On a better lower bound for the frequentist probability of coverage of Bayesian credible intervals in restricted parameter spaces [1]

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Abstract. For estimating a lower restricted parametric function in the framework of Marchand and Strawderman (2006), we show how $1-\alpha \times 100\%$ Bayesian credible intervals can be constructed so that the frequentist probability of coverage is no less than $1-\frac{3\alpha}{2}$. As in Marchand and Strawderman (2013), the findings are achieved through the specification of the *spending function* of the Bayes credible interval and apply to an "equal-tails" modification of the HPD procedure. By exploiting an assumption of logconcavity, we obtain the tighter lower bound $1-\frac{3\alpha}{2}$ ($> \frac{1-\alpha}{1+\alpha}$, for $\alpha < 1/3$), as in Marchand et al. (2008) for the HPD procedure with the assumption of symmetry. Key examples include lower bounded scale parameters from Gamma, Weibull, and Fisher distributions, with the latter also applicable to random effects analysis of variance (e.g., Zhang and Woodroofe, 2002).

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1. Introduction

1.1. Matching frequentist probability of coverage and Bayesian credibility in unrestricted parameter space problems

Frequentist coverage probability is an interesting and informative measure of the efficiency of a Bayes credible set procedure, in particular when the latter is generated through a default or non-informative prior. Of course, it has long been known that there are certain situations (e.g., Lindley, 1958) where a Bayes $1-\alpha$ credible set can be chosen to have exact probability coverage $1-\alpha$. Examples of such procedures include basic tools in the statistician's arsenal such as the z and t intervals $\bar{x} \pm z_{\alpha/2}\sigma/\sqrt{n}$ and $\bar{x} \pm t_{\alpha/2}s/\sqrt{n}$ with exact frequentist coverage probability and exact Bayes credibility $1-\alpha$, arising for samples from a $N(\mu, \sigma^2)$ population, and the non-informative priors $\pi(\mu) = 1$ and $\pi(\mu, \sigma) = \frac{1}{\sigma}$ respectively. There are a vast class of location, scale, or location-scale family inference problems (see Marchand and Strawderman (2006, 2013) for an enumeration of specific examples) where there is a match between the credibility and frequentist probability coverage of Bayes confidence intervals, and which will relate to the contributions of this paper.

Example 1. *Consider an observable X with Lebesgue density $f(x;\theta), x \in \mathcal{X}$, $\theta \in \Theta \subset \mathbb{R}^p$ and the problem of estimating a parametric function $\tau(\theta)$ ($\mathbb{R}^p \to \mathbb{R}$). Assume there exists a pivot of the form $T(X,\theta) = \frac{a_1(X)-\tau(\theta)}{a_2(X)}$; $a_2(\cdot) > 0$; such that $-T(X,\theta)$ has cdf G and Lebesgue density g. Observe at this point that if c, d are such that $G(d) - G(c) = 1-\alpha$, then the confidence interval*

$$I(X) = [\, a_1(X) + c a_2(X), a_1(X) + d a_2(X) \,] \tag{1}$$

has frequentist probability of coverage $P_\theta(I(X) \ni \tau(\theta)) = 1-\alpha$, for all $\theta \in \Theta$.

Further assume that the decision problem is invariant under a group $\mathcal{G}$ of transformations and that the pivot satisfies the invariance requirement $T(x,\theta) = T(gx, \bar{g}\theta)$, for all $x \in \mathcal{X}$, $\theta \in \Theta$, $g \in \mathcal{G}$, $\bar{g} \in \bar{\mathcal{G}}$, with $\mathcal{X}$, Θ, G, and $\bar{G}$ being isomorphic. Relative to this group structure, consider the Haar *right invariant prior π_H.* [2] *A key feature of this choice of prior is that*

$$T(x,\theta)|x \ =^d \ T(X,\theta)|\theta\,,\ \textit{for all } x,\theta,\ \textit{when } \theta \sim \pi_H\,, \tag{2}$$

in other words the posterior distribution of $T(x,\theta)$ is free of x and matches the pivotal distribution of $T(X,\theta) \sim G$ (see Marchand and Strawderman, 2006, Corollary 1, for more details).

Now given property (2), the confidence interval in (1) has credibility

$$P(\tau(\theta) \in I(x)|x) \ = P(-T(x,\theta) \in [c,d]|x) = 1-\alpha\,,\ \textit{for all } x,$$

which matches indeed the frequentist probability of coverage.

1.2. Unmatching and challenges in the presence of parametric restrictions

Now, consider the context of Example 1 with the parametric restriction $\tau(\theta) \geq a$ for some known a. Clearly, the truncation of $I(X) \cap [a,\infty)$ preserves frequentist probability of coverage $1-\alpha$ for the restricted parameter space $\{\theta : \tau(\theta) \geq a\}$ but it is not a Bayes credible set anymore. As investigated by Mandelkern (2002) for estimating the mean μ of a $N(\mu,\sigma^2)$ distribution with known σ^2, several frequentist based and Bayesian options remain but they differ. Namely, the $(1-\alpha)\times 100\%$ highest posterior density (HPD) Bayes credible set associated with the prior $\pi_0(\theta) = \pi_H(\theta)\,\mathbb{I}_{[a,\infty)}(\tau(\theta))$, i.e. the truncation of π_H on the restricted parameter space, has frequentist probability of coverage which fluctuates about its credibility $1-\alpha$. However, the HPD procedure does not fare poorly as a frequentist procedure for large $1-\alpha$ as witnessed by the lower bound $\frac{1-\alpha}{1+\alpha}$ on its frequentist probability of coverage due to Roe and Woodroofe (2000, known σ^2) and Zhang and Woodroofe (2003, unknown σ^2), as well as the better lower bound $1-\frac{3\alpha}{2}$ (for $\alpha < 1/3$, known σ^2) obtained by Marchand et al. (2008).

[2] This satisfies the property $\pi_H(A\,\bar{g}) = \pi_H(A)$ for every measurable subset A of Θ, and for every $g \in G$. Such a measure π_H exists and is unique up to a multiplicative constant for locally compact groups such as location, scale, and location-scale. We refer to Berger (1985) or Eaton (1989) for detailed treatments of invariance and Haar invariant measures.

In a generalization of the above, Marchand and Strawderman (2006) introduced the unified framework of Example 1 and showed for unimodal and symmetric densities g that the Bayes $(1-\alpha)\times 100\%$ HPD credible set of $\tau(\theta)$, associated with the truncated prior π_0 has frequentist probability of coverage greater than $\frac{1-\alpha}{1+\alpha}$ for all values θ lying in the restricted parameter space $\{\theta : \tau(\theta) \geq a\}$.

Marchand and Strawderman (2013) further extended the applicability of the $\frac{1-\alpha}{1+\alpha}$ lower bound for frequentist probability of coverage without the symmetry assumption on g by focussing on different choices of a $(1-\alpha)\times 100\%$ Bayes credible set associated with prior π_0. To this end, they introduced the notion of a spending function as a descriptor of the Bayes credible set (Definition 1), and gave conditions on the spending function guaranteeing frequentist probability of coverage bounded below by $\frac{1-\alpha}{1+\alpha}$ (see Theorem 1). Notably, such Bayes credible sets include an "equal-tails" procedure (see Definition 2) which corresponds to the HPD credible set in the symmetric case.

Despite the wide applicability of the $\frac{1-\alpha}{1+\alpha}$ lower bound for the probability of coverage, it remains somewhat conservative as illustrations show (e.g., see above references), and as addressed by Marchand et al. (2008) who established the tighter lower bound $1-\frac{3\alpha}{2}$ (for $\alpha < 1/3$) for frequentist probability of coverage of the HPD credible set in the normal case with known variance, and more generally for location families with logconcave density g. The main contribution in this paper is the establishment of the $1-\frac{3\alpha}{2}$ lower bound for the frequentist probability of coverage of a class of $(1-\alpha)\times 100\%$ Bayes credible sets, and for location and scale families that verify a logconcavity assumption but without an assumption of symmetry. Pivotal scale family examples include Gamma, Weibull, and Fisher models, with the latter also arising in random effects analysis of variance (see Zhang and Woodroofe, 2002). In contrast to Marchand et al. (2008), our results apply to a class of Bayes credible sets even in the non-symmetric case, as well as for a much larger class of densities. In contrast to Marchand and Strawderman (2013), we obtain a better lower bound for a subclass of Bayes credible sets but we do require a log-concavity assumption (e.g., Example 2). Following definitions and preliminary results in Section 2, a sequence of results in Section 3 lead to the main result of the manuscript. Illustrations and concluding remarks follow in Sections 4 and 5 respectively.

2. Definitions, assumptions, and preliminary results

We assume hereafter a location family structure in the context of Example 1 (i.e., $a_2(\cdot) = 1$), and we set $Y = a_1(X)$ so that we have a pivot of the form

$$T(X, \theta) = Y - \tau(\theta)\,. \tag{3}$$

We represent the distribution and density functions of $-T(X, \theta)$ by G and g respectively, and we assume that g is logconcave (implying its unimodality) with a mode at 0 (the latter assumed without loss of generality as we can always rewrite the pivot as $Y - m - (\tau(\theta) - m)$ when the mode is equal to $m \neq 0$). Equivalently, the logconcavity means that the location family of densities $g(t - \tau(\theta))\,,\ t \in \mathbb{R}\,,$ of Y has an increasing monotone likelihood ratio (mlr) in Y with parameter $\tau(\theta)$. Summarizing the above, we have for further reference.

Assumption 1. *We have a model density $f(x;\theta)$; $x \in \mathcal{X}$, $\theta \in \Theta$; for an observable X, with both X and θ being vectors, and we seek to estimate a parametric function $\tau(\theta)$ ($\mathbb{R}^p \to \mathbb{R}$) with the constraint $\tau(\theta) \geq 0$. We assume there exists a pivot $T(X, \theta) = a_1(X) - \tau(\theta) = Y - \tau(\theta)\,,$ such that $-T(X, \theta)$ has cdf G, Lebesgue density g taken to be logconcave with a mode at 0. We further assume that the decision problem is invariant under a group $\mathcal{G}$ of transformations and that the pivot satisfies the invariance requirement $T(x, \theta) = T(gx, \bar{g}\theta)$, for all $x \in \mathcal{X}$, $\theta \in \Theta$, $g \in \mathcal{G}$, $\bar{g} \in \bar{\mathcal{G}}$, with $\mathcal{X}$, Θ, G, and $\bar{G}$ being isomorphic.*

Example 2. *The set-up includes:* **(i)** *simple location parameter families $X \sim f_0(x - \theta)$, such as normal $N(\theta, 1)$ distributions, with $g = f_0$ logconcave and pivot $X - \theta$; and* **(ii)** *simple scale parameter families $X \sim \frac{1}{\theta}\, f_1(\frac{x}{\theta})\, \mathbb{I}_{(0,\infty)}(x)$, such as Gamma, Fisher, Weibull distributions, with $Y = a_1(X) = \log(X)$, $\tau(\theta) = \log(\theta)$ and such that the density of $\log(\theta) - \log X$, given by $e^{-t}\, f_1(e^{-t})$, is logconcave in t.* [3] *We refer to Marchand and Strawderman (2013) for many further details and examples. The Gamma and Fisher examples are noteworthy from a technical angle, as the resulting*

[3] In terms of f_1, this is equivalent to $\frac{u f_1'(u)}{f_1(u)}$ decreasing in u which is weaker than logconcave f_1. A good illustration are Fisher densities of the form $f_1(u) = \frac{\Gamma(r+s)}{\Gamma(r)\,\Gamma(s)} \frac{u^{r-1}}{(1+u)^{r+s}}\, \mathbb{I}_{(0,\infty)}(u)\,,\ r, s > 0$, which are not log-concave for $r > 1$ but yet satisfy our conditions.

density of the pivot is not symmetric (with the exception of the Fisher densities with $r = s$) and the results of Marchand et al. (2008) do not apply. Furthermore, from a practical point of view, several models generate a sufficient statistic which is Gamma distributed (e.g., Rahman and Gupta, 1993), and the Fisher case arises in random effects analysis of variance, as described for instance by Zhang and Woodroofe (2002), who were also motivated by the study of the frequentist probability of coverage of Bayesian credible sets.

Remark 1. *The findings of this paper are applicable as well to an upper bound restriction $\tau(\theta) \leq a$, such as an upper bounded location or scale parameter. Indeed, in such cases, we can set $T'(X,\theta) = Y' - \tau'(\theta) = -T(X,\theta)$, with $Y' = -Y$ and $\tau'(\theta) = -\tau(\theta)$, and use results applicable to estimate $\tau'(\theta)$ under the lower bound restriction $\tau'(\theta) \geq a' = -a$ with $-T'(X,\theta)$ having cdf equal to $1-G(-t)$ and the logconcavity assumption on the corresponding pdf preserved.*

We consider throughout the uniform prior $\pi_0(\theta) = \mathbb{I}_{(0,\infty)}(\tau(\theta))$, i.e., the truncation of Example 1's π_H onto the restricted parameter space, associated Bayesian intervals with credibility $1-\alpha$, and the frequentist probability of coverage $C(\theta) = P_\theta(I_{\pi_0}(X) \ni \tau(\theta))$ evaluated on the restricted parameter space $\{\theta : \tau(\theta) \geq 0\}$. We obtain conditions on the choice of interval for which the frequentist probability of coverage is bounded uniformly below by $1 - \frac{3\alpha}{2}$. This is achieved through consideration of *spending functions* $\alpha(Y)$ and associated Bayes credible sets $I_{\pi_0,\alpha(\cdot)}(Y)$. To this end, we recall previously established results of Marchand and Strawderman (2013), applicable for the specific case of Assumption 1, and with details provided for sake of completeness.

Definition 1. *For the uniform prior $\pi_0(\theta) = \mathbb{I}_{(0,\infty)}(\tau(\theta))$ and credibility $1-\alpha$, a spending function $\alpha(\cdot) : \mathbb{R} \to [0,\alpha]$ is such that, for all y, $P_{\pi_0}(\tau(\theta) \geq u(y)|y) = \alpha(y)$, $P_{\pi_0}(\tau(\theta) \leq l(y)|y) = \alpha - \alpha(y)$, and $[l(y), u(y)]$ is a $(1-\alpha)\times 100\%$ Bayesian credible interval for $\tau(\theta)$.*

Lemma 1. *Under Assumption 1, for the uniform prior $\pi_0(\theta) = \mathbb{I}_{(0,\infty)}(\tau(\theta))$ and a given spending function $\alpha(\cdot)$, the associated Bayes credible set $I_{\pi_0,\alpha(\cdot)}(y)$ is given by:*

$$l_{\alpha(\cdot)}(y) = y + G^{-1}\{G(-y) + (\alpha - \alpha(y))(1 - G(-y))\} \text{ and } u_{\alpha(\cdot)}(y) = y + G^{-1}\{1 - \alpha(y)(1 - G(-y))\}. \quad (4)$$

Proof. Since $\pi_0 \equiv \pi_H$ truncated on $[0, \infty)$, we have for $t > 0$ and by making use of property (2):

$$P_{\pi_0}(\tau(\theta) \leq t|y) = \frac{P_{\pi_H}(0 \leq \tau(\theta) \leq t|y)}{P_{\pi_H}(0 \leq \tau(\theta)|y)} = \frac{P_{\pi_H}(-y \leq \tau(\theta) - y \leq t - y|y)}{P_{\pi_H}(-y \leq \tau(\theta) - y|y)} = \frac{G(t-y) - G(-y)}{1 - G(-y)}. \tag{5}$$

Applying Definition 1 with this posterior cdf yields the endpoints given in (4) □

In what follows, the choice of the subclass $\mathcal{C}$ for analysis is inspired, as was the case in Marchand and Strawderman (2013), in order to set the lower endpoint of $I_{\pi_0,\alpha(\cdot)}(y)$ equal to 0 for a range of y values replicating this feature of the HPD credible set when g is symmetric, and approximating it otherwise. Moreover, the specific "equal-tails" choice introduced by Marchand and Strawderman (2013) coincides exactly with the HPD credible set for symmetric g.

Definition 2. *(a) Relative to the uniform prior π_0 and to a credibility coefficient $1-\alpha$, we define $\mathcal{C}$ as the subclass of Bayesian credible intervals such that*

$$\mathcal{C} = \left\{ I_{\pi_0,\alpha(\cdot)} : \alpha(y) = \alpha \textit{ for all } y \textit{ such that } y \leq a_0 = -G^{-1}(\frac{\alpha}{1+\alpha}) \right\}.$$

(b) The "equal-tails" credible set is $I_{\pi_0,\alpha_{eq}(\cdot)} \in \mathcal{C}$ is given by (4) with

$$\alpha_{eq}(y) = min \left\{ \alpha, \frac{\alpha}{2} + \frac{G(-y)}{2(1-G(-y))} \right\}.^{4} \tag{6}$$

Remark 2. *With $\alpha(y) = \alpha$ for $y \leq a_0$, Bayesian credible intervals $I_{\pi_0,\alpha(\cdot)} \in \mathbb{C}$ produces estimates $[0, y + G^{-1}(1 - \alpha(1 - G(-y)))]$ for $y \leq a_0$, as given in (4).*

The next result, providing conditions on the spending function $\alpha(\cdot)$ for which a Bayes credible set in $\mathcal{C}$ has frequentist probability of coverage bounded below by $\frac{1-\alpha}{1+\alpha}$, holds in a more general setting but is expressed here in the context of Assumption 1.

Theorem 1. *(Marchand and Strawderman, 2013) Under Assumption 1, the frequentist probability of coverage of a Bayesian credible interval $I_{\pi_0,\alpha(\cdot)}(Y)$ contained in $\mathcal{C}$, is bounded below by $\frac{1-\alpha}{1+\alpha}$ for*

[4] It is constructed by starting with equal-tails under the pivotal distribution G and transferring these tails to the posterior distribution of $\tau(\theta)$. It does not refer to $\alpha(y) = \alpha/2$. See Marchand and Strawderman (2013) for further details.

all θ such that $\tau(\theta) \geq 0$ whenever $I_{\pi_0,\alpha(\cdot)} = I_{\pi_0,\alpha_{eq}(\cdot)}$, and more generally whenever the spending function $\alpha(\cdot)$ is such that:

$$\frac{(1-\alpha)\,G(-y) + \frac{\alpha^2}{1+\alpha}}{1-G(-y)} \leq \alpha(y) \leq (\frac{\alpha}{1+\alpha})\frac{1}{1-G(-y)} \;, \text{for all } y \text{ such that } y > -G^{-1}(\frac{\alpha}{1+\alpha})\,. \quad (7)$$

3. Main results

Although Theorem 1 is quite general, the lower bound $\frac{1-\alpha}{1+\alpha}$ remains conservative as seen namely by numerical evaluations. Moreover in cases where g is symmetric, logconcave, and $\alpha < 1/3$, the better lower bound for coverage $1 - \frac{3\alpha}{2}$ holds (see Marchand et al., 2008). A sequence of results in this section lead to the $1 - \frac{3\alpha}{2}$ lower bound on the frequentist probability of coverage for a subclass of Bayes credible sets in $\mathcal{C}$ that includes the "equal-tails" procedure without the assumption of symmetry. The $1 - \frac{3\alpha}{2}$ lower bound for the frequentist probability of coverage of such a subclass of Bayes credible sets is thus applicable to a large class of situations as specified by Assumption 1, including those of Example 2, and as further illustrated in Section 4. Various intermediate properties, some of which make use of the logconcavity assumption for g as well as the following Assumption 2, are derived and relate to the lower and upper endpoints of $I_{\pi_0,\alpha(\cdot)}$ as well as to the behaviour of the frequentist probability of coverage $C_{\alpha(\cdot)}(\theta)$.

Assumption 2. **(i)** *The spending function $\alpha(y)$ is a continuous function of y, and $\alpha(y)(1-G(-y))$ is non-increasing in y for $y \geq a_0$.* **(ii)** *$a_0 = -G^{-1}(\frac{\alpha}{1+\alpha}) \geq 0$.*

Remark 3. *Conditions* **(i)** *and* **(ii)** *are both technical conditions used in the proofs but they are far from being overly restrictive. Condition* **(i)** *relates to the choice of the Bayes credible set and is always met by the "equal-tails" procedure (and thus HPD credible set in the symmetric case) as well as other Bayes credible sets that we have encountered (see Examples 3 and 5). Recalling that we have set the mode equal to 0 without loss of generality in setting-up our pivot (Assumption 1, Condition* **(ii)***) is equivalent to $G(mode) \geq \frac{\alpha}{1+\alpha}$, which encompasses all pivotal distributions G with the exception of those with a small fraction associated with the right-tail probability given that the* *credibility $1-\alpha$ is typically large and assumed here to be at least equal to $\frac{2}{3}$ (e.g., as an example for $\alpha = 0.10$, condition* **(ii)** *is satisfied as long as this right-tail probability exceeds $\frac{1}{11}$).*

The following result relates to monotonicity properties of the endpoints $l_{\alpha(\cdot)}$ and $u_{\alpha(\cdot)}$ in (4) of the Bayes credible set $I_{\pi_0,\alpha(\cdot)}$.

Lemma 2. *Under Assumptions 1 and 2, for a spending function $\alpha(\cdot)$, we have*

(a) $u_{\alpha(\cdot)}(y) - y$ is non-decreasing in y for $y \geq a_0$,

(b) $l_{\alpha(\cdot)}(y)$ is non-decreasing in y for $y \geq a_0$,

(c) $u_{\alpha(\cdot)}(y)$ is non-decreasing in y.

Proof. **(a)** We have $u_{\alpha(\cdot)}(y) - y = G^{-1}\{1-\alpha(y)(1-G(-y))\}$, which is indeed non-decreasing given Assumption 2, and given that G^{-1} is non-decreasing.
(b) First, observe that for $y \geq a_0 \geq 0$, we have from (4)

$$\begin{aligned} l_{\alpha(\cdot)}(y) - y &= G^{-1}\{G(-y) + (\alpha - \alpha(x))(1-G(-y))\} \\ &\leq G^{-1}(G(-a_0)) = -a_0 \leq 0. \end{aligned}$$

The above, along the with the unimodality (about 0) of g and the non-negativity of $l_{\alpha(\cdot)}$ tell us that

$$g(-y) \leq g(l_{\alpha(\cdot)}(y) - y)\,. \tag{8}$$

Finally, a direct evaluation using (4), along with Assumption 2(i) and (8), imply that

$$\begin{aligned} \frac{\partial}{\partial y} l_{\alpha(\cdot)}(y) &= 1 - \frac{(1-\alpha)g(-y) - \frac{\partial}{\partial y}\left(\alpha(y)(1-G(-y))\right)}{g(l_{\alpha(\cdot)}(y)-y)} \\ &\geq 1 - \frac{(1-\alpha)g(-y)}{g(l_{\alpha(\cdot)}(y)-y)} \geq 1-(1-\alpha) = \alpha \geq 0\,. \end{aligned}$$

(c) Given part (a), it remains to show that $u_{\alpha(\cdot)}(y)$ increases in $y < a_0$. Observe that, for such values of y, $u_{\alpha(\cdot)}(y)$ is the quantile of order $1-\alpha$ of the posterior distribution of $\tau(\theta)$ which has density $\frac{g(\tau(\theta))-y)}{1-G(-y)}\,\mathbb{I}_{[0,\infty)}(\tau(\theta))$, in accordance with (5). Since the logconcavity of g implies that this family of densities has an increasing monotone likelihood ratio in $\tau(\theta)$ with parameter y, the corresponding quantiles are therefore increasing and we have indeed that $u_{\alpha(\cdot)}(y)$ increases in y, $y \leq a_0$. □

Remark 4. *As discussed by Marchand et al. (2008) for the symmetric case, if the density g of the pivot $-(y-\tau(\theta))$ decreases at $+\infty$ at a rate not faster than exponential, then it follows that* $\lim_{y\to-\infty} u_\alpha(y) = a > 0$. *As seen in Lemma 3 below, this along with the monotonicity properties of Lemma 2 implies that the interval $I_{\pi_0,\alpha(\cdot)}$ can only fail to cover $\tau(\theta) \in [0,a]$ if the lower endpoint $l_\alpha(y)$ is $\geq \tau(\theta)$. To justify the above assertion, consider the posterior survivor function*

$$P(\tau(\theta) > t|y) \;=\; \frac{\bar{G}(t-y)}{G(-y)}\,,\; t>0, \tag{9}$$

and observe that for densities g of the form $g(u) = e^{-bu+R(u)}$ for some $b>0$ and $R(\cdot)$ such that $\lim_{u\to\infty} R(u) = 0$, *we have from the above*

$$\lim_{y\to-\infty} P(\tau(\theta) > t|y) \;=\; \lim_{u\to\infty} \frac{g(t+u)}{g(u)} \;=\; e^{-bt}\,. \tag{10}$$

In terms of Bayes credible intervals $I_{\pi_0,\alpha(\cdot)} \in \mathcal{C}$, the above tells us that $I_{\pi_0,\alpha(\cdot)}(y) \to [0, \frac{-\log\alpha}{b}]$ as $y\to-\infty$, i.e., $a = \frac{-\log\alpha}{b} > 0$. Examples include the Gamma(r,θ) and Fisher(r,s,θ) models with $b=r$ in both cases. Finally, we refer to Marchand et al. (2008) for further discussion on situations where a is equal to 0.

The following quantities relate to the inverses of the endpoints $l_{\alpha(\cdot)}$ and $u_{\alpha(\cdot)}$, and are useful to evaluate the frequentist probability of coverage.

Definition 3. *We define*

(i) $x_0(z) = u^{-1}_{\alpha(\cdot)}(z) - z$ *for* $a \leq z \leq y_0$,

(ii) $x_1(z) = l^{-1}_{\alpha(\cdot)}(z) - z$ *for* $z \geq 0$,

(iii) $x_2(z) = u^{-1}_{\alpha(\cdot)}(z) - z$ *for* $z \geq y_0$,

with $a = \lim_{y\to-\infty} u_{\alpha(\cdot)}(y)$, $y_0 = u_\alpha(a_0)$, and a_0 as defined in Assumption 2.

Remark 5. *We point out here that Lemma 2's representation of $I_{\pi_0,\alpha(\cdot)}$ implies that $y_0 = a_0 + G^{-1}(\frac{1}{1+\alpha})$.*

The next lemma provides a convenient expression for the frequentist probability of coverage $C_{\alpha(\cdot)}(\theta)$, as well as useful properties of the quantities $x_0(\tau(\theta))$, $x_1(\tau(\theta))$ and $x_2(\tau(\theta))$.

Lemma 3. *Under Assumptions 1 and 2, we have*

(A)

$$C_{\alpha(\cdot)}(\theta) = \begin{cases} 1 - G(-x_1(\tau(\theta))) & \text{if } \tau(\theta) \in [0, a] \\ G(-x_0(\tau(\theta))) - G(-x_1(\tau(\theta))) & \text{if } \tau(\theta) \in (a, y_0] \\ G(-x_2(\tau(\theta))) - G(-x_1(\tau(\theta))) & \text{if } \tau(\theta) \in (y_0, \infty), \end{cases} \tag{11}$$

where $a = \lim_{y\to-\infty} u_{\alpha(\cdot)}(y)$ *and* $y_0 = u_{\alpha(\cdot)}(a_0)$.

(B) *The quantities* $x_0(\cdot)$, $x_1(\cdot)$, *and* $x_2(\cdot)$ *satisfy the equations*

$$G(-x_0(z)) = 1 - \alpha\left(1 - G(-x_0(z) - z)\right), \tag{12}$$

$$G(-x_1(z)) = \alpha + (1-\alpha)G\left(-x_1(z) - z\right) - \alpha(x_1(z) + z)\left(1 - G(-x_1(z) - z)\right), \tag{13}$$

$$G(-x_2(z)) = 1 - \alpha(x_2(z) + z)\left(1 - G(-x_2(z) - z)\right). \tag{14}$$

Furthermore,

(i) $x_0(\cdot)$ *is non-decreasing with* $x_0(z) \in (-\infty, -G^{-1}(\frac{1}{1+\alpha})]$ *for* $z \in (a, y_0]$;

(ii) *For the "equal-tails" case* $\alpha \equiv \alpha_{eq}$, *and more generally for cases where* $(1-\alpha)G(-y) - \alpha(y)(1-G(-y))$ *decreases in* $y \geq a_0$, $x_1(\cdot)$ *is non-decreasing with* $x_1(z) \in [-G^{-1}(\frac{\alpha}{1+\alpha}), y_1 = -G^{-1}(\alpha - \lim_{z\to\infty} \alpha(z))]$ *for* $z \geq 0$;

(iii) $x_2(\cdot)$ *is non-increasing with* $x_2(z) \in [-G^{-1}(\frac{1}{1+\alpha}), y_2 = G^{-1}(1 - \lim_{z\to\infty} \alpha(z))]$ *for* $z \geq y_0$.

Proof of (A). For $a < \tau(\theta) \leq y_0$, we have

$$\begin{aligned} C_{\alpha(\cdot)}(\theta) &= \mathbb{P}_\theta\left(l_{\alpha(\cdot)}(Y) \leq \tau(\theta) \leq u(Y)\right) \\ &= \mathbb{P}_\theta\left(u^{-1}_{\alpha(\cdot)}(\tau(\theta)) \leq Y \leq l^{-1}_{\alpha(\cdot)}(\tau(\theta))\right) \\ &= \mathbb{P}_\theta\left(\tau(\theta) - l^{-1}_{\alpha(\cdot)}(\tau(\theta)) \leq \tau(\theta) - Y \leq \tau(\theta) - u^{-1}_{\alpha(\cdot)}(\tau(\theta))\right) \\ &= G(-x_0(\tau(\theta))) - G(-x_1(\tau(\theta))), \end{aligned}$$

given Lemma 2's monotonicity properties for the endpoints $l_{\alpha(\cdot)}$ and $u_{\alpha(\cdot)}$. The other parts of (11) follow analogously.

Proof of (B). Equation (12) is a rewriting of $u_{\alpha(\cdot)}(y)$ in (1) for values of $y = u_{\alpha(\cdot)}^{-1}(z)$ such that $\alpha(y) = \alpha$, i.e., $y \leq a_0$. Equations (13) and (14) are obtained in a similar fashion. For the other parts, we have the following.

(i) Part (c) of Lemma 2 tells us that $x_0(z) + z = u_{\alpha(\cdot)}^{-1}(z)$ increases in $z \in (a, y_0)$ which implies that the rhs of (12) decreases in z, whence the non-decreasing property of x_0. Additionally, we have that $a = \lim_{y \to -\infty} u_{\alpha(\cdot)}(y)$ which implies $\lim_{z \to a^+} x_0(z) = -\infty$; and $u_{\alpha(\cdot)}(a_0) = y_0$ which implies $x_0(y_0) = u_{\alpha(\cdot)}^{-1}(y_0) - y_0 = a_0 - y_0 = -G^{-1}(\frac{1}{1+\alpha})$ (see Remark 5).

(ii) The non-decreasing in y assumption for $(1-\alpha)G(-y) - \alpha(y)(1-G(-y))$ is readily verified for the "equal-tails" α_{eq} in (6) and also implies that the rhs of (13) decreases in $l_\alpha^{-1}(z) = x_1(z) + z$. In turn, since $l_\alpha^{-1}(z)$ increases in z (Lemma 2), we must have by (13) that $x_1(z)$ increases in z. Additionally, we have $x_1(0) = l_{\alpha(\cdot)}^{-1}(0) - 0 = a_0$, and $y_1 = \lim_{z \to \infty} x_1(z) = -G^{-1}(\alpha - \lim_{z \to \infty} \alpha(z))$ by making use of (13).

(iii) By virtue of Assumption 2, the rhs of (14) is increasing in $u_{\alpha(\cdot)}^{-1}(z) = x_2(z) + z$ so that is also increasing in z given Lemma 2. The decreasing property of x_2 follows at once from (14). Additionally, we have $x_2(y_0) = x_0(y_0) = a_0 - y_0 = -G^{-1}(\frac{1}{1+\alpha})$, and from (14), we obtain directly $y_2 = \lim_{z \to \infty} x_2(z) = -G^{-1}(1 - \lim_{z \to \infty} \alpha(z))$. □

By virtue of Lemma 3, we obtain the following properties for the frequentist probability of coverage $C_{\alpha(\cdot)}(\theta)$, $\tau(\theta) \geq 0$.

Corollary 1. *Under Assumptions 1 and 2, we have for the frequentist probability of coverage $C_{\alpha(\cdot)}$ of $I_{\pi_0, \alpha_{eq}(\cdot)}$, and more generally for $I_{\pi_0, \alpha(\cdot)}$ in $\mathcal{C}$ such that $(1-\alpha)G(-y) - \alpha(y)(1-G(-y))$ decreases in $y \geq a_0$,*

(a) *For $\tau(\theta) \geq y_0 = a_0 + G^{-1}(\frac{1}{1+\alpha})$, $C_{\alpha(\cdot)}(\theta)$ is non-decreasing in $\tau(\theta)$ and bounded above by $1-\alpha$,*

(b) *For $\tau(\theta) \in [0,a]$, $C_{\alpha(\cdot)}(\theta)$ is a non-decreasing function of $\tau(\theta)$ and bounded below by $C_\alpha(0) = \frac{1}{1+\alpha}$.*

(c) *$C_{\alpha(\cdot)}(\theta) \leq 1 - \alpha + \lim_{y\to\infty} \alpha(y)$ for $\tau(\theta) \geq 0$.*

Proof. (a) The non-decreasing property of $C_{\alpha(\cdot)}(\theta)$ follows directly from Lemma 3's representation $G(-x_2(\tau(\theta))) - G(-x_1(\tau(\theta)))$ for the coverage, and given that $G(\cdot)$ and $x_1(\cdot)$ are increasing on $\mathbb{R}$, while $x_2(\cdot)$ is decreasing on $\tau(\theta) \geq y_0$. Furthermore, from Lemma 3, we have

$$\begin{aligned} C_{\alpha(\cdot)}(\theta) &= G(-x_2(\tau(\theta))) - G(-x_1(\tau(\theta))) \\ &\leq G(-y_2) - G(-y_1) = (1 - \lim_{y\to\infty} \alpha(y)) - (\alpha - \lim_{y\to\infty} \alpha(y)) \\ &= 1 - \alpha. \end{aligned}$$

(b) For $\tau(\theta) \in [0,a]$, we have from Lemma 3 that $C_{\alpha(\cdot)}(\theta) = 1 - G(-x_1(\tau(\theta)))$. Since x_1 is non-decreasing (Lemma 3), it follows that, for $\tau(\theta) \in [0,a]$, $C_{\alpha(\cdot)}(\theta)$ is non-decreasing and bounded below by $1 - G(-x_1(0))) = 1 - G(-a_0) = \frac{1}{1+\alpha}$.

(c) Given part **(a)**, it remains to prove the result for $\tau(\theta) \leq y_0$. Using the properties of Lemma 3 for such θ's, we obtain

$$\begin{aligned} C_{\alpha(\cdot)}(\theta) &\leq 1 - G(-x_1(\tau(\theta))) \\ &\leq 1 - G(-y_1) \\ &= 1 - G\left(G^{-1}(\alpha - \lim_{z\to\infty} \alpha(z))\right) \\ &= 1 - \alpha + \lim_{z\to\infty} \alpha(z). \end{aligned}$$ □

Now, we show with the next results culminating in Theorem 3 that the frequentist probability of coverage $C_{\alpha_{eqt}(\cdot)}$ of the "equal-tails" credible interval $I_{\pi_0,\alpha(\cdot)}$ is bounded below by $1 - \frac{3\alpha}{2}$. We will require the following inequality.

Lemma 4. *Let g be a logconcave density on $\mathbb{R}$ with cdf G. Whenever $a_0 = -G^{-1}(\frac{\alpha}{1+\alpha}) \geq 0$, then we have the inequality*

$$G(-2a_0) \leq \frac{2\alpha^2}{1-\alpha^2}\,, \text{ for } \alpha \in (0, \frac{1}{3}]. \tag{15}$$

Proof. We make use of the logconcavity of G which is a consequence of the logconcavity of g (e.g., Bagnoli and Bergstrom, 2005). With the representation $G(-a_0) = \frac{\alpha}{1+\alpha}$, inequality (15) is equivalent to

$$\begin{aligned} \frac{G(-2a_0)}{G(-a_0)} &\leq \frac{2G(-a_0)}{1-2G(-a_0)} \\ \iff H(-a_0) &= \frac{2\,G^2(-a_0)}{G(-2a_0)} - 1 + 2G(-a_0) \geq 0\,. \end{aligned}$$

Observe that for $\alpha \in (0, 1/3)$, $-a_0$ ranges from $-G^{-1}(1/4)$ to $-\infty$, with $-G^{-1}(1/4) > 0$. Since $H(-G^{-1}(1/4)) = \frac{1/8}{G(-2a_0)} - 1 + \frac{2}{4} \geq \frac{1/8}{G(-a_0)} - \frac{1}{2} = 0$, it will suffice to show that $H(y)$ increases in y for $y > y_0$. Taking a derivative, we have indeed

$$\frac{1}{4}\frac{d}{dy}H(y) = \frac{g(y)}{2} + \frac{G(y)}{G(2y)}\left(\frac{g(y)}{G(y)} - G(y)\,\frac{g(2y)}{G(2y)}\right) > \frac{G(y)}{G(2y)}\left(\frac{g(y)}{G(y)} - \frac{g(2y)}{G(2y)}\right) \geq 0\,, \qquad (16)$$

since $\frac{g(2y)}{G(2y)} < \frac{g(y)}{G(y)}$ for all $y > 0$ given the logconcavity of G. □

Theorem 2. *Let* $z_1 = G^{-1}(1 - \frac{3\alpha}{2} + \frac{\alpha}{1+\alpha}) - G^{-1}(\frac{1-\alpha}{2(1+\alpha)})$. *Under Assumptions 1 and 2, we have* $C_{\alpha_{eq}(\cdot)}(\theta) \geq 1 - \frac{3\alpha}{2}$ *for all* θ *such that* $\tau(\theta) \in [0, z_1]$.

Proof. First, it is the case that $G(-x_0(z_1)) = 1 - \frac{3\alpha}{2} + \frac{\alpha}{1+\alpha}$, as can be inferred by working with (12). For $\tau(\theta) \leq z_1$, we have using Lemma 3's properties of x_0 and x_1 and the above property of z_1

$$\begin{aligned} C_{\alpha(\cdot)}(\theta) &= G(-x_0(\tau(\theta))) - G(-x_1(\tau(\theta))) \\ &\geq G(-x_0(z_1)) - G(-x_1(0)) \\ &= 1 - \frac{3\alpha}{2}\,. \end{aligned}$$ □

Before pursuing, we point out that $z_1 \leq y_0$, for $\alpha \in (0, \frac{1}{3}]$, since we have $G(-x_0(z_1)) = 1 - \frac{3\alpha}{2} + \frac{\alpha}{1+\alpha} \geq \frac{1}{1+\alpha} = G(-x_0(y_0))$, which implies $z_1 \leq y_0$ given that x_0 is non-decreasing (Lemma 3).

Theorem 3. *Under Assumptions 1 and 2, the Bayesian "equal-tails" credible interval* $I_{\pi_0,\alpha_{eq}(\cdot)}$ *has minimum frequentist probability of coverage* $C_{\alpha_{eq}(\cdot)}(\theta)$ *greater than* $1 - \frac{3\alpha}{2}$ *for all* θ *such that* $\tau(\theta) \geq 0$.

Proof. Given that $C_{\alpha_{eq}(\cdot)}(\theta) \geq 1 - \frac{3\alpha}{2}$ for $0 \leq \tau(\theta) \leq z_1$ (Theorem 2) and that $C_{\alpha_{eq}(\cdot)}(\theta)$ is a non-decreasing function of $\tau(\theta)$ for $\tau(\theta) \geq y_0$ (Corollary 1), it will suffice to show that $C_{\alpha_{eq}(\cdot)}(\theta) \geq 1 - \frac{3\alpha}{2}$ for all $\tau(\theta) \in [z_1, y_0]$. Exploiting the properties of Lemma 3, we have

$$\begin{aligned} C_{\alpha(\cdot)}(\theta) &= G(-x_0(\tau(\theta))) - G(-x_1(\tau(\theta))) \\ &\geq G(-x_0(y_0)) - G(-x_1(\tau(\theta))) \\ &\geq \frac{1}{1+\alpha} - G(-x_1(z_1)). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & G(-x_1(z_1)) \leq \frac{1}{1+\alpha} - (1 - \frac{3\alpha}{2}) \\ \Leftrightarrow\ & \frac{\alpha}{2} + \frac{1-\alpha}{2} G(-x_1(z_1) - z_1) \leq \frac{\alpha(1+3\alpha^2)}{2(1+\alpha)} \text{ (by (13) with } \alpha(\cdot) = \alpha_{eq}(\cdot)) \\ \Leftrightarrow\ & G(-x_1(z_1) - z_1) \leq \frac{2\alpha^2}{1-\alpha^2} \\ \Leftarrow\ & G(-2a_0) \leq \frac{2\alpha^2}{1-\alpha^2} \text{ (since } x_1(z_1) + z_1 \geq x_1(a_0) + a_0 \geq 2a_0), \end{aligned}$$

and the result follows by virtue of Lemma 4. □

Our last analytical result establishes that the lower bound $1 - \frac{3\alpha}{2}$ for frequentist coverage probability holds for the Bayesian credible intervals included in $\mathcal{C}$ which satisfy $I_{\alpha(\cdot)} \geq I_{\alpha_{eq}(\cdot)}$.

Lemma 5. *For Bayesian credible intervals $I_{\pi_0, \alpha_i(\cdot)} \in \mathbb{C}$, for $i = 1, 2$, such that $\alpha_1(y) \geq \alpha_2(y)$ for all $y \geq a_0$, we have under Assumptions 1 and 2*

$$inf_{\theta:\tau(\theta)\geq 0} \{C_{\alpha_1(\cdot)}(\theta)\} \geq inf_{\theta:\tau(\theta)\geq 0} \{C_{\alpha_2(\cdot)}(\theta)\}.$$

Proof. First, since $C_{\alpha(\cdot)}(\theta)$ is increasing on $\tau(\theta) \in [y_0, \infty]$ (Corollary 1) and as y_0 does not depend on $\alpha(\cdot)$, we have $inf_{\tau(\theta)\geq 0}\{C_{\alpha(\cdot)}(\theta)\} = inf_{\tau(\theta)\leq y_0}\{C_{\alpha(\cdot)}(\theta)\}$ and it suffices to show that

$$inf_{\tau(\theta)\leq y_0}\{C_{\alpha_1(\cdot)}(\theta)\} \geq inf_{\tau(\theta)\leq y_0}\{C_{\alpha_2(\cdot)}(\theta)\}.$$

Now, from (11), we have for $\tau(\theta) \leq y_0$ and $i = 1, 2$

$$C_{\alpha_i(\cdot)}(\theta) = (min(1, G(-x_0(\tau(\theta)))) - G(-x_{1,\alpha_i(\cdot)}(\tau(\theta))), \tag{17}$$

where we emphasized that $x_{1,\alpha(\cdot)}(\tau(\theta))$ depends on $\alpha(\cdot)$, while $x_0(\tau(\theta))$ does not (see Definition 3). It thus follows that the ordering $\alpha_1(\cdot) \geq \alpha_2(\cdot)$ implies

$$\begin{aligned}
& l_{\alpha_1(\cdot)}(y) \leq l_{\alpha_2(\cdot)}(y), \ \forall y, \\
\Rightarrow \ & l^{-1}_{\alpha_1}(\tau(\theta)) \geq l^{-1}_{\alpha_2}(\cdot)(\tau(\theta)), \ \forall \tau(\theta) \leq y_0, \\
\Rightarrow \ & x_{1,\alpha_1(\cdot)}(\tau(\theta)) \geq x_{1,\alpha_2(\cdot)}(\tau(\theta)), \ \forall \tau(\theta) \leq y_0, \\
\Rightarrow \ & -G(-x_{1,\alpha_1(\cdot)}(\tau(\theta))) \geq -G(-x_{1,\alpha_2(\cdot)}(\tau(\theta))), \ \forall \tau(\theta) \leq y_0, \\
\Rightarrow \ & C_{\alpha_1(\cdot)}(\theta) \geq C_{\alpha_2(\cdot)}(\theta), \ \text{given (17)}, \quad \text{and for } \tau(\theta) \leq y_0. \quad \square
\end{aligned}$$

Corollary 2. *Under Assumptions 1 and 2, a Bayesian credible interval $I_{\pi_0,\alpha(\cdot)}$ satisfying the conditions of Theorem 1, with $\alpha(\cdot) \geq \alpha_{eq}(\cdot)$, has minimum frequentist probability of coverage bounded below by* $1 - \frac{3\alpha}{2}$.

Proof. The result is obtained directly from Lemma 5 and Theorem 3. $\square$

We have thus obtained a class of Bayesian credible intervals with frequentist probability of coverage bounded below by $1 - \frac{3\alpha}{2}$. In cases where the underlying pivotal density g is symmetric, this adds to the result applying to I_{HPD} (see [2]). For general cases with g logconcave and $G(0) \geq \frac{\alpha}{1+\alpha}$, the findings are new. In the next section, we will present examples which illustrate the results.

4. Examples

For the illustrations and discussion below, the following Bayesian credible intervals $I_{\pi_0,\alpha(\cdot)} \in \mathcal{C}$ are considered. Recall that they produces the same estimates for $y \leq a_0$ (see Remark 2).

(i) The "equal-tails" Bayesian credible interval $I_{\pi_0,\alpha_{eq}(\cdot)}(Y)$ given by the spending function in (6). Applying Lemma 1, we obtain the endpoints $l_{\pi_0,\alpha_{eq}(\cdot)}(y) = y + G^{-1}\left(\frac{\alpha+(1-\alpha)G(-y)}{2}\right)$ and $u_{\pi_0,\alpha_{eq}(\cdot)}(y) = y + G^{-1}\left(1 - \frac{\alpha+(1-\alpha)G(-y)}{2}\right)$ for $y \geq a_0$.

(ii) The Bayesian credible interval $I_{\pi_0,\alpha_{upper}(\cdot)}(Y)$ is defined by the largest spending function in (7), i.e.,

$$\alpha_{upper}(y) = min\left\{\alpha, \frac{\alpha}{(1+\alpha)(1-G(-y))}\right\}.$$

Applying Lemma 1, the endpoints of $I_{\pi_0,\alpha_{upper}(\cdot)}(Y)$ become for $y \geq a_0$: $l_{\pi_0,upper}(y) = y + G^{-1}\left((1-\alpha)G(-y) + \frac{\alpha^2}{1+\alpha}\right)$ and $u_{\pi_0,upper}(y) = y + G^{-1}\left(\frac{1}{1+\alpha}\right)$.

(iii) The Bayesian credible interval $I_{\pi_0,\alpha_{lower}(\cdot)}(Y)$ defined by the spending function

$$\alpha_{lower}(y) = min\left\{\alpha, \frac{(1-\alpha)G(-y) + \frac{\alpha^2}{1+\alpha}}{1-G(-y)}\right\};$$

(iv) The Bayesian credible interval $I_{\pi_0,opt}(Y) = [l_{\pi_0,opt)}(Y), u_{\pi_0,opt)}(Y)]$, defined as the choice $I_{\pi_0,\alpha(\cdot)} \in \mathcal{C}$, which for all $y \geq a_0$, minimizes the length $u_{\pi_0,\alpha(\cdot)}(y) - l_{\pi_0,\alpha(\cdot)}(y)$. The corresponding spending function is thus defined implicitly as $\alpha_{opt}(y) = \mathbb{P}_{\pi_0}(\tau(\theta) \geq u_{\pi_0,opt}(y)|y)$.

(v) As well for further reference, the HPD Bayesian credible interval and spending function are denoted respectively by $I_{\pi_0,HPD}(Y)$ and α_{HPD}.

Theorem 1 tells us that Bayesian credible intervals (i) to (iv) have, under Assumption 1 (and log-concavity is not required) minimum frequentist coverage probability bounded below by $\frac{1-\alpha}{1+\alpha}$ for estimating $\tau(\theta) \geq 0$. On the other hand, under Assumptions 1 and 2, Theorem 3 and Corollary 2 tell us that the lower bound $1 - \frac{3\alpha}{2}$ for frequentist coverage probability in the case of $I_{\pi_0,\alpha_{eq}(\cdot)}(Y)$ and $I_{\pi_0,\alpha_{upper}(\cdot)}(Y)$, as well as any other $I_{\pi_0,\alpha(\cdot)} \in \mathcal{C}$ with $\alpha(y) \geq \alpha_{eq}(y)$ for all $y \geq a_0$.

The choice to consider $I_{\pi_0,\alpha_{upper}(\cdot)}(Y)$ is inspired by Lemma 5 which implies the following optimality property.

Corollary 3. *Among Bayesian credible intervals $I_{\pi_0,\alpha(\cdot)} \in \mathbb{C}$, $I_{\pi_0,\alpha_{upper}(\cdot)}(Y)$ maximizes the minimum frequentist coverage probability $C_\alpha(\theta)$ for θ such that $\tau(\theta) \geq 0$.*

When g is symmetric, $I_{\pi_0,\alpha_{eq}(\cdot)}(Y)$ and $I_{\pi_0,HPD}(Y)$ coincide, Assumption 2 is satisfied, and the lower bound $1 - \frac{3\alpha}{2}$ applies under Assumption 1. In general though without symmetry, $I_{\pi_0,HPD}(Y)$ does not belong in $\mathbb{C}$ and the Bayesian credible interval $I_{\pi_0,opt}(y)$ is an adaptation (to $\mathbb{C}$) of the HPD credible interval which forces $\alpha(y) = \alpha$ for $y \leq a_0$, and which coincides with $I_{\pi_0,HPD}(y)$ for values y such that $\alpha_{HPD}(y) \in [\alpha_{lower}(y), \alpha_{upper}(y)]$.

We now turn to specific examples. In each case, Assumption 1 is satisfied.

Example 3. *Consider a Gamma model $X \sim Gamma(r = 5, \theta)$ with $\theta \geq 1$, and $1 - \alpha = 0.95$. The density of the pivot $-T(X, \theta) = -(log(\frac{X}{r}) - log(\theta))$ is given by $g(t) = \frac{r^r}{\Gamma(r)} e^{-r(t+e^{-t})}$. Since $log(g(t))$ is concave and $a_0 = -G^{-1}(\frac{\alpha}{1+\alpha}) \approx 0.613 \geq 0$ (with $\alpha = 0.05$ and $r = 5$), Theorem 3 and Corollary 2 apply for estimating $\tau(\theta) = log(\theta) \geq 0$.*

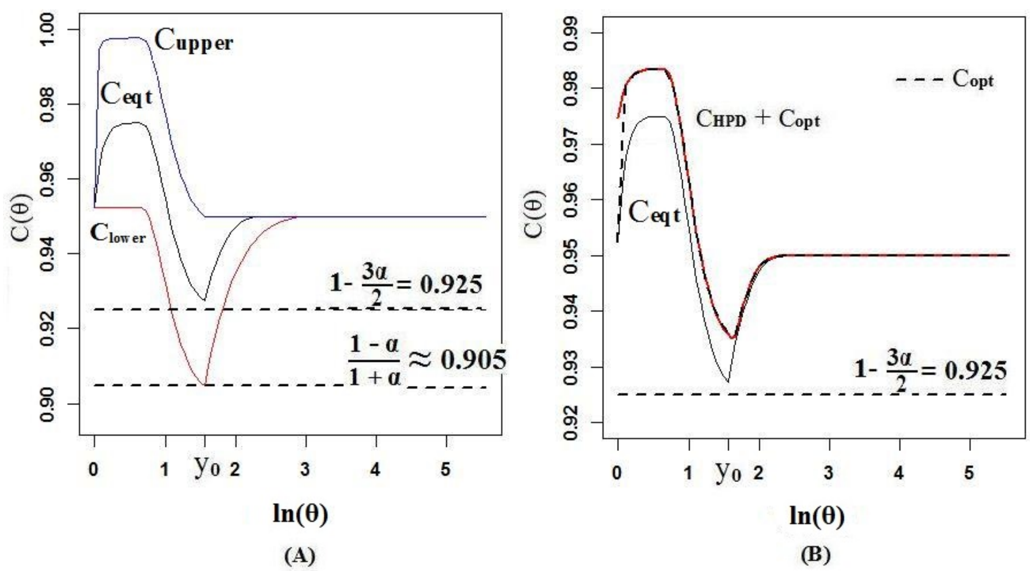

Figure 1: Frequentist coverage probability of Bayesian credible intervals for $X \sim$ Gamma$(5, \theta)$; $\theta \geq 1$, 1-α=0.95.

Here are some observations in relationship to Figures 1 and 2.

(a) Observe that the frequentist coverage of $I_{\pi_0,\alpha_{eq}(\cdot)}(Y)$, $I_{\pi_0,\alpha_{upper}(\cdot)}(Y)$, and $I_{\pi_0,\alpha_{lower}(\cdot)}(Y)$ is non-decreasing for $log(\theta) \geq y_0$ in accordance with part (a) of Corollary 1. Moreover, notice that

the applicability of the lower bound $1-\frac{3\alpha}{2}$ *for* $I_{\pi_0,\alpha_{eq}(\cdot)}$ *and* $I_{\pi_0,\alpha_{upper}(\cdot)}$*, but not for* $I_{\pi_0,\alpha_{lower}(\cdot)}$.

(b) *The value of* $a=\lim_{y\to-\infty}u_\alpha(y)$ *is addressed in Remark 4 and is given here by* $a=-\frac{log(\alpha)}{r}=-\frac{log(0.05)}{5}\approx 0.599$. *In accordance with part (b) of Corollary 1, we see that the coverage is non-decreasing on the range* $[0,a]$. *The frequentist coverage of* $I_{\pi_0,\alpha_{lower}(\cdot)}(Y)$ *is actually constant on* $[0,a]$ *in general. To see this, first compute the lower bound* $l_{\pi_0,\alpha_{lower}(\cdot)}(y)$ *by making use of Lemma 1 obtaining* $l_{\pi_0,\alpha_{lower}(\cdot)}(y)=y+G^{-1}(\frac{\alpha}{1+\alpha})$. *From this, we obtain* $l^{-1}_{\alpha_{lower}(\cdot)}(z)=z-G^{-1}(\frac{\alpha}{1+\alpha})$ *and* $x_1(z)=l^{-1}_{\alpha_{lower}(\cdot)}(z)-z=-G^{-1}(\frac{\alpha}{1+\alpha})$*, which is constant in* z. *We thus infer from (11) that* $C_{\alpha_{lower}(\cdot)}(\theta)=1-G(-x_1(z))=\frac{1}{1+\alpha}$ *for* $\tau(\theta)\in[0,a]$.

(c) *Part (c) of Corollary 1 also gives the upper bound* $1-\alpha+\lim_{y\to\infty}\alpha(y)$ *for frequentist coverage of probability. As illustrated in Figure 1, for* $I_{\pi_0,\alpha_{eq}(\cdot)}(Y)$ *and* $1-\alpha=0.95$*, we obtain an upper bound of* 0.975*, while for* $I_{\pi_0,\alpha_{upper}(\cdot)}(Y)$ *the corresponding upper bound is equal to* $0.95+\frac{0.05}{1.05}\approx 0.9976$. *Moreover,* $I_{\pi_0,\alpha_{upper}(\cdot)}(Y)$ *has globally very high coverage surpassing* $1-\alpha=0.95$ *for small values of* $\tau(\theta)$*, and otherwise quite close to* $1-\alpha=0.95$. *Part of this behavior is consistent of course with Corollary 3, but the overall good frequentist coverage of this Bayesian credible interval, replicated for other examples such as Example 5 below, is nevertheless surprising.*

(d) *Figure 2 shows how close* α_{HPD} *and* α_{opt} *are, while part (B) of Figure 1 reproduces the same information as seen through the frequentist probability of coverage. In this case as well, observe that both* α_{HPD} *and* α_{opt} *exceed* α_{eq} *(this is only empirical) with corresponding coverage probability exceeding* $1-\frac{3\alpha}{2}$ *for all* $\theta\geq 1$.

Example 4. *Consider* $X\sim Weibull(r,\theta)$*,* r *known and* $\theta\geq 1$. *The cumulative distribution function(cdf) of the pivot* $T(X,\theta)=-(log(X)-log(\theta))$ *is* $G(t)=e^{-e^{-rt}}$ *and its density is given by* $g(t)=re^{-rt-e^{-rt}}$. *Since* $log(g(t))$ *is concave and* $G(0)=\frac{1}{e}\geq\frac{\alpha}{1+\alpha}$ *when* $\alpha\leq\frac{1}{e-1}\approx 0.582$*, Theorem 3 and Corollary 2 apply. The frequentist probability of coverage* $C_{\alpha_{eq}(\cdot)}(\theta)$ *is thus greater than* $1-\frac{3\alpha}{2}$ *for all* $\tau(\theta)=log(\theta)\geq 0$. *We refer to Ghashim (2013) for numerical illustrations, which turn out to be somewhat similar to Example 3.*

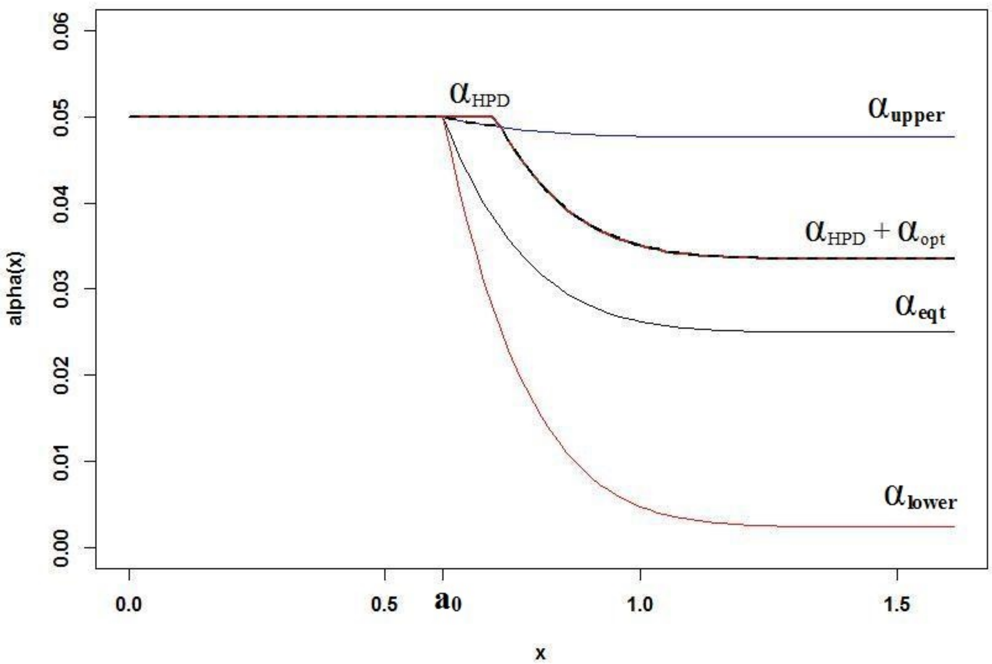

Figure 2: Spending functions $\alpha(\cdot)$ for Gamma model (5,θ), $\theta \geq 1$, and 1-α=0.95.

Example 5. *Consider the normal model $\mathcal{N}(\theta, 1)$, $\theta \geq 0$, with location parameter and the linear pivot is $T(X, \theta) = X - \theta$ with density given by $g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ which is logconcave. From the fact that $G(0) = \frac{1}{2} \geq \frac{\alpha}{1+\alpha}$ for all $\alpha \in [0, 1]$, we can apply Theorem 3 and Corollary 2 yielding a lower bound of $1 - \frac{3\alpha}{2}$ for the frequentist probability of coverage of $I_{\pi_0, \alpha_{eq}(\cdot)(Y)} \equiv I_{\pi_0, HPD}(Y)$, as well as for $I_{\pi_0, \alpha_{upper}(\cdot)}(Y)$ for $\theta \geq 0$. The former result is due to Marchand and al. (2008), but the latter result is new. As for the Gamma example, the frequentist probability of coverage of $I_{\pi_0, \alpha_{upper}(\cdot)}(Y)$ will have high infimum (Corollary 3), and the actual coverage is quite impressive and either above or quite close to the credibility $1 - \alpha = 0.90$ for all $\theta \geq 0$.*

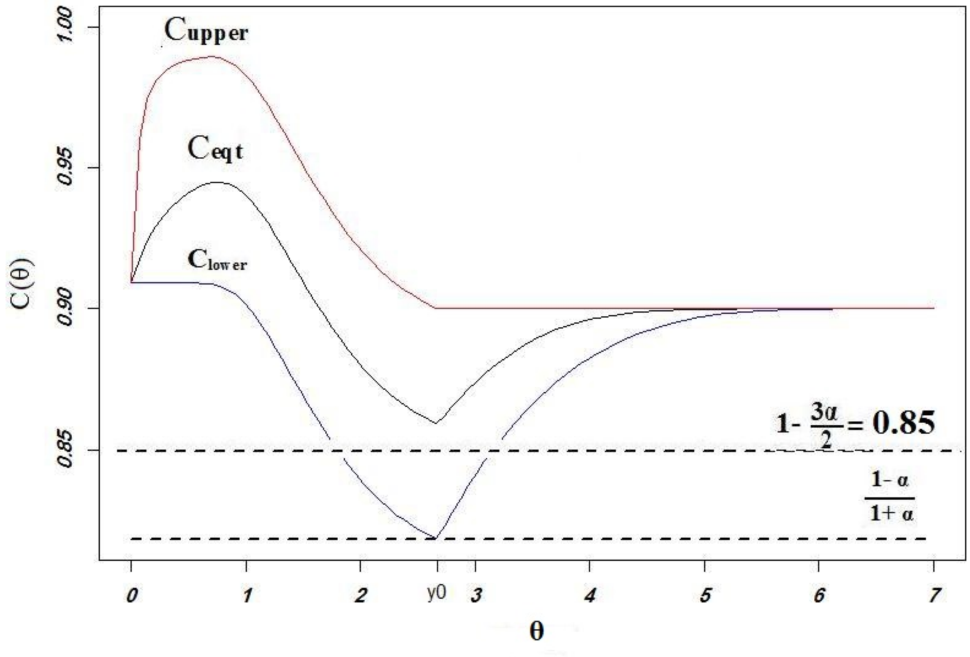

Figure 3: Frequentist coverage probability of Bayesian credible intervals for the Normal model $N(\theta, 1)$ with $\theta \geq 0$ and 1-α=0.90.

5. Concluding remarks

For estimating a lower restricted parameter in the framework of Marchand and Strawderman (2006), such as lower bounded Gamma or Fisher scale parameters, we have shown how $(1-\alpha)\times 100\%$ Bayesian credible intervals can be constructed so that the frequentist coverage probability is no less than $1-\frac{3\alpha}{2}$ when evaluated on the restricted parameter space. In a continuation of Marchand and Strawderman (2013), the emphasis is on the actual choice or determination of the Bayesian credible interval endpoints as represented by the spending function. Our findings, both theoretical and empirical, clearly show how such a choice matters and influences the performance as measured by the criteria of frequentist coverage probability. Among the Bayesian credible intervals with coverage lower bounded by $1 - \frac{3\alpha}{2}$, the "equal-tails" procedure is conceptually simple and matches the HPD credible set when the underlying pivot is symmetric. Another choice given by the largest spending function satsifying the conditions of Theorem 1 has higher infimum coverage theoretically, and numerical evaluations show that it comes close to having infimum coverage equal to its credibility. The findings do require a key logconcavity assumption but it seems likely that a similar lower bound for coverage holds in other situations. In this respect, a key example is the case of estimating a lower bounded normal mean with unknown variance (e.g., Zhang and Woodroofe, 2003; Marchand and Strawderman, 2006, 2013) where the pivotal distribution is Student and thus not logconcave, and where the best established lower bound is $\frac{1-\alpha}{1+\alpha}$.

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