Orthogonality in Partial Abelian Monoids and Applications

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Abstract. In this paper we introduce the notion of orthogonality for arbitrary families of elements of a partial abelian monoid, we consider the special case of an effect algebra and we study, as an application, the existence of the support of an uniform semigroup valued measure on an effect algebra.

1. Introduction

The basic algebraic structure considered in our paper is a partial abelian monoid, which is a partial universal algebra endowed with a nullary operation and an associative and commutative binary partial operation. It is worth noticing that these structures are studied from the algebraic standpoint in the papers [10], [24], [39] and [40] and more extensively in the monography [38].

Our interest in a partial abelian monoid L is due to the fact that, for arbitrary families of elements of L, it can be defined an orthogonality structure with nice properties and allowing to say rigorously when L is $\mathfrak{X}$-complete for every infinite cardinal number $\mathfrak{X}$. We consider, in particular, the orthogonality when L is an effect algebra, we extend easily to L the notion of difference set introduced for orthoalgebras in [14] and [32] and we generalize its main properties. Since L carry also orthogonally additive measures and provide new results in the rapidly developing field of non-commutative measure theory (see [9] and [18]), we consider, as an application, the existence of the support of a positive measure on L with values in an ordered Hausdorff uniform semigroup.

The paper is organized as follows.

In Section 2 we give an account of the basic theory of partial abelian monoids with several non-trivial examples and a brief presentation of some of its substructures: effect algebras, orthoalgebras, orthomodular posets, orthomodular lattices and Boolean algebras. Section 3 studies orthogonal suites in a partial abelian monoid using some ideas of Bourbaki [5] and we deduce several properties, some of them appearing for an special case in [12], and another, like the generalized associative law and the generalized commutative law, appearing in [5] for a commutative semigroup. In Section 4 we present the orthogonality for infinite families of elements of a partial abelian monoid L whose algebraic preordering is antisymmetric, we establish its main properties and we discuss also the $\mathfrak{X}$-completeness of L for

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any infinite cardinal number $\mathfrak{X}$. Section 5 concerns the orthogonality in an effect algebra and, since the compatibility is very difficult to verify in an effect algebra, we use instead an extension of the notion of difference set defined in [14] and [32] for an orthoalgebra, we generalize its main properties and we deduce a relation when the effect algebra is $\mathfrak{X}$-complete that allow us to generalize the important Theorem 4.14 of [21]. Finally, in Section 6, after presenting a basic theory of uniform semigroup and some pertinent classes of measures, we define following [33] the notion of support of a positive measure on an effect algebra with values in an ordered Hausdorff uniform semigroup and we establish a twofold generalization of the main results of [29] for an ordered metrizable topological group in the first statement.

2. Preliminaries

For the terminology, notation and details concerning orderings, lattices, universal algebras and abelian groups we refer to [3], [8], [17], [19], [20] and [34].

Let L be a set containing at least one element 0 called *zero* of L, let $\perp$ be a symmetric binary relation on L and let $(a,b) \to a \oplus b$ be a partial binary operation on L of domain $\perp$. Then the partial universal algebra $\langle L; \oplus, 0\rangle$ is called a *partial abelian monoid* if the following properties hold:

(Commutative Law) If $a, b \in L$ and $a \perp b$, then $a \oplus b = b \oplus a$.
(Associative Law) If $a, b, c \in L$, $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(Identity Law) If $a \in L$, then $a \perp 0$ and $a \oplus 0 = a$.

We note that the identity law implies that $\perp$ is not empty. If $a, b \in L$ and $a \perp b$, we say that a and b are *orthogonal*. If the hypotheses of the associative law are verified, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L. Clearly, every commutative monoid $\langle M; +, 0\rangle$ is a partial abelian monoid where $\perp = M \times M$ and $\oplus = +$. For nice examples of commutative monoids see [22, Section 8.4, pp. 259–261].

Henceforth, $L = \langle L; \oplus, 0\rangle$ denotes a partial abelian monoid.

L can be endowed with its *algebraic preordering* $\leq_{\text{alg}}$ defined by the formula

$$a \leq_{\text{alg}} b \iff \text{There exists } x \in L \text{ such that } a \perp x \text{ and } a \oplus x = b.$$

Clearly, 0 is the least element of $\langle L; \leq_{\text{alg}}\rangle$.

Lemma 2.1. *Let $a, b, c, d \in L$. Then*

a) $a \leq_{\text{alg}} b$ *and* $c \perp b \Rightarrow c \perp a$ *and* $a \oplus c \leq_{\text{alg}} b \oplus c$.
b) $a \leq_{\text{alg}} b$, $c \leq_{\text{alg}} d$ *and* $b \perp d \Rightarrow a \perp c$ *and* $a \oplus c \leq_{\text{alg}} b \oplus d$.

Proof. a) Since $a \leq_{\text{alg}} b$, there exists $x \in L$ such that $a \perp x$ and $a \oplus x = b$. Then $c \perp (a \oplus x)$. By the associative law, we have $c \perp a$, $(c \oplus a) \perp x$ and $c \oplus (a \oplus x) = (c \oplus a) \oplus x$, and therefore $c \oplus b = (c \oplus a) \oplus x$. Hence $c \oplus a \leq_{\text{alg}} c \oplus b$.

b) By a double application of part a). □

Let $u \in L$. Then we say that

(1) u is a *unit* in L if $a \leq_{\text{alg}} u$ for every $a \in L$.

(2) u is *cancellable* in L if, for all $a, b \in L$ such that $a \perp u$, $b \perp u$ and $a \oplus u = b \oplus u$, we have $a = b$.
(3) u is *directly finite* in L if, for every $a \in L$ such that $a \perp u$ and $a \oplus u = u$, we have $a = 0$. (This is weaker than to say that u is cancellable in L.)

With respect to the partial abelian monoid L we say that

(1) L is *unital* if L possess a distinguished unit denoted by 1, and then $a \leq_{\text{alg}} 1$ for every $a \in L$.
(2) L is *cancellative* if every element of L is cancellable in L.
(3) L is *conical* if, for all $a, b \in L$ such that $a \perp b$ and $a \oplus b = 0$, we have $a = 0$ (and therefore $b = 0$).

Lemma 2.2. *Assume that L is conical and every element of L is directly finite. Then*

a) *$\langle L; \leq_{\text{alg}} \rangle$ is a partially ordered set.*
b) *If $a, b, c, d \in L$, $a \leq_{\text{alg}} b$, $c \leq_{\text{alg}} d$, $b \perp d$ and $a \oplus c = b \oplus d$, then $a = b$ and $c = d$.*

Proof. a) It follows from [24, Lemma 2.3].

b) It follows from Lemma 2.1 b) that $a \perp c$. Since $a \leq_{\text{alg}} b$ and $c \leq_{\text{alg}} d$, there exist $x, y \in L$ such that $a \perp x$, $c \perp y$, $a \oplus x = b$ and $c \oplus y = d$. Since $a \oplus c = b \oplus d$, we have $a \oplus c = (a \oplus x) \oplus (c \oplus y)$ and using the commutative and associative laws, we can write $a \oplus c = (x \oplus y) \oplus (a \oplus c)$. Since every element of L is directly finite, we get $x \oplus y = 0$ and therefore $x = y = 0$ because L is conical. So $a = b$ and $c = d$. □

Remark 2.3. We note that Lemma 2.1 b) and Lemma 2.2 b) contain [21, Lemma 2.1].

Corollary 2.4. *If L is conical and every element of L is directly finite, then L possesses at most one unit.*

Assume that L is cancellative. Then if $a, b \in L$ and $a \leq_{\text{alg}} b$ there exists a unique element $c \in L$ such that $a \perp c$ and $a \oplus c = b$. We write $c = b - a$ and we call c the *difference* between b and a. Clearly, $a - a = 0$ and $a - 0 = a$ for every $a \in L$. Moreover, if $a, b \in L$ and $a \leq_{\text{alg}} b$, then $a \perp (b - a)$ and $a \oplus (b - a) = b$.

Lemma 2.5. *Assume that L is cancellative and let $a, b, c \in L$. Then*

a) *$a \perp b$ and $a \oplus b \leq_{\text{alg}} c \Rightarrow b \leq_{\text{alg}} c$ and $a \leq_{\text{alg}} c - b$.*
b) *$a \leq_{\text{alg}} b$ and $c \leq_{\text{alg}} b - a \Rightarrow a \leq_{\text{alg}} b - c$.*
c) *$a \perp b \Rightarrow (a \oplus b) - a = b$.*

Proof. a) Since $b \leq_{\text{alg}} (a \oplus b)$, we have $b \leq_{\text{alg}} c$. Moreover, since $a \oplus b \leq_{\text{alg}} c$, there exists $x \in L$ such that $x \perp (a \oplus b)$ and $(a \oplus b) \oplus x = c$. By the commutative law, it follows that $(b \oplus a) \oplus x = c$. Then the associative law implies that $a \perp x$, $b \perp (a \oplus x)$ and $b \oplus (a \oplus x) = c$. Hence $a \oplus x = c - b$ and therefore $a \leq_{\text{alg}} c - b$.

b) Since $c \leq_{\text{alg}} b - a$, there exists $y \in L$ such that $y \perp c$ and $y \oplus c = b - a$. But $b = a \oplus (b - a)$. So $b = a \oplus (y \oplus c)$. By the associative law, we have $a \perp y$, $c \perp (a \oplus y)$ and $b = (a \oplus y) \oplus c$. Then $c \leq_{\text{alg}} b$ and $b - c = a \oplus y$ and this implies that $a \leq_{\text{alg}} b - c$.

c) Since $a \leq_{\text{alg}} (a \oplus b)$, we have $a \oplus b = a \oplus ((a \oplus b) - a)$ and by cancellation we get $b = (a \oplus b) - a$. $\square$

Example 2.6. Consider the real field $\langle \mathbb{R}; +, \cdot, 0, 1\rangle$ endowed with its usual total ordering $\leq$. Let

$$\perp = \{(a, b) \in \mathbb{R} \times \mathbb{R} : a = 0 \text{ or } b = 0\} \cup (]0, 1] \times]0, 1])$$

and define

$$a \oplus b = \frac{a + b}{1 + a \cdot b} \quad \text{if} \quad a \perp b.$$

Then it is easy to verify that $\langle \mathbb{R}; \oplus, 0\rangle$ is a partial abelian monoid which is conical and every element of $\mathbb{R}$ different of 1 is cancellable in $\langle \mathbb{R}; \oplus, 0\rangle$. Since

$$a, b \in \mathbb{R} \text{ and } a \leq_{\text{alg}} b \iff a = 0 \text{ or } a = b \text{ or } (a, b \in]0, 1] \text{ and } a < b)$$

it follows that $\langle \mathbb{R}; \leq_{\text{alg}}\rangle$ is a partially ordered set and $\langle \mathbb{R}; \oplus, 0\rangle$ is not unital.

Example 2.7. Consider the abelian group $\langle \mathbb{R}; +, 0\rangle$ endowed with its usual total ordering $\leq$ and let $+\infty$ be a symbol not belonging to $\mathbb{R}$. Let $M = \mathbb{R} \cup \{+\infty\}$ and we extend the binary operation $+$ on M by putting $a + (+\infty) = (+\infty) + a = +\infty$ for all $a \in M$. Then $M = \langle M; +, 0\rangle$ is a commutative monoid which is unital with unit $+\infty$ and every element of $\mathbb{R}$ is cancellable in M, but M is not conical.

Example 2.8. Let $G = \langle G; +, 0\rangle$ be a non trivial abelian group and let $L(G)$ be the set of all subgroups of G. Under the relation of inclusion $\subseteq$, $L(G)$ is a partially ordered set. It is actually a lattice if $L(G)$ is endowed with the two binary operations defined by the formulae:

$$S_1 \wedge S_2 = S_1 \cap S_2$$
$$S_1 \vee S_2 = \{a + b : a \in S_1 \text{ and } b \in S_2\}.$$

It is well known that $\langle L(G); \wedge, \vee\rangle$ is a complete modular lattice with least element 0 and greatest element G (see [3, Theorem 11, p. 13] and [17, p. 5]).

Define

$$\perp = \{(S_1, S_2) \in L(G) \times L(G) : S_1 \wedge S_2 = 0\}$$

and

$$S_1 \oplus S_2 = S_1 \vee S_2 \text{ if } S_1 \perp S_2.$$

Then, using [38, Proposition 8.1] it is easy to show that $\langle L(G); \oplus, 0\rangle$ is a partial abelian monoid which is conical and every element of $L(G)$ is directly finite. Hence Lemma 2.2 implies that $\langle L(G); \leq_{\text{alg}}\rangle$ is a partially ordered set (clearly, the relation of inclusion $\subseteq$ is weaker than $\leq_{\text{alg}}$). We consider two important cases:

(1) If every finite set of elements of G generates a cyclic subgroup, then it follows from Ore's Theorem [30, Theorem 4, p. 267] that the partial abelian monoid $\langle L(G); \oplus, 0\rangle$ is cancellative, but not necessarily unital (this happens, for example, when G is the full rational group).
(2) If the order of each element of G is a square-free integer, then it follows from Kertész's Theorem [27] that the partial abelian monoid $\langle L(G); \oplus, 0\rangle$ is unital with unit G, but not necessarily cancellative (this happens, for example, when G is the additive group $\mathbb{Z}(2) \times \mathbb{Z}(2)$).

Example 2.9. Let K be any field and let V be a vector space over K of dimension greater than or equal to 2. Consider the set $L(V)$ of all subspaces of V which is partially ordered by the inclusion relation $\subseteq$. Then $L(V)$ endowed with the two binary operations defined by the formulae:

$$S_1 \wedge S_2 = S_1 \cap S_2$$
$$S_1 \vee S_2 = \{x + y : x \in S_1 \text{ and } y \in S_2\}$$

is a complete modular lattice with least element 0 and greatest element V. If we define

$$\perp = \{(S_1, S_2) \in L(V) \times L(V) : S_1 \wedge S_2 = 0\}$$

and

$$S_1 \oplus S_2 = S_1 \vee S_2 \text{ if } S_1 \perp S_2,$$

then it is easy to show that $\langle L(V); \oplus, 0\rangle$ is a partial abelian monoid which is conical and every element of $L(V)$ is directly finite, and therefore $\langle L(V); \leq_{\text{alg}}\rangle$ is a partially ordered set. Also every line in V is not cancellable in $L(V)$ and by [4, §3, n° 3, Proposition 5, p. 37] it follows that $\langle L(V); \oplus, 0\rangle$ is unital with unit V.

Now suppose that $L = \langle L; \oplus, 0\rangle$ is a partial abelian monoid.

We say that L is an *effect algebra* if L is conical, cancellative and unital. For example, if $\langle L; \wedge, \vee\rangle$ is a distributive lattice with least element 0 and greatest element 1, then the derived partial abelian monoid $\langle L; \oplus, 0\rangle$ is an effect algebra.

We have the following simple result:

Proposition 2.10 ([39, Lemma 1.4])**.** *For every partial abelian monoid L, the following conditions are equivalent:*

i) *L is an effect algebra.*
ii) *There exists an element 1 in L, different of 0, and satisfying the following axioms:*
   (Orthosupplementation Law) *For every $a \in L$ there exists a unique $b \in L$ such that $a \perp b$ and $a \oplus b = 1$.*
   (Zero-One Law) *If $a \in L$ and $a \perp 1$, then $a = 0$.*

Let L be an effect algebra. If $a \in L$ then the unique element $b \in L$ such that $a \perp b$ and $a \oplus b = 1$ is called the *orthosupplement* of a and denoted by a'. It is clear that the orthosupplement has the following properties:

(1) $1' = 0$.
(2) $(a')' = a$ for all $a \in L$.
(3) If $a, b \in L$ and $a \leq_{\text{alg}} b$, then $b' \leq_{\text{alg}} a'$.
(4) Let $a, b \in L$. Then $a \perp b$ if and only if $a \leq_{\text{alg}} b'$.

For a list of nice examples of effect algebras we refer to [13, 4. Examples] and for some of its properties we refer also to [1] and [12]. Another interesting example of an effect algebra is the following: Let I be a non-empty index set and, for every $i \in I$, let $L_i = \langle L_i; \oplus, 0, 1\rangle$ be an effect algebra. Form the Cartesian product $L = \mathsf{X}_{i\in I} L_i$ and let $\pi_i : L \to L_i$ be the canonical projection onto the i-th coordinate set. Put

$\perp = \{(a,b) \in L \times L : \pi_i(a) \perp \pi_i(b)$ for all $i \in I\}$ and define $\oplus : \perp \to L$ by the formula

$$\pi_i(a \oplus b) = \pi_i(a) \oplus \pi_i(b)$$

for all $i \in I$. If $\pi_i(a) = 0$ and $\pi_i(b) = 1$ for all $i \in I$, we put $a = 0$ and $b = 1$. It is easy to see that $\langle L; \oplus, 0, 1\rangle$ is an effect algebra called the Cartesian product of the family $(L_i)_{i\in I}$ and it is denoted by $\prod_{i\in I} L_i$.

An *orthoalgebra* is an effect algebra L in which the zero-one law is replaced by the stronger

(Consistence Law) If $a \in L$ and $a \perp a$, then $a = 0$.

For examples and details concerning orthoalgebras we refer to [14], [29] and [32].

An *orthomodular poset* is an effect algebra L that satisfies the following axiom

(Coherence Law) For $a, b, c \in L$, the conditions $a \perp b$, $a \perp c$ and $b \perp c$ imply $(a \oplus b) \perp c$.

Since the coherence law is stronger than the consistence law, every orthomodular poset is an orthoalgebra.

An *orthomodular lattice* is an orthoalgebra L such that $\langle L; \leq_{\text{alg}}\rangle$ is a lattice.

It is easy to verify that every orthomodular lattice is an orthomodular poset.

A *centerium* is an orthoalgebra L that satisfies the following axiom

(Law of Compatibility) For all $a, b \in L$ there exist $a_1, a_2, c \in L$ such that $a_1 \perp a_2$, $a_1 \perp c$, $a_2 \perp c$, $a = a_1 \oplus c$ and $b = a_2 \oplus c$.

A *Boolean algebra* is an orthomodular poset L that is also a centerium. It follows from [31, Corollary 1.3.14] that L is a Boolean algebra if and only if L is a distributive orthomodular lattice.

For terminology and details concerning orthomodular posets, orthomodular lattices and Boolean algebras we refer to [2], [25] and [31].

3. Orthogonal suites

We shall denote by $\mathbb{N}$ the set of all natural numbers $0, 1, 2, \ldots$ and by $\leq_0$ the usual total ordering on $\mathbb{N}$. Moreover, if $p, q \in \mathbb{N}$ and $p \leq_0 q$ we denote by $[p, q]_0$ the order interval $\{k \in \mathbb{N} : p \leq_0 k \leq_0 q\}$ in $\langle \mathbb{N}; \leq_0\rangle$.

In this Section $L = \langle L; \oplus, 0\rangle$ denotes a partial abelian monoid.

Let $(a_i)_{i\in I}$ be a finite non-empty family of elements of L and let $n = \text{card}(I)$. From the $n!$ different total orderings on I we choose one of them denoted by $\leq$. The finite family $(a_i)_{i\in I}$ of elements of L together with the totally ordered set $\langle I; \leq\rangle$ define an algebraic system called a *suite* in L of n elements and it is denoted by $(a_i)_{i\in\langle I;\leq\rangle}$. Also we can define $i_0 = \min I$ and $i_p = \min(I \setminus \{i_0, i_1, \ldots, i_{p-1}\})$ for all $p \in [1, n-1]_0$ if $n \geq_0 2$.

A suite in L of n elements $(a_i)_{i\in\langle I;\leq\rangle}$ is said to be *orthogonal* in L either $n = 1$ or, when $n \geq_0 2$, if

$$a_{i_0} \perp a_{i_1} \text{ and } a_{i_p} \perp (\cdots((a_{i_0} \oplus a_{i_1}) \oplus a_{i_2}) \cdots a_{i_{p-1}})$$

for all $p \in [2, n-1]_0$. Further, the element s of L defined respectively by the formulae

$$s = a_{i_0} \text{ and } s = (\cdots((a_{i_0} \oplus a_{i_1}) \oplus a_{i_2}) \cdots a_{i_{n-2}}) \oplus a_{i_{n-1}}$$

is called the $\oplus$-*join* of the orthogonal suite $(a_i)_{i \in \langle I; \leq \rangle}$ and it is denoted by $\bigoplus_{i \in \langle I; \leq \rangle} a_i$ or $\bigoplus_{i \in I} a_i$ for simplicity.

For example, consider the partial abelian monoid $L = \langle L(V); \oplus, 0 \rangle$ where V is a real infinite-dimensional vector space. Let $I = \langle I; \leq \rangle$ be a finite totally ordered set with $\mathrm{card}(I) \geq_0 3$ and let $(S_i)_{i \in I}$ be a family of elements of L. Then the suite $(S_i)_{i \in \langle I; \leq \rangle}$ is orthogonal in L if and only if, for every $i \in I$, if S is the subspace of V spanned by $\bigcup_{j \in I \setminus \{i\}} S_j$, we have $S \cap S_i = 0$. In this case, $\bigoplus_{i \in I} S_i$ is the subspace of V spanned by $\bigcup_{i \in I} S_i$.

Since the void subset $\emptyset$ of any partially ordered set $\langle J; \leq \rangle$ is totally ordered, it follows that $(a_i)_{i \in \langle \emptyset; \leq \rangle}$ is a suite in L of 0 elements. Then it is convenient to accept also that $\bigoplus_{i \in \langle \emptyset; \leq \rangle} a_i = 0$.

An important special case of a suite in L is the following (see [12, Definition 4.1] and [38, p. 288]): Let $p, q \in \mathbb{N}$ be such that $p \leq_0 q$, let $I = [p, q]_0$ and let $(a_i)_{i \in I}$ be a family of elements of L. Since $\mathrm{card}(I) = q - p + 1$, $(a_i)_{i \in \langle I; \leq_0 \rangle}$ is a suite in L of $q - p + 1$ elements which it will be denoted by $(a_i)_{p \leq_0 i \leq_0 q}$. In the case where $(a_i)_{p \leq_0 i \leq_0 q}$ is an orthogonal suite in L, the corresponding $\oplus$-join is denoted by $\bigoplus_{i=p}^{q} a_i$ or $a_p \oplus a_{p+1} \oplus \cdots \oplus a_q$.

Remark 3.1. Let $I = \langle I; \leq \rangle$ be a finite totally ordered set with $\mathrm{card}(I) = n \geq_0 2$ and let $(a_i)_{i \in I}$ be a family of elements of L. Write $i_0 = \min I$, $i_p = \min(I \setminus \{i_0, i_1, \ldots, i_{p-1}\})$ for all $p \in [1, n-1]_0$ and $b_k = a_{i_k}$ for all $k \in [0, n-1]_0$. It is clear that $(a_i)_{i \in \langle I; \leq \rangle}$ is an orthogonal suite in L if and only if $(b_k)_{0 \leq_0 k \leq_0 n-1}$ is an orthogonal suite in L. In either case, we have $\bigoplus_{i \in I} a_i = \bigoplus_{k=0}^{n-1} b_k$.

We begin with an obvious lemma which we use sometimes without explicit mention:

Lemma 3.2. *Let $I = \langle I; \leq \rangle$ be a finite totally ordered set with $\mathrm{card}(I) = n \geq_0 2$, let $i_{n-1} = \max I$, let $I' = I \setminus \{i_{n-1}\}$ and let $(a_i)_{i \in I}$ be a family of elements of L. Then the following conditions are equivalent:*

i) *$(a_i)_{i \in \langle I; \leq \rangle}$ is an orthogonal suite in L of n elements.*
ii) *$(a_i)_{i \in \langle I'; \leq \rangle}$ is an orthogonal suite in L of $n-1$ elements and $a_{i_{n-1}} \perp \left(\bigoplus_{i \in I'} a_i\right)$.*

Further, $\bigoplus_{i \in I} a_i = \left(\bigoplus_{i \in I'} a_i\right) \oplus a_{i_{n-1}}$.

The following result gives the generalized associative law for the partial binary operation $\oplus$:

Theorem 3.3. *Let $p \in \mathbb{N} \setminus \{0\}$, let $I = \langle I; \leq \rangle$ be a finite non-empty totally ordered set which is the union of non-empty sets I_k ($k \in [1, p]_0$) verifying the following condition:*

$$i \in I_k, j \in I_l \text{ and } 1 \leq_0 k <_0 l \leq_0 p \text{ imply } i < j \tag{1}$$

and let $(a_i)_{i \in I}$ be a family of elements of L. If $(a_i)_{i \in \langle I; \leq \rangle}$ is an orthogonal suite in L, then

a) $(a_i)_{i\in\langle I_k;\leq\rangle}$ *is an orthogonal suite in* L *for all* $k\in[1,p]_0$.
b) $\left(\bigoplus_{i\in I_k} a_i\right)_{1\leq_0 k\leq_0 p}$ *is an orthogonal suite in* L.
c) $\bigoplus_{i\in I} a_i = \bigoplus_{k=1}^{p}\left(\bigoplus_{i\in I_k} a_i\right)$.

Proof. We proceed by induction on $n=\operatorname{card}(I)$.

Let $n=1$. Since every set I_k is non-empty, it follows that $p=1$, and the Theorem is trivial.

Let $n\geq_0 2$ and suppose that the Theorem is true for every index set of cardinal $n-1$ with a decomposition satisfying (1). We distinguish two cases:

I) The set I_p contains a unique element i_{n-1}.

Then $i_{n-1}=\max I$ and putting $I'=I_1\cup I_2\cup\ldots\cup I_{p-1}$ we have $I'=I\setminus\{i_{n-1}\}$. Hence Lemma 3.2 assures that $(a_i)_{i\in\langle I';\leq\rangle}$ is an orthogonal suite in L of $n-1$ elements such that $a_{i_{n-1}}\perp\left(\bigoplus_{i\in I'} a_i\right)$ and $\bigoplus_{i\in I} a_i=\left(\bigoplus_{i\in I'} a_i\right)\oplus a_{i_{n-1}}$. Since the decomposition of I' satisfies the condition (1) the induction hypothesis implies that $(a_i)_{i\in\langle I_k;\leq\rangle}$ is an orthogonal suite in L for all $k\in[1,p-1]_0$, $\left(\bigoplus_{i\in I_k} a_i\right)_{1\leq_0 k\leq_0 p-1}$ is also an orthogonal suite in L and $\bigoplus_{i\in I'} a_i=\bigoplus_{k=1}^{p-1}\left(\bigoplus_{i\in I_k} a_i\right)$. Then, concerning the orthogonal suite $(a_i)_{i\in\langle I;\leq\rangle}$, the properties a) and b) are trivially verified. Since $a_{i_{n-1}}\perp\left(\bigoplus_{i\in I'} a_i\right)$ and $\bigoplus_{i\in I_p} a_i=a_{i_{n-1}}$ the property c) follows from Lemma 3.2.

II) The set I_p contains more than one element.

Let $i_{n-1}=\max I$. Then $i_{n-1}=\max I_p$. Let $I'=\{i\in I: i<i_{n-1}\}$ and let $I'_p=I'\cap I_p$. So $\operatorname{card}(I')=n-1$, $(a_i)_{i\in\langle I';\leq\rangle}$ is an orthogonal suite in L, $I'=I_1\cup\ldots\cup I_{p-1}\cup I'_p$ and this decomposition satisfies the condition (1). Then the induction hypothesis implies that $(a_i)_{i\in\langle I'_p;\leq\rangle}$ and $(a_i)_{i\in\langle I_k;\leq\rangle}$ $(k\in[1,p-1]_0)$ are orthogonal suites in L such that, for $b_k=\bigoplus_{i\in\langle I_k;\leq\rangle} a_i$ $(k\in[1,p-1]_0)$ and $b_p=\bigoplus_{i\in I'_p} a_i$, we have that $(b_k)_{1\leq_0 k\leq_0 p}$ is an orthogonal suite in L and $\bigoplus_{i\in I'} a_i=\bigoplus_{k=1}^{p} b_k$. So $\bigoplus_{i\in I'} a_i=\left(\bigoplus_{k=1}^{p-1}\left(\bigoplus_{i\in I_k} a_i\right)\right)\oplus\left(\bigoplus_{i\in I'_p} a_i\right)$. Then with respect to the orthogonal suite $(a_i)_{i\in\langle I;\leq\rangle}$ in L, since $a_{i_{n-1}}\perp\left(\bigoplus_{k=1}^{p-1}\left(\bigoplus_{i\in I_k} a_i\right)\right)\oplus\left(\bigoplus_{i\in I'_p} a_i\right)$, $I_p=I'_p\cup\{i_{n-1}\}$ and $i_{n-1}=\max I_p$, the associative law and Lemma 3.2 imply that $(a_i)_{i\in\langle I_p;\leq\rangle}$ is an orthogonal suite in L and therefore $\left(\bigoplus_{i\in I_k} a_i\right)_{1\leq_0 k\leq_0 p}$ is also an orthogonal suite in L with $\bigoplus_{i\in I} a_i=\bigoplus_{k=1}^{p}\left(\bigoplus_{i\in I_k} a_i\right)$, and therefore the properties a), b) and c) are verified. □

Corollary 3.4. *Let* $I=\langle I;\leq\rangle$ *be a finite totally ordered set with* $\operatorname{card}(I)=n\geq_0 2$, *let* $i_0=\min I$ *and let* $I'=I\setminus\{i_0\}$. *If* $(a_i)_{i\in\langle I;\leq\rangle}$ *is an orthogonal suite in* L *of* n *elements, then* $(a_i)_{i\in\langle I';\leq\rangle}$ *is an orthogonal suite in* L *of* $n-1$ *elements such that* $a_{i_0}\perp\left(\bigoplus_{i\in I'} a_i\right)$ *and* $\bigoplus_{i\in I} a_i=a_{i_0}\oplus\left(\bigoplus_{i\in I'} a_i\right)$.

Theorem 3.5. *Let* $I=\langle I;\leq\rangle$ *be a finite totally ordered set with* $\operatorname{card}(I)\geq_0 3$, *let* I_1 *and* I_2 *be two non-empty subsets of* I *such that* $I_1\cap I_2=\emptyset$ *and let* $(a_i)_{i\in I}$ *be a family of elements of* L. *If* $(a_i)_{i\in\langle I_1;\leq\rangle}$ *and* $(a_i)_{i\in\langle I_2;\leq\rangle}$ *are orthogonal suites in* L *such that* $\bigoplus_{i\in I_1} a_i\perp\bigoplus_{i\in I_2} a_i$, *then* $(a_i)_{i\in\langle I_1\cup I_2;\leq\rangle}$ *is an orthogonal suite in* L *and* $\bigoplus_{i\in I_1\cup I_2} a_i=\left(\bigoplus_{i\in I_1} a_i\right)\oplus\left(\bigoplus_{i\in I_2} a_i\right)$.

Proof. Write $b = \bigoplus_{i\in I_1} a_i$, $i_0 = \min I_2$ and $i_p = \min(I_2 \setminus \{i_0, i_1, \dots, i_{p-1}\})$ for all $p \in [1, n-1]_0$ where $n = \mathrm{card}(I_2)$. Then Remark 3.1 implies that $\bigoplus_{i\in I_2} a_i = a_{i_0} \oplus a_{i_1} \oplus a_{i_2} \oplus \cdots \oplus a_{i_{n-2}} \oplus a_{i_{n-1}}$. Then using the associative law and Corollary 3.4 we can write in a successive process

$$\begin{aligned}\Big(\bigoplus\nolimits_{i\in I_1} a_i\Big) \oplus \Big(\bigoplus\nolimits_{i\in I_2} a_i\Big) &= b \oplus (a_{i_0} \oplus (a_{i_1} \oplus \cdots \oplus a_{i_{n-1}}))\\ &= (b \oplus a_{i_0}) \oplus (a_{i_1} \oplus (a_{i_2} \oplus \cdots \oplus a_{i_{n-2}} \oplus a_{i_{n-1}}))\\ &= ((b \oplus a_{i_0}) \oplus a_{i_1}) \oplus (a_{i_2} \oplus (a_{i_3} \oplus \cdots \oplus a_{i_{n-2}} \oplus a_{i_{n-1}}))\\ \cdots &= (\cdots((b \oplus a_{i_0}) \oplus a_{i_1}) \oplus \cdots \oplus a_{i_{n-2}}) \oplus a_{i_{n-1}}))\end{aligned}$$

and we get that $(a_i)_{i\in\langle I_1\cup I_2;\leq\rangle}$ is an orthogonal suite in L such that $\bigoplus_{i\in I_1\cup I_2} a_i = \big(\bigoplus_{i\in I_1} a_i\big) \oplus \big(\bigoplus_{i\in I_2} a_i\big)$. □

Corollary 3.6. *Let $I = \langle I; \leq\rangle$ be a finite totally ordered set with* $\mathrm{card}(I) = n \geq_0 3$, *let $j \in I$, let $I' = I \setminus \{j\}$ and let $(a_i)_{i\in I}$ be a family of elements of L. If $(a_i)_{i\in\langle I';\leq\rangle}$ is an orthogonal suite in L of $n-1$ elements and $a_j \perp \bigoplus_{i\in I'} a_i$, then $(a_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L of n elements and $\bigoplus_{i\in I} a_i = \big(\bigoplus_{i\in I'} a_i\big) \oplus a_j$.*

Proposition 3.7. *Let $I = \langle I; \leq\rangle$ be a finite totally ordered set with* $\mathrm{card}(I) = n \geq_0 3$, *let $j \in I$, let $I' = I\setminus\{j\}$ and let $(a_i)_{i\in I}$ be a family of elements of L. If $(a_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L of n elements, then $(a_i)_{i\in\langle I';\leq\rangle}$ is an orthogonal suite in L of $n-1$ elements and $\bigoplus_{i\in I} a_i = \big(\bigoplus_{i\in I'} a_i\big) \oplus a_j$.*

Proof. Write $i_0 = \min I$ and $i_p = \min(I \setminus \{i_0, i_1, \dots, i_{p-1}\})$ for all $p \in [1, n-1]_0$.

If $j = i_{n-1}$ (resp. $j = i_0$) then the result follows from Lemma 3.2 (resp. commutative law and Corollary 3.4).

Suppose that $i_0 < j < i_{n-1}$. Then there exists $h \in [1, n-2]_0$ such that $j = i_h$. If we put $I_1 = \{i_0, i_1, \dots, i_{h-1}\}$, $I_2 = \{i_h\}$ and $I_3 = \{i_{h+1}, i_{h+2}, \dots, i_{n-1}\}$, then I_1, I_2, I_3 are non-empty sets, $I = I_1 \cup I_2 \cup I_3$ and this decomposition satisfies the condition (1) of Theorem 3.3, and it follows that $(a_i)_{i\in\langle I_k;\leq\rangle}$ is an orthogonal suite in L for all $k \in [1, 3]_0$, $\big(\bigoplus_{i\in I_k} a_i\big)_{1\leq_0 k\leq_0 3}$ is also an orthogonal suite in L and $\bigoplus_{i\in I} a_i = \big(\bigoplus_{i\in I_1} a_i\big) \oplus a_{i_h} \oplus \big(\bigoplus_{i\in I_3} a_i\big)$. Then the associative and commutative laws imply that $\bigoplus_{i\in I_1} a_i \perp \bigoplus_{i\in I_3} a_i$. Since $I' = I_1 \cup I_3$ and $I_1 \cap I_3 = \emptyset$, it follows from Theorem 3.5 that $(a_i)_{i\in\langle I';\leq\rangle}$ is an orthogonal suite in L and clearly $\bigoplus_{i\in I} a_i = \big(\bigoplus_{i\in I'} a_i\big) \oplus a_{i_h}$. □

Corollary 3.8. *Let $I = \langle I; \leq\rangle$ be a finite totally ordered set with* $\mathrm{card}(I) = n \geq_0 3$. *Let J be a non-empty subset of I and let $(a_i)_{i\in I}$ be a family of elements of L. If $(a_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L, then $(a_i)_{i\in\langle J;\leq\rangle}$ is also an orthogonal suite in L and $\bigoplus_{i\in J} a_i \leq_{\mathrm{alg}} \bigoplus_{i\in I} a_i$.*

The following result gives the generalized commutative law for the binary partial operation $\oplus$:

Theorem 3.9. *Let $(a_i)_{i\in I}$ be a finite family of elements of L with* $\mathrm{card}(I) = n \geq_0 2$. *Then for any two total orderings on L, the orthogonality of the corresponding suites holds simultaneously and the $\oplus$-joins have the same value.*

Proof. We proceed by induction on n. Clearly for $n = 2$ the Theorem is immediate.

Let $n \in \mathbb{N}$ be such that $n \geq_0 3$ and suppose that the Theorem is true for all index set of cardinal $n - 1$.

Let $\leq'$ be a total ordering on I such that $(a_i)_{i\in\langle I;\leq'\rangle}$ is an orthogonal suite in L. Let $\leq''$ be another total ordering on I and suppose that j is the greatest element of $\langle I;\leq''\rangle$. Put $I' = I \setminus \{j\}$. Then Proposition 3.7 implies that $(a_i)_{i\in\langle I';\leq'\rangle}$ is an orthogonal suite in L of $n-1$ elements such that $a_j \perp \bigoplus_{i\in\langle I';\leq'\rangle} a_i$ and $\bigoplus_{i\in\langle I;\leq'\rangle} a_i = \left(\bigoplus_{i\in\langle I';\leq'\rangle} a_i\right) \oplus a_j$. Moreover, the induction hypothesis implies that $(a_i)_{i\in\langle I';\leq''\rangle}$ is an orthogonal suite in L and $\bigoplus_{i\in\langle I';\leq''\rangle} a_i = \bigoplus_{i\in\langle I';\leq'\rangle} a_i$. So $a_j \perp \bigoplus_{i\in\langle I';\leq''\rangle} a_i$ and Lemma 3.2 implies that $(a_i)_{i\in\langle I;\leq''\rangle}$ is an orthogonal suite in L and $\bigoplus_{i\in\langle I;\leq''\rangle} a_i = \left(\bigoplus_{i\in\langle I';\leq''\rangle} a_i\right) \oplus a_j = \bigoplus_{i\in\langle I;\leq'\rangle} a_i$. □

Corollary 3.10. *Let $I = \langle I;\leq\rangle$ be a finite totally ordered set with* $\mathrm{card}(I) \geq_0 2$*, let σ be a bijection from I onto I, let $\leq'$ be the total ordering on I defined by the formula: $i <' j$ if $\sigma^{-1}(i) < \sigma^{-1}(j)$ and let $(a_i)_{i\in I}$ be a family of elements of L. If $(a_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L, then $(a_i)_{i\in\langle I;\leq'\rangle}$ is an orthogonal suite in L and $\bigoplus_{i\in\langle I;\leq\rangle} a_i = \bigoplus_{i\in\langle I;\leq'\rangle} a_i$.*

Theorem 3.11. *Let $p \in \mathbb{N}\setminus\{0\}$, let $I = \langle I;\leq\rangle$ be a finite non-empty totally ordered set which is the union of non-empty pairwise disjoint sets I_k ($k \in [1,p]_0$) and let $(a_i)_{i\in I}$ be a family of elements of L. If $(a_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L, then*

a) *$(a_i)_{i\in\langle I_k;\leq\rangle}$ is an orthogonal suite in L for all $k \in [1,p]_0$.*
b) *$\left(\bigoplus_{i\in I_k} a_i\right)_{1\leq_0 k\leq_0 p}$ is an orthogonal suite in L.*
c) *$\bigoplus_{i\in I} a_i = \bigoplus_{k=1}^{p} \left(\bigoplus_{i\in I_k} a_i\right)$.*

Proof. It follows from Theorem 3.9 if we choose a total ordering $\leq'$ on I such that the decomposition of $\langle I;\leq'\rangle$ satisfies the condition (1) of Theorem 3.3. □

Theorem 3.12. *Let $I = \langle I;\leq\rangle$ be a finite totally ordered set with* $\mathrm{card}(I) = n \geq_0 2$ *and let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be two families of elements of L such that $a_i \perp b_i$ for all $i \in I$. If $(a_i \oplus b_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L, then $(a_i)_{i\in\langle I;\leq\rangle}$ and $(b_i)_{i\in\langle I;\leq\rangle}$ are orthogonal suites in L such that $\bigoplus_{i\in I} a_i \perp \bigoplus_{i\in I} b_i$ and $\bigoplus_{i\in I}(a_i \oplus b_i) = \left(\bigoplus_{i\in I} a_i\right) \oplus \left(\bigoplus_{i\in I} b_i\right)$.*

Proof. We proceed by induction on n.

First we prove the Theorem for $n = 2$. Write $I = \{i_0, i_1\}$ where $i_0 < i_1$. Since $a_{i_1}, b_{i_1} \leq_{\mathrm{alg}} (a_{i_1} \oplus b_{i_1})$ and $(a_{i_0} \oplus b_{i_0}) \perp (a_{i_1} \oplus b_{i_1})$ it follows from Lemma 2.1 a) that $(a_{i_0} \oplus b_{i_0}) \perp a_{i_1}$ and $(a_{i_0} \oplus b_{i_0}) \perp b_{i_1}$. Then the associative and commutative laws imply that $a_{i_0} \perp a_{i_1}$, $b_{i_0} \perp b_{i_1}$, $(a_{i_0} \oplus a_{i_1}) \perp b_{i_0}$ and $(a_{i_0} \oplus b_{i_0}) \oplus a_{i_1} = (a_{i_0} \oplus a_{i_1}) \oplus b_{i_0}$. Since $(a_{i_0} \oplus b_{i_0}) \perp (a_{i_1} \oplus b_{i_1})$ we get $(a_{i_0} \oplus a_{i_1}) \oplus b_{i_0} \perp b_{i_1}$ and therefore $(a_{i_0} \oplus a_{i_1}) \perp (b_{i_0} \oplus b_{i_1})$ with $(a_{i_0} \oplus b_{i_0}) \oplus (a_{i_1} \oplus b_{i_1}) = (a_{i_0} \oplus a_{i_1}) \oplus (b_{i_0} \oplus b_{i_1})$.

Now let $n \in \mathbb{N}$ be such that $n \geq_0 3$ and suppose that the Theorem is true for every index set of cardinal $n - 1$.

Assume that $\mathrm{card}(I) = n$, $a_i \perp b_i$ for all $i \in I$ and $(a_i \oplus b_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L. Choose $j \in I$ and let $I' = I \setminus \{j\}$. Then by Proposition 3.7 it follows that

$(a_i \oplus b_i)_{i\in\langle I',\leq\rangle}$ is an orthogonal suite in L of $n-1$ elements such that $(a_j \oplus b_j) \perp \bigoplus_{i\in I'}(a_i \oplus b_i)$ and $\bigoplus_{i\in I}(a_i \oplus b_i) = \big(\bigoplus_{i\in I'}(a_i \oplus b_i)\big) \oplus (a_j \oplus b_j)$. Then the induction hypothesis implies that $(a_i)_{i\in\langle I';\leq\rangle}$ and $(b_i)_{i\in\langle I';\leq\rangle}$ are orthogonal suites in L such that $\bigoplus_{i\in I'} a_i \perp \bigoplus_{i\in I'} b_i$ and $\bigoplus_{i\in I'}(a_i \oplus b_i) = \big(\bigoplus_{i\in I'} a_i\big) \oplus \big(\bigoplus_{i\in I'} b_i\big)$. To complete the proof it suffices to apply the special case $n = 2$ and Corollary 3.6. □

Corollary 3.13. *Let $I = \langle I;\leq\rangle$ be a finite totally ordered set with* $\mathrm{card}(I) = n \geq_0 2$ *and let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be two families of elements of L such that $b_i \leq_{\mathrm{alg}} a_i$ for all $i \in I$. If $(a_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L, then $(b_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L and $\bigoplus_{i\in I} b_i \leq_{\mathrm{alg}} \bigoplus_{i\in I} a_i$.*

Proof. Since $b_i \leq_{\mathrm{alg}} a_i$ for all $i \in I$, we get $b_i \perp (a_i - b_i)$ and $b_i \oplus (a_i - b_i) = a_i$ for all $i \in I$. Moreover, since $(a_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L, it follows from Theorem 3.12 that $(b_i)_{i\in\langle I;\leq\rangle}$ and $(a_i - b_i)_{i\in\langle I;\leq\rangle}$ are orthogonal suites in L such that $\bigoplus_{i\in I} b_i \perp \bigoplus_{i\in I}(a_i - b_i)$ and $\bigoplus_{i\in I} a_i = \big(\bigoplus_{i\in I} b_i\big) \oplus \big(\bigoplus_{i\in I}(a_i - b_i)\big)$. Whence $\bigoplus_{i\in I} b_i \leq_{\mathrm{alg}} \bigoplus_{i\in I} a_i$. □

Theorem 3.14. *Let $n \in \mathbb{N} \setminus \{0\}$ and let $(a_i)_{1\leq_0 i\leq_0 n}$ be an orthogonal suite in L. Then there exists a chain $b_0 \leq_{\mathrm{alg}} b_1 \leq_{\mathrm{alg}} b_2 \leq_{\mathrm{alg}} \cdots \leq_{\mathrm{alg}} b_n$ in L such that $b_0 = 0$ and $b_n = \bigoplus_{i=1}^{n} a_i$.*

Proof. Put $b_0 = 0$ and let $p \in [1, n]_0$. By Corollary 3.8 it follows that $(a_i)_{1\leq_0 i\leq_0 p}$ is an orthogonal suite in L. Then if we define $b_p = \bigoplus_{i=1}^{p} a_i$ we get $b_p \leq_{\mathrm{alg}} b_{p+1}$ for all $p \in [1, n-1]_0$, and therefore $b_0 \leq_{\mathrm{alg}} b_1 \leq_{\mathrm{alg}} b_2 \leq_{\mathrm{alg}} \cdots \leq_{\mathrm{alg}} b_n$. □

Theorem 3.15. *Assume that L is cancellative. Let $n \in \mathbb{N} \setminus \{0\}$, let $b_0 \leq_{\mathrm{alg}} b_1 \leq_{\mathrm{alg}} b_2 \leq_{\mathrm{alg}} \cdots \leq_{\mathrm{alg}} b_n$ be a chain in L and let $a_p = b_p - b_{p-1}$ for all $p \in [1, n]_0$. Then $(a_p)_{1\leq_0 p\leq_0 n}$ is an orthogonal suite in L such that $\bigoplus_{p=1}^{n} a_p = b_n - b_0$.*

Proof. We proceed by induction on n.

For $n = 1$ the Theorem is trivial.

Let $n \in \mathbb{N} \setminus \{0\}$ and suppose that the Theorem is true for n.

Let $b_0 \leq_{\mathrm{alg}} b_1 \leq_{\mathrm{alg}} b_2 \leq_{\mathrm{alg}} \cdots \leq_{\mathrm{alg}} b_n \leq_{\mathrm{alg}} b_{n+1}$ be a chain in L and let $a_p = b_p - b_{p-1}$ for all $p \in [1, n+1]_0$. Then the induction hypothesis implies that $(a_p)_{1\leq_0 p\leq_0 n}$ is an orthogonal suite in L such that $\bigoplus_{p=1}^{n} a_p = b_n - b_0$. Since $b_0 \leq_{\mathrm{alg}} b_n \leq_{\mathrm{alg}} b_{n+1}$, we have $b_n - b_0 \perp b_0$, $b_n = (b_n - b_0) \oplus b_0$, $b_{n+1} - b_n \perp b_n$ and $b_{n+1} = (b_{n+1} - b_n) \oplus b_n$. Then $b_n - b_0 \leq_{\mathrm{alg}} b_n$ and Lemma 2.1 a) implies that $b_{n+1} - b_n \perp b_n - b_0$. So $a_{n+1} \perp \bigoplus_{p=1}^{n} a_p$ and Lemma 3.2 implies that $(a_p)_{1\leq_0 p\leq_0 n+1}$ is an orthogonal suite in L and $\bigoplus_{p=1}^{n+1} a_p = \Big(\bigoplus_{p=1}^{n} a_p\Big) \oplus a_{n+1} = (b_n - b_0) \oplus (b_{n+1} - b_n)$. Since $b_{n+1} = b_n \oplus (b_{n+1} - b_n) = (b_0 \oplus (b_n - b_0)) \oplus (b_{n+1} - b_n) = b_0 \oplus ((b_n - b_0) \oplus (b_{n+1} - b_n))$, we get $b_{n+1} - b_0 = (b_n - b_0) \oplus (b_{n+1} - b_n)$ and therefore the Theorem is true for $n + 1$. □

Remark 3.16. It is clear that Corollaries 3.10 and 3.13 and Theorems 3.3, 3.5 and 3.11 improve [12, Theorem 4.2]. Also, if E is a commutative monoid, then Theorem 3.3 (resp. 3.9) improves [5, §1, n° 3, Théorèm 1] (resp. [5, §1, n° 5, Théorèm 3]).

4. Infinite orthogonal families

In this Section we assume that $L = \langle L; \oplus, 0\rangle$ is a partial abelian monoid such that $\langle L; \leq_{\mathrm{alg}}\rangle$ is a partially ordered set. This implies, in particular, that L is conical.

We extend first the notion of orthogonality to countably infinite families of elements of L.

Let M be an infinite subset of $\mathbb{N}$ and let $(a_i)_{i\in M}$ be a family of elements of L. Then the countably infinite family $(a_i)_{i\in M}$ of elements of L together with the totally ordered set $\langle M; \leq_0\rangle$ define an algebraic system called a *sequence* in L and denoted by $(a_i)_{i\in\langle M;\leq_0\rangle}$.

A sequence in L $(a_i)_{i\in\langle M;\leq_0\rangle}$ is said to be *orthogonal* in L if, for every $n \in \mathbb{N}$, the suite $(a_i)_{i\in\leq\langle M\cap[0,n]_0;\leq_0\rangle}$ is orthogonal in L. It is clear that if $(a_i)_{i\in\langle M;\leq_0\rangle}$ is an orthogonal sequence in L, then for every infinite subset N of M, the sequence in L $(a_i)_{i\in\langle N;\leq_0\rangle}$ is also orthogonal in L.

If $(a_i)_{i\in\langle M;\leq_0\rangle}$ is an orthogonal sequence in L and there exists $\bigvee\big(\bigoplus_{i\in M\cap[0,n]_0} a_i \mid n \in \mathbb{N}\big)$ in $\langle L; \leq_{\mathrm{alg}}\rangle$, we denote this element by $\bigoplus_{i\in M} a_i$ and we call it the $\oplus$-*join* of the orthogonal sequence $(a_i)_{i\in\langle M;\leq_0\rangle}$.

For example, consider the partial abelian monoid $L = \langle \mathbb{R}; \oplus, 0\rangle$ of the Example 2.6, and let $a_i = \frac{1}{2}$ for all $i \in \mathbb{N}$. Then it is clear that $(a_i)_{i\in\langle\mathbb{N};\leq_0\rangle}$ is an orthogonal sequence in L. Consider the family $(k_j)_{j\in\mathbb{N}}$ of elements of $\mathbb{N}\setminus\{0\}$ such that $k_0 = 2$ and $k_{j+1} = 3k_j - 1$ for all $j \in \mathbb{N}$. Then it is easy to verify by induction on n that $\bigoplus_{i=0}^{n} a_i = 1 - \frac{1}{k_n}$ for all $n \in \mathbb{N}$. Since $\bigvee(1 - \frac{1}{k_n} \mid n \in \mathbb{N}) = 1$ in $\langle L; \leq\rangle$ and $\leq_{\mathrm{alg}}\big|_{]0,1]} = \leq\big|_{]0,1]}$, we conclude that the $\oplus$-join of the orthogonal sequence $(a_i)_{i\in\langle\mathbb{N};\leq_0\rangle}$ is equal to 1.

The following simple result allow us to consider only orthogonal sequences in $L\setminus\{0\}$:

Lemma 4.1. *Let $(a_i)_{i\in\langle M;\leq_0\rangle}$ be a sequence in L such that $N = \{i \in M : a_i \neq 0\}$ is an infinite subset of M. Then $(a_i)_{i\in\langle M;\leq_0\rangle}$ is an orthogonal sequence in L if and only if $(a_i)_{i\in\langle N;\leq_0\rangle}$ is an orthogonal sequence in $L\setminus\{0\}$. Further, $\bigoplus_{i\in M} a_i = \bigoplus_{i\in N} a_i$ if one of these $\oplus$-joins exist in $\langle L; \leq_{\mathrm{alg}}\rangle$.*

Proof. Write $P = \{i \in M : a_i = 0\}$. The Lemma is trivial if $P = \emptyset$.

Suppose $P \neq \emptyset$ and let $n \in \mathbb{N}$ be such that $M \cap [0, n]_0$ is a non-empty set. Since $M \cap [0, n]_0 = (N \cap [0, n]_0) \cup (P \cap [0, n]_0)$, the Lemma follows from Theorems 3.5 and 3.11 noting that $\bigoplus_{i\in M\cap[0,n]_0} a_i = \bigoplus_{i\in N\cap[0,n]_0} a_i$. □

We say that L is σ-*complete* if the $\oplus$-join of every orthogonal sequence in L exists in $\langle L; \leq_{\mathrm{alg}}\rangle$.

Corollary 4.2. *If the $\oplus$-join of every orthogonal sequence in $L\setminus\{0\}$ exists in $L\setminus\{0\}$, then L is σ-complete.*

For example, consider the partial abelian monoid $L = \langle \mathbb{R}; \oplus, 0\rangle$ of the Example 2.6. We shall show that L is σ-complete. In fact, let $(a_i)_{i\in\langle\mathbb{N};\leq_0\rangle}$ be an orthogonal sequence in L. By Corollary 4.2 we may suppose that $a_i \neq 0$ for all $i \in \mathbb{N}$. Since

$x, y \in L \setminus \{0\}$ and $x \perp y$ imply $x, y \in]0,1]$ and $x \oplus y \in]0,1]$, it follows that $a_i \in]0,1]$ for all $i \in \mathbb{N}$. So $(\bigoplus_{i=0}^{n} a_i)_{n\in\mathbb{N}}$ is an increasing family of elements of $\langle]0,1]; \leq_{\mathrm{alg}}\rangle$ and therefore an increasing family of elements of $\langle]0,1]; \leq\rangle$ because $\leq_{\mathrm{alg}}\big|_{]0,1]} = \leq\big|_{]0,1]}$. Then the $\oplus$-join of the orthogonal sequence $(a_i)_{i\in\langle\mathbb{N};\leq_0\rangle}$ exists in $\langle L; \leq_{\mathrm{alg}}\rangle$.

The following result generalizes [24, Theorem 4.9] for $m = \aleph_0$:

Theorem 4.3. *Consider the following two statements:*

i) *There exists the supremum* $\bigvee(b_i \mid i \in \mathbb{N})$ *for all chain* $b_0 \leq_{\mathrm{alg}} b_1 \leq_{\mathrm{alg}} b_2 \leq_{\mathrm{alg}} \cdots \leq_{\mathrm{alg}} b_n \leq_{\mathrm{alg}} \cdots$ *in* L.
ii) L *is* σ*-complete.*

Then i) *implies* ii). *Further, if* L *is cancellative, then* ii) *implies* i).

Proof. The implication i) $\Rightarrow$ ii) follows from Theorem 3.14 and the implication ii) $\Rightarrow$ i) follows from Theorem 3.15. □

We extend now the notion of orthogonality to arbitrary infinite families of elements of L.

It is well known that to every set X a cardinal number is assigned; it is called the *cardinality* of X and it is denoted by $|X|$ or $\mathrm{card}(X)$ as in the case of finite sets. We denote by $\leq_c$ the usual well ordering on every non-empty set of cardinal numbers.

If X is a non-empty set, we denote by $\mathcal{F}(X)$ the set of all finite subsets of X.

Let I be an infinite index set and let $(a_i)_{i\in I}$ be a family of elements of L. We say that $(a_i)_{i\in I}$ is an *orthogonal family* in L if, for every $F \in \mathcal{F}(I) \setminus \{\emptyset\}$ and every total ordering $\leq$ on F, the suite $(a_i)_{i\in\langle F;\leq\rangle}$ is orthogonal in L. In this case, if we fix the set F, it follows from Theorem 3.9 that the corresponding $\oplus$-join $\bigoplus_{i\in F} a_i$ does not depend of the choice of the total ordering $\leq$ on F and we can write $\mathrm{s}(F) = \bigoplus_{i\in F} a_i$ for all $F \in \mathcal{F}(I)$ where $\mathrm{s}(\emptyset) = 0$. Moreover, if there exists the supremum $\bigvee(\mathrm{s}(F) \mid F \in \mathcal{F}(I))$ in $\langle L; \leq_{\mathrm{alg}}\rangle$, we call it the $\oplus$*-join* of the orthogonal family $(a_i)_{i\in I}$ in L and we denote it by $\bigoplus_{i\in I} a_i$. So $\bigoplus_{i\in I} a_i = \bigvee\big(\bigoplus_{i\in F} a_i \mid F \in \mathcal{F}(I)\big)$. Clearly, every infinite subfamily of an orthogonal family in L is also orthogonal in L.

Let $\mathfrak{X}$ be an infinite cardinal number. We say that L is $\mathfrak{X}$*-complete* if the $\oplus$-join of every orthogonal family in L $(a_i)_{i\in I}$ with $\mathrm{card}(I) \leq_c \mathfrak{X}$ exists in $\langle L; \leq_{\mathrm{alg}}\rangle$. We say that L is *complete* if L is $\mathfrak{X}$-complete for every infinite cardinal $\mathfrak{X}$.

Clearly, L is σ-complete if and only if L is $\aleph_0$-complete. Hence the partial abelian monoid of the Example 2.6 is $\aleph_0$-complete. Consider also the partial abelian monoid $L = \langle L(V); \oplus, 0\rangle$ of the Example 2.9 where V is an infinite-dimensional real vector space. We shall show first that the partial orderings $\leq_{\mathrm{alg}}$ and $\subseteq$ on $L(V)$ coincide. Clearly, $S_1, S_2 \in L(V)$ and $S_1 \leq_{\mathrm{alg}} S_2$ imply $S_1 \subseteq S_2$. To establish the converse, let us suppose that $S_1, S_2 \in L(V)$ and $S_1 \subseteq S_2$. Since L is unital there exists $S \in L(V)$ such that $S \wedge S_1 = 0$ and $S \oplus S_1 = V$. Choose $S_3 = S \wedge S_2$. Then $S_3 \in L(V)$ and

clearly $S_1 \wedge S_3 = 0$. Moreover, since $L(V)$ is a modular lattice, we get

$$\begin{aligned} S_1 \oplus S_3 &= S_1 \vee (S \wedge S_2) \\ &= (S_1 \vee S) \wedge S_2 \\ &= V \wedge S_2 \\ &= S_2 \end{aligned}$$

and therefore $S_1 \leq_{\text{alg}} S_2$. Now we shall show that L is complete. Let $\mathfrak{X}$ be an infinite cardinal number and let $(S_i)_{i\in I}$ be an orthogonal family in L with $\text{card}(I) \leq_c \mathfrak{X}$. Then, for every $F \in \mathcal{F}(I)\setminus\{\emptyset\}$ and every total ordering $\leq$ on F, the suite $(S_i)_{i\in\langle F;\leq\rangle}$ is orthogonal in L whose $\oplus$-join is the subspace $S(F)$ of V generated by $\bigcup_{i\in F} S_i$. Since $L(V)$ is a complete lattice, the supremum $\bigvee(S(F) \mid F \in \mathcal{F}(I))$ exists in $\langle L(V); \subseteq\rangle$ and therefore in $\langle L(V); \leq_{\text{alg}}\rangle$.

Theorem 4.4. *Assume that L is cancellative. Let I be an infinite index set and let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be two orthogonal families in L such that $a_i \leq_{\text{alg}} b_i$ for all $i \in I$. If there exist $\bigoplus_{i\in I} a_i$ and $\bigoplus_{i\in I} b_i$ and they are equal, then $a_i = b_i$ for all $i \in I$.*

Proof. For every $i \in I$ there exists $c_i \in L$ such that $a_i \perp c_i$ and $a_i \oplus c_i = b_i$. Fix an index j in I. Let $F \in \mathcal{F}(I)$, let $F_0 = F \cup \{j\}$ and let $\leq$ be a total ordering in F_0. Then the suites $(a_i)_{i\in\langle F_0;\leq\rangle}$ and $(b_i)_{i\in\langle F_0;\leq\rangle}$ are orthogonal in L. Therefore, from Theorem 3.12, it follows that $(c_i)_{i\in\langle F_0;\leq\rangle}$ is also an orthogonal suite in L with $\bigoplus_{i\in F_0} a_i \perp \bigoplus_{i\in F_0} c_i$ and $\left(\bigoplus_{i\in F_0} a_i\right) \oplus \left(\bigoplus_{i\in F_0} c_i\right) = \bigoplus_{i\in F_0} b_i$. Since $c_j \leq_{\text{alg}} \bigoplus_{i\in F_0} c_i$ and $\bigoplus_{i\in F} a_i \leq_{\text{alg}} \bigoplus_{i\in F_0} a_i$, it follows from Lemma 2.1 b) that $c_j \perp \bigoplus_{i\in F} a_i$ and $c_j \oplus \left(\bigoplus_{i\in F} a_i\right) \leq_{\text{alg}} \bigoplus_{i\in F_0} b_i \leq_{\text{alg}} \bigoplus_{i\in I} b_i$. Hence Lemma 2.5 a) implies that $\bigoplus_{i\in F} a_i \leq_{\text{alg}} \left(\bigoplus_{i\in I} b_i\right) - c_j$ for all $F \in \mathcal{F}(I)$. So $\bigoplus_{i\in I} a_i \leq_{\text{alg}} \left(\bigoplus_{i\in I} b_i\right) - c_j$, and from Lemma 2.5 b) we get that $c_j \leq_{\text{alg}} \left(\bigoplus_{i\in I} b_i\right) - \left(\bigoplus_{i\in I} a_i\right)$ and therefore $c_j = 0$ for all $j \in I$. So $a_j = b_j$ for all $j \in I$. □

Theorem 4.5. *Assume that L is cancellative. Let I be an infinite index set which is the union of disjoint infinite sets I_1 and I_2 and let $(a_i)_{i\in I}$ be a family of elements of L verifying the conditions:*

i) *The families $(a_i)_{i\in I_1}$ and $(a_i)_{i\in I_2}$ are orthogonal in L.*
ii) *There exist $\bigoplus_{i\in I_1} a_i$ and $\bigoplus_{i\in I_2} a_i$ in $\langle L; \leq_{\text{alg}}\rangle$.*
iii) $\bigoplus_{i\in I_1} a_i \perp \bigoplus_{i\in I_2} a_i$.

Then the family $(a_i)_{i\in I}$ is orthogonal in L having $\left(\bigoplus_{i\in I_1} a_i\right) \oplus \left(\bigoplus_{i\in I_2} a_i\right)$ as $\oplus$-join.

Proof. Write $a = \bigoplus_{i\in I_1} a_i$ and $b = \bigoplus_{i\in I_2} a_i$.

Let $F \in \mathcal{F}(I) \setminus \{\emptyset\}$. We may suppose that $F_1 = F \cap I_1$ and $F_2 = F \cap I_2$ are non-empty. Let $\leq$ be a total ordering on F. Then $(a_i)_{i\in\langle F_1;\leq\rangle}$ and $(a_i)_{i\in\langle F_2;\leq\rangle}$ are orthogonal suites in L and, using the hypothesis iii), Lemma 2.1 b) implies that $\bigoplus_{i\in F_1} a_i \perp \bigoplus_{i\in F_2} a_i$. Hence it follows from Theorem 3.5 that the suite $(a_i)_{i\in\langle F;\leq\rangle}$ is orthogonal in L and $\bigoplus_{i\in F} a_i = \left(\bigoplus_{i\in F_1} a_i\right) \oplus \left(\bigoplus_{i\in F_2} a_i\right)$. So $(a_i)_{i\in I}$ is an orthogonal family in L such that $\bigoplus_{i\in F} a_i \leq_{\text{alg}} (a \oplus b)$ by Lemma 2.1 b) for all $F \in \mathcal{F}(I)$. Let $c \in L$ be an upper bound of the set $\{\bigoplus_{i\in F} a_i : F \in \mathcal{F}(I)\}$ in $\langle L; \leq_{\text{alg}}\rangle$.

Let $F_1 \in \mathcal{F}(I_1) \setminus \{\emptyset\}$, $F_2 \in \mathcal{F}(I_2) \setminus \{\emptyset\}$ and let $\leq$ be a total ordering on $F_1 \cup F_2$. So $\bigoplus_{i \in F_1 \cup F_2} a_i \leq_{\mathrm{alg}} c$ and by Theorem 3.5 we have $\left(\bigoplus_{i \in F_1} a_i\right) \oplus \left(\bigoplus_{i \in F_2} a_i\right) \leq_{\mathrm{alg}} c$. Then, if we fix F_2, Lemma 2.5 a) implies that $\bigoplus_{i \in F_1} a_i \leq_{\mathrm{alg}} c - \left(\bigoplus_{i \in F_2} a_i\right)$ for all $F_1 \in \mathcal{F}(I_1)$ and therefore $a \leq_{\mathrm{alg}} c - \bigoplus_{i \in F_2} a_i$. So $\bigoplus_{i \in F_2} a_i \leq_{\mathrm{alg}} c - a$ for all $F_2 \in \mathcal{F}(I_2)$ and consequently $b \leq_{\mathrm{alg}} c - a$. Since $a \perp b$ and $a \oplus (c - a) = c$, Lemma 2.1 a) implies that $a \oplus b \leq_{\mathrm{alg}} c$. Hence the $\oplus$-join of the orthogonal family $(a_i)_{i \in I}$ exists in $\langle L; \leq_{\mathrm{alg}} \rangle$ and it is equal to $a \oplus b$. □

Corollary 4.6. *Assume that L is cancellative and complete. Let I be an infinite index set, let $(a_i)_{i \in I}$ be an orthogonal family in L and let J be an infinite subset of I such that $I \setminus J$ is also an infinite set. If $a = \bigoplus_{j \in J} a_j$, $b = \bigoplus_{k \in I \setminus J} a_k$ and $c = \bigoplus_{i \in I} a_i$, then $a \perp b$ and $c = a \oplus b$.*

Proof. It suffices by Theorem 4.5 to show that $a \perp b$.

Let $F_1 \in \mathcal{F}(J) \setminus \{\emptyset\}$ and let $F_2 \in \mathcal{F}(I \setminus J) \setminus \{\emptyset\}$. Take $F = F_1 \cup F_2$ and let $\leq$ be a total ordering on F. By Theorem 3.11 we get $\bigoplus_{i \in F_1} a_i \perp \bigoplus_{i \in F_2} a_i$ and $\left(\bigoplus_{i \in F_1} a_i\right) \oplus \left(\bigoplus_{i \in F_2} a_i\right) = \bigoplus_{i \in F} a_i \leq_{\mathrm{alg}} c$. Then Lemma 2.5 a) implies that $\bigoplus_{i \in F_1} a_i \leq_{\mathrm{alg}} c - \bigoplus_{i \in F_2} a_i$ for all $F_1 \in \mathcal{F}(J)$ and therefore $a \leq_{\mathrm{alg}} c - \bigoplus_{i \in F_2} a_i$. So $\bigoplus_{i \in F_2} a_i \leq_{\mathrm{alg}} c - a$ for all $F_2 \in \mathcal{F}(I \setminus J)$, and consequently $b \leq_{\mathrm{alg}} c - a$. Since $a \perp (c - a)$, Lemma 2.1 a) implies that $a \perp b$. □

Theorem 4.7. *Assume that L is σ-complete. Let I be a set of cardinal $\aleph_0$ and let $(a_i)_{i \in I}$ be an orthogonal family in L. If φ is a bijection from $\mathbb{N}$ onto I and $b_j = a_{\varphi(j)}$ for all $j \in \mathbb{N}$, then the sequence $(b_j)_{j \in \langle \mathbb{N}; \leq_0 \rangle}$ is orthogonal in L and $\bigoplus_{j \in \mathbb{N}} b_j = \bigoplus_{i \in I} a_i$.*

Proof. We shall show first that the sequence $(b_j)_{j \in \langle \mathbb{N}; \leq_0 \rangle}$ is orthogonal in L. Let $n \in \mathbb{N}$ and let $F' = \{\varphi(j) : j \in [0, n]_0\}$. Then F' is a finite non-empty subset of I. Define a total ordering $\leq'$ on F' putting $\varphi(j) <' \varphi(k)$ if $j, k \in [0, n]_0$ and $j <_0 k$. Then $(a_i)_{i \in \langle F'; \leq' \rangle}$ is an orthogonal suite in L. Since $\varphi(0) = \min F'$ and $\varphi(k) = \min(F' \setminus \{\varphi(0), \varphi(1), \ldots, \varphi(k-1)\})$ for all $k \in [1, n]_0$ and $b_j = a_{\varphi(j)}$ for all $j \in [0, n]_0$, it follows from Remark 3.1 that $(b_j)_{0 \leq_0 j \leq_0 n}$ is an orthogonal suite in L, and therefore the sequence $(b_j)_{j \in \langle \mathbb{N}; \leq_0 \rangle}$ is orthogonal in L.

Moreover

$$\bigoplus_{j=0}^{n} b_j = \bigoplus_{i \in F'} a_i \leq_{\mathrm{alg}} \bigoplus_{i \in I} a_i$$

for all $n \in \mathbb{N}$. Whence $\bigoplus_{j \in \mathbb{N}} b_j \leq_{\mathrm{alg}} \bigoplus_{i \in I} a_i$.

To show the other inequality, let $F \in \mathcal{F}(I) \setminus \{\emptyset\}$ and let $\leq$ be a total ordering on F. Then $\varphi^{-1}(F)$ is a finite non-empty subset of $\mathbb{N}$. Let $n = \max \varphi^{-1}(F)$ in $\langle \mathbb{N}; \leq_0 \rangle$. Put $F' = \{\varphi(j) : j \in [0, n]_0\}$. Then $F' \in \mathcal{F}(I)$ and $F' \supseteq F$. Define as above the total ordering $\leq'$ on F'. Then, using Theorem 3.9, Corollary 3.8 and Remark 3.1, we get

$$\bigoplus_{i \in \langle F; \leq \rangle} a_i = \bigoplus_{i \in \langle F; \leq' \rangle} a_i \leq_{\mathrm{alg}} \bigoplus_{i \in \langle F'; \leq' \rangle} a_i = \bigoplus_{j=0}^{n} b_j \leq_{\mathrm{alg}} \bigoplus_{j \in \mathbb{N}} b_j$$

for all $F \in \mathcal{F}(I)$, and therefore $\bigoplus_{i \in I} a_i \leq_{\mathrm{alg}} \bigoplus_{j \in \mathbb{N}} b_j$. □

5. Orthogonality in effect algebras

In this Section, $L = \langle L; \oplus, 0, 1\rangle$ denotes an effect algebra.

The following result will be useful for our presentation:

Lemma 5.1. *Let $a, b, c \in L$. Then*

a) $c \leq_{\text{alg}} a <_{\text{alg}} b \Rightarrow a - c <_{\text{alg}} b - c$ *and* $(b - c) - (a - c) = b - a$.
b) $a \leq_{\text{alg}} b \leq_{\text{alg}} c \Rightarrow a \leq_{\text{alg}} c - (b - a)$ *and* $(c - (b - a)) - a = c - b$.

Proof. a) Since $a = c \oplus (a - c)$ and $b = c \oplus (b - c)$, we get $c \oplus (a - c) <_{\text{alg}} c \oplus (b - c)$ and by [12, Theorem 2.5] we get $a - c <_{\text{alg}} b - c$. On the other hand, since $b = a \oplus (b - a)$, the associative law implies that $b = (c \oplus (a - c)) \oplus (b - a) = c \oplus ((a - c) \oplus (b - a))$ and therefore $b - c = (a - c) \oplus (b - a)$. So $(b - c) - (a - c) = b - a$.

b) Since $b = a \oplus (b - a) \leq_{\text{alg}} c$, we get by Lemma 2.5 a) that $b - a \leq_{\text{alg}} c$ and $a \leq_{\text{alg}} c - (b - a)$. Then by [12, Lemma 4.8 (iv)] and part a), we get $(c - (b - a)) - a = (c - a) - (b - a) = c - b$. □

If $a \in L \setminus \{0\}$ and $a \perp a$, we say that a is an *isotropic element* of L. Clearly, L is an orthoalgebra if and only if it contains no isotropic elements.

Let $a \in L \setminus \{0\}$. We define $0a = 0$ and $1a = a$. If $n \in \mathbb{N} \setminus \{0, 1\}$ and $(a_i)_{1 \leq_0 i \leq_0 n}$ is an orthogonal suite in L of n elements such that $a_i = a$ for all $i \in [1, n]_0$ we write

$$na = \bigoplus_{i=1}^{n} a_i$$

and we say that na is *defined*. If $n \in \mathbb{N} \setminus \{0, 1\}$, na is defined and na is not orthogonal to a, then n is called the *isotropic index* of a. If na is defined for all $n \in \mathbb{N} \setminus \{0, 1\}$, we say that a has an *infinite isotropic index*, and, in this case if $a_i = a$ for all $i \in \mathbb{N}$, then the sequence $(a_i)_{i \in \langle \mathbb{N}; \leq_0 \rangle}$ is orthogonal in L.

For example, in $L = \mathbb{R}^+[0, 1]$ every isotropic element has a finite isotropic index $\geq_0 2$. For an example of an effect algebra with elements of infinite isotropic index, see [13, Example 4.6].

A non-empty subset L_1 of L is said to be a *sub-effect algebra* of L if the following axioms hold:

(1) 0 and 1 belong to L_1.
(2) If $a \in L_1$, then $a' \in L_1$.
(3) If $a, b \in L_1$ and $a \perp b$, then $a \oplus b \in L_1$.

Clearly, in the induced structure, L_1 is an effect algebra in its own right. As such, if L_1 is a Boolean algebra (resp. orthomodular poset), we call it a *Boolean subalbegra* (resp. *sub-orthomodular poset*) of L.

For example, if $L = \mathbb{R}^+[0, 1]$, then $L_1 = \mathbb{Q} \cap [0, 1]$ is a sub-effect algebra of L and $L_2 = \{0, 1\}$ is a Boolean subalgebra of L.

A non-empty subset C of L is say to be a *pairwise orthogonal set* in L if $a, b \in C$ and $a \neq b$ imply $a \perp b$.

For example, if $L = \mathbb{R}^+[0, 1]$ and $C = \{0, \frac{1}{\sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}}\}$, then C is a pairwise orthogonal set in L.

Assume that L is an orthoalgebra and let C be a non-empty subset of L. Recall that

(1) C is a *compatible set* in L if there exists a Boolean subalgebra L_1 of L such that $C \subseteq L_1$.
(2) C is a *jointly orthogonal set* in L if C is a pairwise orthogonal and compatible set in L.

For example, if L is an orthoalgebra, $n \in \mathbb{N} \setminus \{0,1\}$ and $(a_i)_{i \leq_0 i \leq_0 n}$ is an orthogonal suite in $L \setminus \{0\}$, then $C = \{a_i : i \in [1,n]_0\}$ is a jointly orthogonal set in L by [29, Theorem 2.21].

Theorem 5.2. *Assume that L is an orthoalgebra and let C be a pairwise orthogonal set in L. Then C is a jointly orthogonal set in L if and only if C is contained in a sub-orthomodular poset of L.*

Proof. The necessity is trivial.

To prove the sufficiency, suppose that there exists a sub-orthomodular poset L_1 of L such that $C \subseteq L_1$. Let $F \in \mathcal{F}(C) \setminus \{\emptyset\}$. Then F is a finite pairwise orthogonal set in L_1 and [21, Corollary 2.11] implies that the sub-orthoalgebra L_2 of L_1 generated by F is Boolean. So L_2 is a Boolean subalgebra of L such that $F \subseteq L_2$, and therefore F is a jointly orthogonal set in L. By [21, Theorem 3.4] it follows that C is a jointly orthogonal set in L. □

Corollary 5.3. *Assume that L is an orthomodular poset. Then a non-empty subset C of L is a jointly orthogonal set in L if and only if C is a pairwise orthogonal set in L.*

Example 5.4. Consider the set $L = \{0,1\}^{\mathbb{N}}$ with zero $\overline{0} = (0,0,0,\ldots,0,\ldots)$ and unit $\overline{1} = (1,1,1,\ldots,1,\ldots)$. For two elements $e = (e_i)_{i\in\mathbb{N}}$ and $f = (f_i)_{i\in\mathbb{N}}$ of L we define $e \perp f$ if $e_i f_i = 0$ for all $i \in \mathbb{N}$ and, in this case, we put $e \oplus f = (e_i + f_i)_{i\in\mathbb{N}}$. Then it is easy to show that $L = \langle L; \oplus, \overline{0}, \overline{1}\rangle$ is an orthomodular poset. For every $n \in \mathbb{N}$ consider the element a_n of L defined by the formula $a_n = (\delta_{ni})_{i\in\mathbb{N}}$ where $\delta_{ni} = 0$ if $i \neq n$ and $\delta_{nn} = 1$. It is clear that $a_n \neq a_m$ if and only if $n \neq m$. We shall show that $C = \{a_n : n \in \mathbb{N}\}$ is a jointly orthogonal set in $L \setminus \{\overline{0}\}$. In fact, let $n, m \in \mathbb{N}$ be such that $n \neq m$. If $i = n$, then $i \neq m$ and therefore $\delta_{mi} = 0$; if $i = m$, then $i \neq n$ and therefore $\delta_{ni} = 0$; if $i \neq n$ and $i \neq m$, then $\delta_{ni} = 0$ and $\delta_{mi} = 0$. Since this implies that $\delta_{ni}\delta_{mi} = 0$ for all $i \in \mathbb{N}$, it follows that $a_n \perp a_m$. Then Corollary 5.3 implies that C is a jointly orthogonal set in L and it is trivial that $a_n \neq \overline{0}$ for all $n \in \mathbb{N}$.

Now we extend the notion of difference set considered in [14] and [32] for orthoalgebras and we establish its main properties.

Let $n \in \mathbb{N} \setminus \{0\}$ and let D be a finite subset of $L \setminus \{0\}$ such that $\mathrm{card}(D) = n$. We say that D is a *difference set* in L of cardinality n if there exists a finite strictly increasing family $(b_i)_{i\in[0,n]_0}$ in $\langle L; \leq_{\mathrm{alg}}\rangle$ such that $D = \{b_i - b_{i-1} : i \in [1,n]_0\}$ and, in this case, we say that the family $(b_i)_{i\in[0,n]_0}$ *yields* the difference set D.

For example, if $L = \mathbb{R}^+[0,1]$ and $D = \{a, 2a, 3a, 4a\}$ where $a = \frac{1}{10}$, then D is a difference set in L of cardinality four, because

$$0 <_{\mathrm{alg}} a <_{\mathrm{alg}} 3a <_{\mathrm{alg}} 6a <_{\mathrm{alg}} 10a$$

in $\langle L; \leq_{\mathrm{alg}}\rangle$ and $a = a - 0$, $2a = 3a - a$, $3a = 6a - 3a$ and $4a = 10a - 6a$.

Lemma 5.5. *Every difference set in L is a pairwise orthogonal set in $L \setminus \{0\}$.*

Proof. Let $n \in \mathbb{N} \setminus \{0\}$ and let D be a difference set in L of cardinality n yielded by the finite strictly increasing family $(b_i)_{i \in [0,n]_0}$ in $\langle L; \leq_{\mathrm{alg}} \rangle$. Then $D = \{b_i - b_{i-1} : i \in [1,n]_0\}$. By Theorem 3.15 it follows that $(b_i - b_{i-1})_{1 \leq_0 i \leq_0 n}$ is an orthogonal suite in $L \setminus \{0\}$ and since $\mathrm{card}(D) = n$ we have $b_i - b_{i-1} \neq b_j - b_{j-1}$ if $i \neq j$. Then if $J = \{i, j\}$, Corollary 3.8 implies that the suite $(b_k - b_{k-1})_{k \in \langle J; \leq_0 \rangle}$ is orthogonal in L, and therefore $b_i - b_{i-1} \perp b_j - b_{j-1}$. □

Lemma 5.6. *Let $n \in \mathbb{N} \setminus \{0,1\}$, let D be a difference set of cardinality n and let $d \in D$. Then $D \setminus \{d\}$ is a difference set in L of cardinality $n-1$.*

Proof. Suppose that D is yielded by the finite strictly increasing family $(b_i)_{i \in [0,n]_0}$ in $\langle L; \leq_{\mathrm{alg}} \rangle$. Then there exists $k \in [1,n]_0$ such that $d = b_k - b_{k-1}$. Define

$$c_j = b_j \text{ if } j \in [0, k-1]_0 \text{ and } c_j = b_{j+1} - d \text{ if } j \in [k, n-1]_0$$

with $[k, n-1]_0 = \emptyset$ if $k = n$. By Lemma 5.1, $(c_j)_{j \in [0,n-1]_0}$ is a finite strictly increasing family in $\langle L; \leq_{\mathrm{alg}} \rangle$ and $D \setminus \{d\} = \{c_j - c_{j-1} : j \in [1, n-1]_0\}$. Hence $D \setminus \{d\}$ is a difference set in L of cardinality $n-1$. □

Corollary 5.7. *Every non-empty subset of a difference set in L is a difference set in L.*

Lemma 5.8. *Let $n \in \mathbb{N} \setminus \{0\}$ and let D be a difference set of cardinality n. If $(b_i)_{i \in [0,n]_0}$ and $(c_j)_{j \in [0,n]_0}$ are finite strictly increasing families in $\langle L; \leq_{\mathrm{alg}} \rangle$ both of which yield the difference set D, then $b_n - b_o = c_n - c_0$.*

Proof. We may suppose that $n \geq_0 2$.

Write $d_i = b_i - b_{i-1}$ and $e_i = c_i - c_{i-1}$ for all $i \in [1,n]_0$. By Theorem 3.15 $(d_i)_{1 \leq_0 i \leq_0 n}$ and $(e_i)_{1 \leq_0 i \leq_0 n}$ are orthogonal suites in $L \setminus \{0\}$ such that $\bigoplus_{i=1}^n d_i = b_n - b_0$ and $\bigoplus_{i=1}^n e_i = c_n - c_0$. Since

$$D = \{d_i : i \in [1,n]_0\} = \{e_i : i \in [1,n]_0\}$$

there exists a bijection σ from $[1,n]_0$ onto $[1,n]_0$ such that $e_i = d_{\sigma(i)}$ for all $i \in [1,n]_0$. Define a total ordering $\leq$ on $[1,n]_0$ putting $i < j$ if $\sigma^{-1}(i) <_0 \sigma^{-1}(j)$. Then by Corollary 3.10 it follows that $(d_i)_{i \in \langle [1,n]_0; \leq \rangle}$ is an orthogonal suite in L and $\bigoplus_{i=1}^n d_i = \bigoplus_{i \in \langle [1,n]_0; \leq \rangle} d_i$. Since $[1,n]_0 = \{\sigma(i) : i \in [1,n]_0\}$ and $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$, we have $\bigoplus_{i \in \langle [1,n]_0; \leq \rangle} d_i = \bigoplus_{i=1}^n d_{\sigma(i)} = \bigoplus_{i=1}^n e_i = c_n - c_0$ and therefore $b_n - b_0 = c_n - c_0$. □

Let $n \in \mathbb{N} \setminus \{0\}$ and let D be a difference set in L of cardinality n yielded by the finite strictly increasing family $(b_i)_{i \in [0,n]_0}$ in $\langle L; \leq_{\mathrm{alg}} \rangle$. We define

$$\bigoplus D = b_n - b_0$$

noting that $\bigoplus D$ is well-defined by Lemma 5.8.

If we consider the void subset $\emptyset$ of L as a difference set in L of cardinality zero, we can define

$$\bigoplus \emptyset = 0.$$

For example, if $L = \mathbb{R}^+[0,1]$ and $D = \{\frac{1}{4}, \frac{1}{3}, \frac{5}{12}\}$, then it is easy to see that D is a difference set in L of cardinality three with $\bigoplus D = 1$.

Lemma 5.9. *Let $n \in \mathbb{N} \setminus \{0,1\}$ and let D be a difference set in L of cardinality n. If $d \in D$, then $(D \setminus \{d\}) \perp d$ and $\bigoplus D = \bigoplus(D \setminus \{d\}) \oplus d$.*

Proof. Let $(b_i)_{i\in[0,n]_0}$ be a finite strictly increasing family in $\langle L; \leq_{\mathrm{alg}}\rangle$ which yields the difference set D in L and let $k \in [1,n]_0$ be such that $d = b_k - b_{k-1}$.

Suppose that $k = n$. Then $D \setminus \{d\} = \{b_i - b_{i-1} : i \in [1, n-1]_0\}$ and therefore $\bigoplus(D \setminus \{d\}) = b_{n-1} - b_0$. Since $b_0 <_{\mathrm{alg}} b_{n-1} <_{\mathrm{alg}} b_n$ it follows from [12, Remark 4.10 (i)] that $b_{n-1} - b_0 \perp b_n - b_{n-1}$ and $(b_{n-1} - b_0) \oplus (b_n - b_{n-1}) = b_n - b_0$. Hence $d \perp \bigoplus(D \setminus \{d\})$ and $\bigoplus(D \setminus \{d\}) \oplus d = \bigoplus D$.

Suppose that $1 \leq_0 k <_0 n$. Then, by the proof of Lemma 5.6, we have

$$D \setminus \{d\} = \{c_j - c_{j-1} : j \in [1, n-1]_0\}$$

where $c_0 = b_0$, $c_1 = b_1$, $\ldots$, $c_{k-1} = b_{k-1}$, $c_k = b_{k+1} - d$, $\ldots$, $c_{n-1} = b_n - d$. So $\bigoplus(D \setminus \{d\}) = (b_n - d) - b_0$. Since $d \perp b_n - d$ and $(b_n - d) - b_0 \leq_{\mathrm{alg}} b_n - d$, we get by Lemma 2.1 a) that $d \perp (b_n - d) - b_0$ and therefore $d \perp \bigoplus(D \setminus \{d\})$. Moreover, by [12, Lemma 4.8 (iv)] it follows that $(b_n - d) - b_0 = (b_n - b_0) - d$ and therefore $\bigoplus(D \setminus \{d\}) \oplus d = \bigoplus D$. □

Lemma 5.10. *Let $n \in \mathbb{N} \setminus \{0\}$, let D be a difference set in L of cardinality n and let d be an element of $L \setminus \{0\}$ such that $d \notin D$ and $d \perp \bigoplus D$. Then $D \cup \{d\}$ is a difference set in L of cardinality $n+1$ and $\bigoplus(D \cup \{d\}) = (\bigoplus D) \oplus d$.*

Proof. Let $(b_i)_{i\in[0,n]_0}$ be a finite strictly increasing family in $\langle L; \leq_{\mathrm{alg}}\rangle$ which yields the difference set D. Then

$$D = \{b_i - b_{i-1} : i \in [1,n]_0\}$$

and $\bigoplus D = b_n - b_0$. Define $c_0 = 0$, $c_1 = b_1 - b_0$, $\ldots$, $c_n = b_n - b_0$ and $c_{n+1} = (b_n - b_0) \oplus d$ because $d \perp \bigoplus D$. Then $c_{i-1} <_{\mathrm{alg}} c_i$ and $c_i - c_{i-1} = b_i - b_{i-1}$ for all $i \in [1,n]_0$ by Lemma 5.1 and $c_n <_{\mathrm{alg}} c_{n+1}$ because $d \neq 0$ and $c_{n+1} - c_n = d$ by Lemma 2.5 c). So $D \cup \{d\} = \{c_i - c_{i-1} : i \in [1, n+1]_0\}$ and since $d \notin D$, $D \cup \{d\}$ is a difference set in L of cardinality $n+1$ such that $\bigoplus(D \cup \{d\}) = (\bigoplus D) \oplus d$. □

Lemma 5.11. *Let $I = \langle I; \leq\rangle$ be a finite total ordered set with* $\mathrm{card}(I) = n \geq_0 2$ *and let $(a_i)_{i\in I}$ be a family of pairwise different elements of $L \setminus \{0\}$. Then $D = \{a_i : i \in I\}$ is a difference set in L of cardinality n if and only if $(a_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L. Furthermore, $\bigoplus D = \bigoplus_{i\in F} a_i$.*

Proof. It remains to prove the necessity. Write $i_1 = \min I$ and $i_p = \min(I \setminus \{i_1, i_2, \ldots, i_{p-1}\})$ for all $p \in [2,n]_0$ and $c_k = a_{i_k}$ for all $k \in [1,n]_0$. By Remark 3.1, $(c_k)_{1\leq_0 k\leq_0 n}$ is an orthogonal suite in $L \setminus \{0\}$ such that $\bigoplus_{k=1}^n c_k = \bigoplus_{i\in I} a_i$. Clearly,

$$D = \{c_k : k \in [1,n]_0\}.$$

Put

$$b_0 = 0 \text{ and } b_k = \bigoplus_{i=1}^k c_i \text{ for all } k \in [1,n]_0.$$

Then $(b_i)_{i\in[0,n]_0}$ is a finite strictly increasing family in $\langle L;\leq_{\mathrm{alg}}\rangle$ such that $c_k = b_k - b_{k-1}$ for $k\in[1,n]_0$. So $D=\{b_k-b_{k-1}:k\in[1,n]_0\}$ is a difference set in L of cardinality n such that $\bigoplus D=\bigoplus_{i\in I}a_i$. □

Lemma 5.12. *If D and E are two difference sets in L such that $D\cap E=\emptyset$ and $\bigoplus D\perp\bigoplus E$, then $D\cup E$ is also a difference set in L such that $\bigoplus(D\cup E)=(\bigoplus D)\oplus(\bigoplus E)$.*

Proof. Let $n,m\in\mathbb{N}\setminus\{0\}$ and suppose that $\mathrm{card}(D)=n$ and $\mathrm{card}(E)=m$. Then we can write $D=\{d_1,d_2,\dots,d_n\}$ and $E=\{e_1,e_2,\dots,e_m\}$ and since $D\cap E=\emptyset$, the $n+m$ elements of $L\setminus\{0\}$ $d_1,d_2,\dots,d_n,e_1,e_2,\dots,e_m$ are pairwise different and its union gives the set $D\cup E$. Moreover, the suites $(d_i)_{1\leq_0 i\leq_0 n}$ and $(e_j)_{1\leq_0 j\leq_0 m}$ are orthogonal in L as was indicated in the proof of Lemma 5.5. Define

$$c_1=d_1,\dots,c_n=d_n,c_{n+1}=e_1,\dots,c_{n+m}=e_m$$

and put $I_1=[1,n]_0$ and $I_2=[n+1,n+m]_0$. Then $[1,n+m]_0=I_1\cup I_2$, $I_1\cap I_2=\emptyset$, $(c_i)_{i\in\langle I_1;\leq_0\rangle}$ and $(c_i)_{i\in\langle I_2;\leq_0\rangle}$ are orthogonal suites in $L\setminus\{0\}$ such that $\bigoplus_{i\in I_1}c_i\perp\bigoplus_{i\in I_2}c_i$. By Theorem 3.5 it follows that $(c_i)_{i\in\langle[1,n+m]_0;\leq_0\rangle}$ is an orthogonal suite in L such that

$$\bigoplus_{i=1}^{n+m}c_i=\left(\bigoplus_{i\in I_1}c_i\right)\oplus\left(\bigoplus_{i\in I_2}c_i\right).$$

Then Lemma 5.11 implies that $D\cup E$ is a difference set in L of cardinality $n+m$ such that $\bigoplus(D\cup E)=(\bigoplus D)\oplus(\bigoplus E)$. □

Theorem 5.13. *Let $\mathfrak{X}$ be an infinite cardinal number. Consider the following statements:*

i) *L is $\mathfrak{X}$-complete.*
ii) *For every pairwise orthogonal set C in $L\setminus\{0\}$ with $\mathrm{card}(C)\leq_c\mathfrak{X}$ and whose finite subsets are difference sets in L there exists the supremum $\bigvee(\bigoplus D\mid D\in\mathcal{F}(C))$ in $\langle L;\leq_{\mathrm{alg}}\rangle$.*

Then i) *implies* ii). *Further, if L is an orthoalgebra, then* ii) *implies* i).

Proof. i) $\Rightarrow$ ii). Let C be a pairwise orthogonal set in $L\setminus\{0\}$ with $\mathrm{card}(C)\leq_c\mathfrak{X}$ and whose finite subsets are difference sets in L. For every $i\in C$ put $a_i=i$. Then $(a_i)_{i\in C}$ is a family of elements of $L\setminus\{0\}$ such that $C=\{a_i:i\in C\}$. Let $D\in\mathcal{F}(C)\setminus\{\emptyset\}$ and let $\leq$ be a total ordering on D. Then $D=\{a_i:i\in D\}$ is a difference set in L, and therefore the suite $(a_i)_{i\in\langle D;\leq\rangle}$ is orthogonal in L. So $(a_i)_{i\in C}$ is an orthogonal family in L and since L is $\mathfrak{X}$-complete, we deduce that the supremum $\bigvee(\bigoplus D\mid D\in\mathcal{F}(C))$ exists in $\langle L;\leq_{\mathrm{alg}}\rangle$.

ii) $\Rightarrow$ i). Let $(a_i)_{i\in I}$ be an orthogonal family in $L\setminus\{0\}$ such that $\mathrm{card}(I)\leq_c\mathfrak{X}$. Then the set $C=\{a_i:i\in I\}$ is a pairwise orthogonal set in $L\setminus\{0\}$ and since L is an orthoalgebra it follows that $(a_i)_{i\in I}$ is a family of pairwise different elements of $L\setminus\{0\}$. Then it follows from Lemma 5.11 that every finite subset of C is a difference set in L and therefore, by the hypothesis, there exists the supremum $\bigvee(\bigoplus D\mid D\in\mathcal{F}(C))$ in $\langle L;\leq_{\mathrm{alg}}\rangle$.

It remains to show that the $\oplus$-join of the orthogonal family $(a_i)_{i\in I}$ is $\bigvee(\bigoplus D\mid D\in\mathcal{F}(C))$. Clearly $\bigvee(\bigoplus D\mid D\in\mathcal{F}(C))$ is an upper bound of the set $\{\bigoplus_{i\in F}a_i:$

$F \in \mathcal{F}(I)\}$. Suppose $u \in L \setminus \{0\}$ and $\bigoplus_{i\in F} a_i \leq_{\mathrm{alg}} u$ if $F \in \mathcal{F}(I)$ and $\leq$ is a total ordering of F. Then $D = \{a_i : i \in F\}$ is a difference set in L such that $\bigoplus D = \bigoplus_{i\in F} a_i$, and therefore $\bigvee(\bigoplus D \mid D \in \mathcal{F}(C)) \leq_{\mathrm{alg}} u$. □

Note that if L is an orthoalgebra and C is a finite non-empty jointly orthogonal set in L, then $D = C \setminus \{0\}$ is a difference set in L, and we can define

$$\bigoplus C = \bigoplus D.$$

Let $\mathfrak{X}$ be an infinite cardinal number. Following [21] an orthoalgebra L is said to be $\mathfrak{X}$-*orthosummable* if, for every jointly orthogonal set C in L with $\mathrm{card}(C) \leq_c \mathfrak{X}$, there exists the supremum $\bigvee(\bigoplus F \mid F \in \mathcal{F}(C))$ in $\langle L; \leq_{\mathrm{alg}}\rangle$. If L is $\aleph_0$-orthosummable we call it a *σ-orthoalgebra*.

Corollary 5.14. *Assume that L is an orthoalgebra and let $\mathfrak{X}$ be an infinite cardinal number. Then the following statements are equivalent:*

i) *L is $\mathfrak{X}$-complete.*
ii) *L is $\mathfrak{X}$-orthosummable*

Proof. It follows from Theorems 5.2 and 5.13 and [32, Theorem 2.1]. □

Theorem 5.15. *Assume that L is an orthoalgebra. Then the following statements are equivalent:*

i) *L is a σ-orthoalgebra.*
ii) *L is σ-complete.*
iii) *Every countable chain in L has a supremum in $\langle L; \leq_{\mathrm{alg}}\rangle$.*

Proof. It follows from Corollary 5.14 and Theorem 4.3. □

Remark 5.16. Theorem 5.15 contains [21, Theorem 4.14].

As an illustration we prove that the effect algebra $L = \mathbb{R}^+[0,1]$ is complete. Let $\mathfrak{X}$ be an infinite cardinal number and let $(a_i)_{i\in I}$ be an orthogonal family in $L \setminus \{0\}$ with $\mathrm{card}(I) \leq_c \mathfrak{X}$. Let $F \in \mathcal{F}(I) \setminus \{\emptyset\}$ and let $\leq$ be a total ordering on F. Then $(a_i)_{i\in\langle F;\leq\rangle}$ is an orthogonal suite in L, and therefore $\bigoplus_{i\in F} a_i = \sum_{i\in F} a_i \leq_{\mathrm{alg}} 1$. So $\sum_{i\in F} a_i \leq 1$ for all $F \in \mathcal{F}(I)$. By [7, Chapitre IV, §7, n° 1, Théorème 1, p. 171], the family $(a_i)_{i\in I}$ is summable in $\mathbb{R}$ and $\bigoplus_{i\in I} a_i = \sum_{i\in I} a_i \leq_{\mathrm{alg}} 1$. So the $\oplus$-join of the orthogonal family $(a_i)_{i\in I}$ exists in $\langle L; \leq_{\mathrm{alg}}\rangle$. Then L is $\mathfrak{X}$-complete.

6. The support of a measure

Let $L = \langle L; \oplus, 0\rangle$ be a partial abelian monoid and let $M = \langle M; +, 0\rangle$ be a commutative monoid. An element μ of M^L is said to be a *measure on L with values in M* if $\mu(0) = 0$ and $\mu(a \oplus b) = \mu(a) + \mu(b)$ whenever $a, b \in L$ and $a \perp b$. We denote by $\mathrm{a}(L, M)$ the set of all measures on L with values in M.

If $\mu \in \mathrm{a}(L, M)$, we note the following additional properties of μ:

(1) μ is *finitely additive*, that is, if $\langle I; \leq\rangle$ is a finite non-empty totally ordered set and $(a_i)_{i\in\langle I;\leq\rangle}$ is an orthogonal suite in L, then $\mu\left(\bigoplus_{i\in I} a_i\right) = \sum_{i\in I} \mu(a_i)$.

(2) Suppose that L is cancellative. If $a, b \in L$ and $a \leq_{\mathrm{alg}} b$, then $\mu(b) = \mu(a) + \mu(b - a)$.

We consider the following interesting examples:

Example 6.1. Consider the partial abelian monoid $L = \langle \mathbb{R}; \oplus, 0 \rangle$ of Example 2.6 and the commutative monoid $M = \langle M; +, 0 \rangle$ of Example 2.7. Define $\mu : L \to M$ by the formulae:

$$\begin{aligned} &\mu(a) = \mathrm{th}^{-1} a \ \text{ if } a \in [0, 1[\\ &\mu(1) = +\infty \\ &\mu(a) = 0 \ \text{ if } a \in \mathbb{R} \setminus [0, 1]. \end{aligned}$$

Then the additive formula for hyperbolic tangent implies that μ is a measure on L with values in M.

Example 6.2. Recall that a *refinement monoid* is a commutative monoid $M = \langle M; +, 0 \rangle$ satisfying the following axiom

> (Refinement Law) For all $a_0, a_1, b_0, b_1 \in M$ such that $a_0 + a_1 = b_0 + b_1$ there exist elements $c_{ij} \in M$ $(i, j = 0, 1)$ with $a_i = c_{i0} + c_{i1}$ and $b_i = c_{0i} + c_{1i}$ for all $i = 0, 1$.

For example, the usual commutative monoid $\langle \mathbb{N} \cup \{+\infty\}; +, 0 \rangle$ is a refinement monoid.

Following [10] an element $\mu \in \mathrm{a}(L, M)$ is a V-*measure* if M is a conical refinement monoid and the following axioms hold:

(1) If $a \in L$ and $\mu(a) = 0$, then $a = 0$.
(2) If $a \in L$, $x_1, x_2 \in M$ and $\mu(a) = x_1 + x_2$, then there exist $a_1, a_2 \in L$ such that $a_1 \perp a_2$, $a = a_1 \oplus a_2$ and $\mu(a_i) = x_i$ for $i = 1, 2$.

For example, if X is any non-empty set, then the counting measure over X is a V-measure (see [11, p. 54]).

Following [3] we call *po-monoid* a commutative monoid $M = \langle M; +, 0 \rangle$ endowed with a partial ordering $\leq$ on M such that $a, b, c \in M$ and $a \leq b$ imply $a + c \leq b + c$, and we write $M = \langle M; +, 0, \leq \rangle$. For example, every ordered abelian group is a *po*-monoid. Moreover, consider the submonoid $\langle \mathbb{R}_+ \cup \{+\infty\}; +, 0 \rangle$ of the commutative monoid given in the Example 2.7. Then $\langle \mathbb{R}_+ \cup \{+\infty\}; +, 0, \leq_{\mathrm{alg}} \rangle$ is a *po*-monoid by Lemma 2.1.

Let $M = \langle M; +, 0, \leq \rangle$ be a *po*-monoid and let $\mu \in \mathrm{a}(L, M)$. Then we say that μ is a *positive measure* if $0 \leq \mu(a)$ for all $a \in L$. Clearly, if μ is a positive measure on L, then μ is an increasing function from $\langle L; \leq_{\mathrm{alg}} \rangle$ into $\langle M; \leq \rangle$.

Unexplained topological terminology is that of [26].

Following [35] a *uniform semigroup* is a commutative monoid $S = \langle S; +, 0 \rangle$ endowed with a uniformity $\mathcal{U}$ for S such that the function $(a, b) \to a + b$ is uniformly continuous from $S \times S$ into S, and it is denoted by $\langle S; +, 0, \mathcal{U} \rangle$.

Examples 6.3. 1) Every abelian topological group is a uniform semigroup by [7, Chapitre III, §3, n° 3, Théorème 2].

2) Consider the commutative monoid $S_\infty = \langle \mathbb{R}_+ \cup \{+\infty\}; +, 0\rangle$ and let $\mathcal{U}_\infty$ be the uniformity for S_∞ generated by the metric p_∞ defined by the formula $p_\infty(x,y) = |\frac{x}{1+x} - \frac{y}{1+y}|$ with the convention $\frac{+\infty}{1+(+\infty)} = 1$. Noting that p_∞ satisfies the following semi-invariant property: $p_\infty(x+y, x'+y') \leq p_\infty(x,x') + p_\infty(y,y')$ for all $x, x', y, y' \in S_\infty$, we can show that $\langle S_\infty; +, 0, \mathcal{U}_\infty\rangle$ is a uniform semigroup. In fact, let $U \in \mathcal{U}_\infty$. We may suppose that $U = U_\epsilon = \{(x,y) : p_\infty(x,y) < \epsilon\}$ for some $\epsilon > 0$. Since the elements of a base for the product uniformity for $S_\infty \times S_\infty$ are of the form

$$W(U,V) = \{(x,y),(x',y')) : (x,x') \in U \text{ and } (y,y') \in V\}$$

it suffices to choose $W = W(U_{\frac{1}{2}\epsilon}, U_{\frac{1}{2}\epsilon})$ to have $f(W) \subseteq U_\epsilon$ where $f((x,y),(x',y')) = (x+y, x'+y')$.

3) Let G be an abelian topological group and let $\mathcal{N}(0)$ be the neighbourhood system of the identity 0. On the set $\mathcal{S} = 2^G \setminus \{\emptyset\}$ consider the binary operation $S_1 + S_2 = \{x + y : x \in S_1 \text{ and } y \in S_2\}$. Then $\mathcal{S} = \langle \mathcal{S}; +, \{0\}\rangle$ is a commutative monoid. For every $V \in \mathcal{N}(0)$ put $U(V) = \{(S_1, S_2) \in \mathcal{S} \times \mathcal{S} : S_i \subseteq S_{3-i}$ for $i = 1, 2\}$. Then it is easy to show that the set $\{U(V) : V \in \mathcal{N}(0)\}$ is a base for a uniformity $\mathcal{U}$ for $\mathcal{S}$ such that $\langle \mathcal{S}; +, \{0\}, \mathcal{U}\rangle$ is a uniform semigroup.

Let $S = \langle S; +, 0, \mathcal{U}\rangle$ be a uniform semigroup. It is well known that there exists a family P of pseudo-metrics p on S which are semi-invariant and if $U(p,\epsilon) = \{(x,y) \in S \times S : p(x,y) < \epsilon\}$, then the set $\{U(p,\epsilon) : p \in P, \epsilon > 0\}$ is a subbase for the uniformity $\mathcal{U}$ (see [15], [36] and [37]). Clearly if τ denotes the topology generated by $\mathcal{U}$, then the topological space (S,τ) is Hausdorff if and only if the condition $p(x,y) = 0$ for all $p \in P$ and $x, y \in S$ imply $x = y$. In this case we say that the uniform semigroup S is Hausdorff.

Let $S = \langle S; +, 0, \mathcal{U}\rangle$ be a Hausdorff uniform semigroup and let τ be the topology generated by $\mathcal{U}$. Following [16], a family $(a_i)_{i\in I}$ of elements of L is *summable* in S if the net $(\sum_{i\in F} a_i)_{F\in\langle\mathcal{F}(I);\supseteq\rangle}$ is τ-convergent and we write $\tau\text{-}\lim_F \sum_{i\in F} a_i = \sum_{i\in I} a_i$. It is easy to prove that a family $(a_i)_{i\in I}$ in S is summable to an element $s \in S$ if and only if, for every $p \in P$ and every $\epsilon > 0$, there exists $F_0 \in \mathcal{F}(I)$ such that $F \in \mathcal{F}(I)$ and $F \supseteq F_0$ imply $p(\sum_{i\in F} a_i, s) < \epsilon$. Moreover, if $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ are summable families in S, then the family $(a_i + b_i)_{i\in I}$ is summable in S and $\sum_{i\in I}(a_i + b_i) = \sum_{i\in I} a_i + \sum_{i\in I} b_i$.

Lemma 6.4. *Let $(a_i)_{i\in I}$ be a family of elements of S and let I_1 and I_2 be non-empty and disjoint sets whose union is I. If the subfamilies $(a_i)_{i\in I_1}$ and $(a_i)_{i\in I_2}$ are summable in S, then $(a_i)_{i\in I}$ is also summable in S and $\sum_{i\in I} a_i = \sum_{i\in I_1} a_i + \sum_{i\in I_2} a_i$.*

Proof. Let $p \in P$ and let $\epsilon > 0$. Put

$$s_k = \sum\nolimits_{i\in I_k} a_i \text{ for } k = 1, 2.$$

Then there exists $F_k^0 \in \mathcal{F}(I_k)$ such that $F_k \in \mathcal{F}(I_k)$ and $F_k \supseteq F_k^0$ imply

$$p\left(\sum\nolimits_{i\in F_k} a_i, s_k\right) < \frac{1}{2}\epsilon \text{ for } k = 1, 2.$$

Let $F_0 = F_1^0 \cup F_2^0$ and let $F \in \mathcal{F}(I)$ be such that $F \supseteq F_0$. Then $F \cap I_k \in \mathcal{F}(I_k)$, $F \cap I_k \supseteq F_k^0$ $(k = 1, 2)$, $F = (F \cap I_1) \cup (F \cap I_2)$ and $(F \cap I_1) \cap (F \cap I_2) = \emptyset$. So

$$\begin{aligned} p\left(\sum\nolimits_{i\in F} a_i, s_1 + s_2\right) &= p\left(\sum\nolimits_{i\in F\cap I_1} a_i + \sum\nolimits_{i\in F\cap I_2} a_i, s_1 + s_2\right) \\ &\le p\left(\sum\nolimits_{i\in F\cap I_1} a_i, s_1\right) + p\left(\sum\nolimits_{i\in F\cap I_2} a_i, s_2\right) \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \\ &= \epsilon. \end{aligned}$$

Hence the family $(a_i)_{i\in I}$ is summable in S and $\sum_{i\in I} a_i = s_1 + s_2$. □

Lemma 6.5. *Let $(x_i)_{i\in I}$ be a summable family in S, let σ be a bijection from an index set K onto I and let $y_k = x_{\sigma(k)}$ for all $k \in K$. Then $(y_k)_{k\in K}$ is a summable family in S and $\sum_{k\in K} y_k = \sum_{i\in I} x_i$.*

Proof. Let $p \in P$ and let $\epsilon > 0$. Put

$$s = \sum\nolimits_{i\in I} x_i.$$

Then there exists $F_0 \in \mathcal{F}(I)$ such that $F \in \mathcal{F}(I)$ and $F \supseteq F_0$ imply $p\left(\sum_{i\in F} x_i, s\right) < \epsilon$. Let $H_0 = \sigma^{-1}(F_0)$. Then $H_0 \in \mathcal{F}(K)$. Let $H \in \mathcal{F}(K)$ be such that $H \supseteq H_0$. Then $\sigma(H) \in \mathcal{F}(I)$ and $\sigma(H) \supseteq \sigma(H_0) = F_0$, and therefore

$$p\left(\sum\nolimits_{i\in\sigma(H)} x_i, s\right) < \epsilon.$$

Write $H = \{k_0, k_1, \ldots, k_n\}$ where $n \in \mathbb{N}$. Then $\sigma(H) = \{\sigma(k_0), \sigma(k_1), \ldots, \sigma(k_n)\}$ and the generalized commutative law for $\langle S; \oplus, 0\rangle$ imply that

$$\sum\nolimits_{j\in H} y_j = \sum\nolimits_{i=0}^{n} y_{k_i} = \sum\nolimits_{i=0}^{n} x_{\sigma(k_i)} = \sum\nolimits_{i\in\sigma(H)} x_i$$

and therefore

$$p\left(\sum\nolimits_{j\in H} y_j, s\right) < \epsilon.$$

Consequently, the family $(y_k)_{k\in K}$ is summable in S and $\sum_{k\in K} y_k = s$. □

Let $L = \langle L; \oplus, 0\rangle$ be a partial abelian monoid, let $S = \langle S; +, 0, \mathcal{U}\rangle$ be a Hausdorff uniform semigroup and let $\mu \in \mathrm{a}(L, S)$. We say that

(1) μ is *s-bounded* if $\tau\text{-}\lim_n \mu(a_n) = 0$ for every orthogonal sequence $(a_n)_{n\in\langle\mathbb{N};\le_0\rangle}$ in L.
(2) μ is *countably additive* if, for every orthogonal sequence $(a_i)_{i\in\langle\mathbb{N};\le_0\rangle}$ such that the $\oplus$-join $\bigoplus_{i\in\mathbb{N}} a_i$ exists in $\langle L; \le_{\mathrm{alg}}\rangle$, the family $(\mu(a_i))_{i\in\mathbb{N}}$ is summable in S and $\mu\left(\bigoplus_{i\in\mathbb{N}} a_i\right) = \sum_{i\in\mathbb{N}} \mu(a_i)$.
(3) μ is *completely additive* if, for every orthogonal family $(a_i)_{i\in I}$ in L such that the $\oplus$-join $\bigoplus_{i\in I} a_i$ exists in $\langle L; \le_{\mathrm{alg}}\rangle$, the family $(\mu(a_i))_{i\in I}$ is summable in S and $\mu\left(\bigoplus_{i\in I} a_i\right) = \sum_{i\in I} \mu(a_i)$.

For a nice example of an element of $\mathrm{a}(L, S)$, where L is an effect algebra and S is a Hausdorff abelian topological group, see [9, Example 4.4 (2)].

Remarks 6.6. (1) It is clear that every element of $\mathrm{a}(L,S)$ which is completely additive is also countably additive.

(2) If S is a Hausdorff abelian topological group and L is a σ-complete effect algebra, then every element of $\mathrm{a}(L,S)$ which is countably additive is also s-bounded (see [18, Lemma 3.1]). But it is not true if S is a uniform semigroup. For example, consider the commutative monoid $\langle \mathbb{N}\cup\{+\infty\};+,0\rangle$ and let $\mathcal{U}$ be the uniformity for $\mathbb{N}\cup\{+\infty\}$ determined by the semi-invariant metric $p(x,y)=|\frac{x}{1+x}-\frac{y}{1+y}|$ with the convention $\frac{+\infty}{1+(+\infty)}=1$. Then $\langle \mathbb{N}\cup\{+\infty\};+,0,\mathcal{U}\rangle$ is a Hausdorff uniform semigroup. Let the usual complete Boolean algebra $L=\langle 2^{\mathbb{N}};\cap,\cup\rangle$ and consider the counting measure μ over $\mathbb{N}$. Then μ is countably additive, but not s-bounded because $\mu(\{n\})=1$ for any $n\in\mathbb{N}$ and $(\{n\})_{n\in\langle\mathbb{N};\leq_0\rangle}$ is an orthogonal sequence in L.

(3) There are also elements of $\mathrm{a}(L,S)$ which are s-bounded but not countably additive, even if L is a complete effect algebra and S is a Hausdorff abelian topological group (see [18, Example 3.2]).

A uniform semigroup $S=\langle S;+,0,\mathcal{U}\rangle$ is said to be *ordered* if there exists a partial ordering $\leq$ on S such that $\langle S;+,0,\leq\rangle$ is a *po*-monoid and we write $S=\langle S;+,0,\mathcal{U},\leq\rangle$. For example, $\langle S_\infty;+,0,\mathcal{U}_\infty,\leq_{\mathrm{alg}}\rangle$ is an ordered uniform semigroup.

Recall that, following [23], an *ordered topological group* is an ordered abelian group which is also a topological group. For nice examples of positive measures on an orthoalgebra with values in an ordered Hausdorff topological group, see [29, Examples 5.1].

In the remainder of this Section, $L=\langle L;\oplus,0,1\rangle$ denotes an effect algebra and $S=\langle S;+,0,\mathcal{U},\leq\rangle$ denotes an ordered Hausdorff uniform semigroup.

If $\mu\in\mathrm{a}(L,S)$, then the subset of L $\ker\mu=\{a\in L:\mu(a)=0\}$ is called the *kernel* of μ and clearly $0\in\ker\mu$.

Let μ be a positive element of $\mathrm{a}(L,S)$. Following [33], we say that

(1) μ is a *Jauch-Piron measure* if every finite subset M of $\ker\mu$ has an upper bound in $\ker\mu$.

(2) μ has a *support* if there exists an element $s\in L$ such that $\ker\mu=[0,s']$. The support of μ, when it exists, is unique, and will be denoted by $\mathrm{supp}(\mu)$.

Example 6.7. For every $i\in\mathbb{N}$ consider the effect algebra $L_i=\mathbb{R}^+[0,1]$ and let $L=\prod_{i\in\mathbb{N}}L_i$ with zero $\overline{0}=(0,0,0,\ldots)$ and unit $\overline{1}=(1,1,1,\ldots)$. Choose the usual ordered Hausdorff uniform semigroup $S=S_\infty$. For every element $a=(a_i)_{i\in\mathbb{N}}$ of L put

$$\mu(a)=\sum_{i=0}^{\infty}a_i.$$

Clearly, μ is a positive measure of $\mathrm{a}(L,S)$. Moreover, since $\ker\mu=\{\overline{0}\}$, it follows that μ is a Jauch-Piron measure with support $\overline{1}$.

Example 6.8. Let H be a complex Hilbert space of dimension not less than 3 with inner product $(\cdot\mid\cdot)$, let I be the identity operator on H, let $B_{sa}(H)$ be the

additive abelian group of all self-adjoint operators on H ordered by the positive cone $B_{sa}(H)^+ = \{A \in B_{sa}(H) : (A\varphi \mid \varphi) \geq 0 \text{ for all } \varphi \in H\}$. Clearly, $0, I \in B_{sa}(H)^+$ and $0 < I$. Then we can define the effect algebra

$$\mathcal{E}(H) = B_{sa}(H)^+[0, I]$$

called the *standard effect algebra* on H. Let $L = \mathcal{G}(H)$ be the sub-effect algebra of $\mathcal{E}(H)$ consisting of all idempotent elements in $\mathcal{E}(H)$. Let $\varphi \in H$ be such that $\|\varphi\| = 1$ and let S be the usual ordered additive topological group of real numbers. For every operator P in L put

$$\mu(P) = (P\varphi \mid \varphi).$$

Then it is easy to see that μ is a positive measure on L with values in S. Moreover,

$$\ker \mu = \{P \in L : (P\varphi \mid \varphi) = 0\}$$

and since

$$\begin{aligned}(P\varphi \mid \varphi) &= (P^2\varphi \mid \varphi)\\ &= (P\varphi \mid P^*\varphi)\\ &= (P\varphi \mid P\varphi)\end{aligned}$$

we get

$$\ker \mu = \{P \in L : P\varphi = 0\}.$$

Let Q be the element of L defined by the formula

$$Q\psi = (\psi \mid \varphi)\varphi \text{ for all } \psi \in H.$$

Let $P \in \ker \mu$. Since

$$\begin{aligned}(QP)\psi &= Q(P\psi)\\ &= (P\psi \mid \varphi)\varphi\\ &= (\psi \mid P\varphi)\varphi\\ &= 0\end{aligned}$$

we get that $(I - Q)P = P$. Then [2, Theorem 4.5, p. 112] implies that $P \leq I - Q$, and therefore

$$\ker \mu = [0, I - Q].$$

So the support of μ exists and it is equal to Q, which is a one-dimensional projector acting in H.

The main results of this Section are the following:

Theorem 6.9. *Assume that L is complete and the uniform space $(S, \mathcal{U})$ is metrizable. If μ is a positive s-bounded countably additive element of* $\mathrm{a}(L, S)$ *with support, then μ is a completely additive Jauch-Piron measure.*

Proof. Write $s = \mathrm{supp}(\mu)$. Let M be a finite subset of $\ker \mu$. So $M \subseteq \ker \mu = [0, s']$, and therefore $\mu(s') = 0$ and $m \in M$ implies $m \leq_{\mathrm{alg}} s'$. Hence μ is a Jauch-Piron measure.

Let $(a_i)_{i\in I}$ be an orthogonal family in $L\setminus\{0\}$. Since L is a complete effect algebra, there exists the $\oplus$-join of the orthogonal family $(a_i)_{i\in I}$ in $\langle L;\leq_{\rm alg}\rangle$. Write $a=\bigoplus_{i\in I}a_i$ and $J=\{i\in I:\mu(a_i)\neq 0\}$. We may suppose that J is infinite. Since the uniform space $(S,\mathcal{U})$ is metrizable, it follows from [26, 13 Metrization Theorem, p. 186] that the uniformity $\mathcal{U}$ has a countable base $\{U_n:n\in\mathbb{N}\}$. Then $\{U_n[0]:n\in\mathbb{N}\}$ is a countable base for the neighbourhood system $\mathcal{N}(0)$ of the identity 0. For every $n\in\mathbb{N}$, let $J_n=\{i\in J:\mu(a_i)\notin U_n[0]\}$. We shall show that every J_n is finite. Suppose that there exists $m\in\mathbb{N}$ such that J_m is infinite. By [6, §6, n$^{\rm o}$ 1, Lemma 1, p. 67] there exists a countably infinite subset K of J_m. Let σ be a bijection from $\mathbb{N}$ onto K. Write $b_k=a_{\sigma(k)}$ for all $k\in\mathbb{N}$. Then $(b_k)_{k\in\mathbb{N}}$ is a subfamily of $(a_i)_{i\in I}$, and therefore $(b_k)_{k\in\langle\mathbb{N};\leq_0\rangle}$ is an orthogonal sequence in $L\setminus\{0\}$. Since μ is s-bounded, it follows that τ-$\lim_k\mu(b_k)=0$, a contradiction with the fact that $\mu(b_k)\notin U_m[0]$ for all $k\in\mathbb{N}$.

Since S is a Hausdorff uniform semigroup, it follows that $\bigcap_{n=0}^{\infty}U_n[0]=\{0\}$, and therefore $J\subseteq\bigcup_{n=0}^{\infty}J_n$. Hence J is a countably infinite set. Write $b=\bigoplus_{i\in J}a_i$ and $c=\bigoplus_{i\in I\setminus J}a_i$. By Corollary 4.6 we have $b\perp c$ and $a=b\oplus c$. So $\mu(a)=\mu(b)+\mu(c)$.

Let $F\in\mathcal{F}(I\setminus J)$. By the finite additivity of μ it follows that

$$\mu\left(\bigoplus\nolimits_{i\in F}a_i\right)=\sum\nolimits_{i\in F}\mu(a_i)=0.$$

Hence $\bigoplus_{i\in F}a_i\leq_{\rm alg}s'$ for all $F\in\mathcal{F}(I\setminus J)$. So $c=\bigvee\left(\bigoplus_{i\in F}a_i\mid F\in\mathcal{F}(I\setminus J)\right)\leq_{\rm alg}s'$. Since μ is positive and $\mu(s')=0$, we have $\mu(c)=0$. Moreover, since $\mu(a_i)=0$ for all $i\in I\setminus J$, it follows that the family $(\mu(a_i))_{i\in I\setminus J}$ is trivially summable in S and $\sum_{i\in I\setminus J}\mu(a_i)=0=\mu(c)$.

On the other hand, let φ be a bijection from $\mathbb{N}$ onto J and put $b_k=a_{\varphi(k)}$ for all $k\in\mathbb{N}$. Then by Theorem 4.7 it follows that $(b_k)_{k\in\langle\mathbb{N};\leq_0\rangle}$ is an orthogonal sequence in $L\setminus\{0\}$ such that $\bigoplus_{k\in\mathbb{N}}b_k=\bigoplus_{j\in J}a_j=b$. Since μ is countably additive, we get that the family $(\mu(b_k))_{k\in\mathbb{N}}$ is summable in S and $\sum_{k\in\mathbb{N}}\mu(b_k)=\mu(b)$. Put $x_k=\mu(b_k)$ for all $k\in\mathbb{N}$ and $\sigma=\varphi^{-1}$. Then σ is a bijection from J onto $\mathbb{N}$. Let $y_j=x_{\sigma(j)}$ for all $j\in J$. So $y_j=\mu(b_{\sigma(j)})=\mu(a_j)$ for all $j\in J$. By Lemma 6.5 it follows that the family $(\mu(a_j))_{j\in J}$ is summable in S and $\sum_{j\in J}\mu(a_j)=\mu(b)$. Then Lemma 6.4 implies that the family $(\mu(a_i))_{i\in I}$ is summable in S and $\sum_{i\in I}\mu(a_i)=\mu(b)+\mu(c)=\mu(a)$ proving that μ is completely additive. □

Theorem 6.10. *Assume that L is complete and let μ be an element of* $\mathrm{a}(L,S)$. *If μ is a positive completely additive Jauch-Piron measure, then μ has a support.*

Proof. We may suppose that there exists $a\in L\setminus\{0\}$ such that $\mu(a)=0$. Consider the partially ordered set $(\mathcal{C},\subseteq)$, where $\mathcal{C}=\{C\in 2^L\setminus\{\emptyset\}:C$ is a pairwise orthogonal set in $L\setminus\{0\}$ whose finite subsets are difference sets in L such that $\mu(b)=0$ for all $b\in C\}$. Since $C=\{a\}\in\mathcal{C}$, the set $\mathcal{C}$ is non-empty. Let $\mathcal{C}_0$ be a totally ordered subset of $\mathcal{C}$ and let $C_0=\bigcup\{C:C\in\mathcal{C}_0\}$. Clearly C_0 is a non-empty subset of $L\setminus\{0\}$ such that $\mu(b)=0$ for all $b\in C_0$. Let D be a finite subset of C_0. Since $\langle\mathcal{C}_0;\subseteq\rangle$ is a totally ordered set, there exists $C\in\mathcal{C}_0$ such that $D\subseteq C$, and therefore D is a difference set in L. To show that C_0 is a pairwise orthogonal set in $L\setminus\{0\}$, let $a_1,a_2\in C_0$ be such that $a_1\neq a_2$. Then there exist $C_1,C_2\in\mathcal{C}_0$

such that $a_1 \in C_1$, and $a_2 \in C_2$. We may suppose that $C_1 \subseteq C_2$. Then $a_1, a_2 \in C_2$ and therefore $a_1 \perp a_2$. Consequently, C_0 is an upper bound of the set $\mathcal{C}_0$ and by Zorn Lemma, the partially ordered set $\langle \mathcal{C}; \subseteq \rangle$ contains a maximal element E_0. For every $i \in E_0$ put $a_i = i$. Then $(a_i)_{i \in E_0}$ is a family of elements of $L \setminus \{0\}$ such that $E_0 = \{a_i : i \in E_0\}$. Since L is complete, the proof of Theorem 5.13 implies that the family $(a_i)_{i \in E_0}$ is orthogonal in L and there exists $s = \bigoplus_{i \in E_0} a_i$ in $\langle L; \leq_{\mathrm{alg}} \rangle$. Since μ is completely additive, the family $(\mu(a_i))_{i \in E_0}$ is summable in S and $\sum_{i \in E_0} \mu(a_i) = \mu(s)$. Since $\mu(a_i) = 0$ for all $i \in E_0$ we get $\mu(s) = 0$, and therefore $s \in \ker \mu$.

We shall show that $s' = \mathrm{supp}(\mu)$. For this we must prove that $\ker \mu = [0, s]$. Let $c \in \ker \mu$. Choose $M = \{s, c\}$. Since μ is a Jauch-Piron measure, there exists $d \in \ker \mu$ such that $s \leq_{\mathrm{alg}} d$ and $c \leq_{\mathrm{alg}} d$. Write $e = d - s$. Then $e \perp s$. Since $a_i \leq_{\mathrm{alg}} s$ for all $i \in E_0$, it follows from Lemma 2.1 a) that $e \perp a_i$ for all $i \in E_0$. Moreover, since μ is positive, $\mu(d) = 0$ and $e \leq_{\mathrm{alg}} d$, it follows that $\mu(e) = 0$. Since $c \leq_{\mathrm{alg}} d$, to prove that $c \leq_{\mathrm{alg}} s$, it suffices to establish that $e = 0$.

Suppose that $e \neq 0$ and let $E = E_0 \cup \{e\}$. Then E is a pairwise orthogonal set in $L \setminus \{0\}$ such that $\mu(b) = 0$ for all $b \in E$. Moreover, let $D \in \mathcal{F}(E)$. Then $D \setminus \{e\} \in \mathcal{F}(E_0)$ is a difference set in L such that $\bigoplus(D \setminus \{e\}) \leq_{\mathrm{alg}} s$ and therefore $\bigoplus(D \setminus \{e\}) \perp e$. By Lemma 5.10 it follows that D is a difference set in L. Then the set E belongs to $\mathcal{C}$ and strictly contains E_0. This contradicts the maximality of E_0. So $e = 0$ and the proof is complete. □

Remark 6.11. It is clear that Theorem 6.10 is a twofold generalization of [29, Theorem 5.7] and Theorem 6.9 is a twofold generalization of [29, Theorem 5.6] for the case of an ordered metrizable topological group. Consequently, they are improvements of an interesting result of Maeda [28, Proposition 1.11, p. 239].

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