\frac{6}{a}} + \mathcal{J}_n(g,b) + \frac{g(ng)^{1/2+\gamma}\mu(A)}{n^2g^2} + \frac{g^{2-\frac{4}{a}}(ng)^{(\frac{1}{2}+\gamma)(1+\beta)}\mu(A)^{1+\beta}}{ng} + g^{3-\frac{6}{a}}(ng)^{(\frac{1}{2}+\gamma)(1+\beta)}\mu(A)^{1+\beta}\right\}\\
&\leq \mathrm{cst}\mu(A)^{1+\beta}\left\{1 + \frac{1+g(ng)^{1/2+\gamma}}{n^2g^2} + \frac{1+g^{2-\frac{4}{a}}(ng)^{(\frac{1}{2}+\gamma)(1+\beta)}}{ng} + g^{3-\frac{6}{a}}(ng)^{(\frac{1}{2}+\gamma)(1+\beta)} + (ng)^{(1+2\gamma)(1+\beta)}\mathcal{J}_n(g,b)\right\}\\
&\leq \mathrm{cst}\mu(A)^{1+\beta}\left\{1 + g^{1-\frac{6}{a}}g^{-2\gamma(1+\beta)-2\beta} + (ng)^{(1+2\gamma)(1+\beta)}\mathcal{J}_n(g,b)\right\}.\end{aligned}$$

Since $ng^5 < \infty$ as $n \to \infty$, an argument identical to that leading to Equation (10) in Bouezmarni et al (2019) allows to deduce that taking $\gamma, \beta \in (0, b/16)$ implies $(ng)^{(1+2\gamma)(1+\beta)}\mathcal{J}_n(g,b) < 1$ for n sufficiently large. This choice of β, γ also entails $g^{1-\frac{6}{a}}g^{-2\gamma(1+\beta)-2\beta} < 1$ as $n \to \infty$, completing the proof.

References

Balacheff S, Dupont G (1980) Normalité asymptotique des processus empiriques tronqués et des processus de rang (cas multidimensionnel mélangeant). In: Nonparametric asymptotic statistics (Proc. Conf., Rouen, 1979) (French), Lecture Notes in Math., vol 821, Springer, Berlin, pp 19–45

Bouezmarni T, Camirand Lemyre F, Quessy JF (2019) On the large-sample behavior of two estimators of the conditional copula under serially dependent data. To appear in Metrika

Dehling H, Philipp W (2002) Empirical Process Techniques for Dependent Data, Birkhäuser Boston, Boston, MA, pp 3–113

Masry E (1996) Multivariate local polynomial regression for time series: uniform strong consistency and rates. Journal of Time Series Analysis 17(6):571

Masry E, Fan J (1997) Local polynomial estimation of regression functions for mixing processes. Scandinavian Journal of Statistics (2):165