Testing Independence Based on Bernstein Empirical Copula and Copula Density

Mohamed Belalia[a], Taoufik Bouezmarni[a], Abderrahim Taamouti[b]

[a]*Département de mathématiques, Université de Sherbrooke, Canada*
[b]*Durham University Business School, UK*

Abstract

In this paper we provide three nonparametric tests of independence between continuous random variables based on Bernstein copula and copula density. The first test is constructed based on functional of Cramér-von Mises of the Bernstein empirical copula. The two other tests are based on Bernstein density copula and use Cramér-von Mises and Kullback-Leiber divergence-type respectively. Furthermore, we study the asymptotic distribution of each of our tests. Finally, we consider a Monte Carlo experiment to investigate the performance of the tests. In particular, we examine their size and power that we compare with those of the classical nonparametric tests that are based on the empirical distribution.

Keywords: Bernstein empirical copula, Copula density Cramér–von Mises statistic, Kullback-Leiber divergence-type, Independence test.

1 Introduction

Testing the dependence between random variables is crucial in statistics, economics, finance and other disciplines. In economics, these tests are very useful to detect and quantify possible economic causal effects that can be of great importance for policy-makers. In finance, identifying the dependence between different asset prices (returns) is essential for risk management and portfolio selection. Standard tests of independence are given by the usual

*Corresponding author

Email addresses: Mohamed.Belalia@usherbrooke.ca (Mohamed Belalia), Taoufik.Bouezmarni@USherbrooke.ca (Taoufik Bouezmarni), Abderrahim.Taamouti@durham.ac.uk (Abderrahim Taamouti)

T-test and F-test that are defined in the context of linear regression models. However, these tests are only appropriate for testing independence in Gaussian models, thus they might fail to capture nonlinear dependence. With the recent great interest in nonlinear dependence, it is not surprising that there has been a search for alternative dependence measures and tests of independence. In this paper we propose three nonparametric tests of independence. These tests are model-free and can be used to detect both linear and nonlinear dependence.

Tests of independence have recently gained momentum. In particular, several statistical procedures have been proposed to test for the independence between two random variables X and Y. In the following, we briefly review some of these tests, and for the sake of simplicity we focus on the case where the law of the variables of interest is continuous. Most classical tests of independence were initially based on some measures of dependence, say ρ, that take the value 0 under the null hypothesis of independence. Once a random sample $\{(X_1, Y_1), ..., (X_n, Y_n)\}$ is collected, an estimation $\hat{\rho}_n$ of ρ is obtained and compared with the value of ρ under the null. One of the desirable properties is that the process $n^{1/2}(\hat{\rho}_n - \rho)$ converges weakly to $N(0, \sigma_0^2)$ as $n \to \infty$, where σ_0 is the limiting variance of $n^{1/2}\hat{\rho}_n$. The most popular example is the test that is based on the Pearson correlation coefficient. Other tests of independence have been constructed using other popular measures of dependence that are based on ranks. Denote by R_i and S_i, for $i = 1, ..., n$, the rank functions for X and Y, respectively. The rank-based measures of dependence do not depend on the marginals. The most used rank-based measures of dependence are Kendall's tau and Spearman's rho. The Kendall's tau measure is defined as $\tau_n = \frac{2}{n(n-1)}(C_n - D_n)$, where C_n is the number of concordant pairs of ranks, and D_n is the number of discordant pairs. The pairs (R_i, S_i) and (R_j, S_j) being concordant if $(R_i - R_j)(S_i - S_j) > 0$ and discordant otherwise. The statistic τ_n is an estimate of $\tau = 2P\left\{(X - Y)(X_1 - Y_1) > 0\right\} - 1$, where (X_1, Y_1) is an independent copy of (X, Y). Under the null hypothesis of independence $\tau = 0$, and it can be shown that $n^{1/2}\left(\tau_n - 0\right) \sim N(0, 4/9)$, as $n \to \infty$, see Prokhorov (2001). Now, the Spearman's Rho, denoted by ρ_n^S, is simply defined as the correlation between the ranks $(R_1, S_1), ..., (R_n, S_n)$. Spearman's Rho is an estimate of the correlation coefficient, say ρ^S, between $U_1 = F(X)$ and $U_2 = G(Y)$, where F and G are the distribution functions of X and Y. Under the null hypothesis of independence $\left(\rho^S = 0\right)$, it can be shown that $n^{1/2}\left(\rho_n^S - 0\right) \sim N(0, 1)$, as $n \to \infty$, see Borkowf (2002).

However, the tests of independence that are based on the abovementioned measures are usually not consistent. In particular, under some alternatives their power functions do not tend to one as the sample size tend to infinity. A typical example to illustrate this inconsistency is given by the following. Let X and Y be two uniformly random variables that are distributed over $(0, 1)$: $X \sim$Unif$(0, 1)$ and set $Y = T(X)$, where T is the tent map, i.e. $T(u) = 2\min(u, 1-u)$. This implies that $Y \sim$Unif$(0, 1)$ and X and Y are strongly dependent. On the contrary, for this example the values of Pearson correlation, Kendall's tau and Spearman's rho are all equal to zero, which wrongly indicate that X and Y are independent. Furthermore, it can be shown than $n^{1/2}\hat{\rho}_n \sim N(0, \sigma^2)$, for some positive constant σ that depends on ρ. As a result, at the significance level 5%, the power of the associated test tends to be equal to $2\Phi(-1.96\frac{\sigma_0}{\sigma})$, where σ_0^2 is the asymptotic variance under the null hypothesis of independence and Φ is the cumulative distribution function of the standard Gaussian. For the particular case $\sigma_0^2 = 1$ and $\sigma^2 = 6/5$, the power of the test which is based on Pearson correlation tends to be equal to 0.1024, instead of 1, as $n \to \infty$.

To overcome the inconsistency problem of the previous tests of independence, Blum et al. (1961) were among the first to use nonparametric test statistics based on the comparison of empirical distribution functions. For bivariate random variables X and Y, Blum et al. (1961) use a Cramér–von Mises distance to compare the joint empirical distribution function of (X, Y); $H(x, y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(X_i \leq x, Y_i \leq y)$, with the corresponding marginal empirical distributions, i.e., $F_n(x) = H_n(x, \infty)$ and $G_n(y) = H_n(\infty, y)$. Their test statistic converges, in the Skorohod space $D([-\infty, +\infty]^2)$, to a process $\mathbb{H}(x, y) = \mathbb{B}\{F(x), G(y)\}$, where F and G are the marginals of X and Y, respectively, and $\mathbb{B}$ is a continuous centred Gaussian process.

Other tests of independence have used copula functions. When the marginal distributions of the components of the random vector $\mathbf{X} = (X_1, ..., X_d)$ are continuous, Sklar (1959) has shown that there exists a unique distribution function C (hereafter copula function) with uniform marginals, such that $H(X_1, ..., X_d) = P(X_1 \leq x_1, ..., X_d \leq x_d) = C\left(F_1(x_1), ..., F_d(x_d)\right)$, for $x_1, ..., x_d \in \mathbb{R}^d$ [1]. Under the null hypothesis of independence between the components of $\mathbf{X}$, the copula function is equal to the independent copula C_π

[1] For more details on the copula theory the readers are referred to an excellent book by Nelsen (2006)

which is defined as $C_\pi(\mathbf{u}) = C_\pi(u_1, ..., u_d) = \prod_{j=1}^{d} u_j$, for $\mathbf{u} \in [0,1]^d$. The work of Blum et al. (1961) and the above characterization of the independence in terms of copula function have inspired Dugué (1975), Deheuvels (1981a,b,c), and recently Ghoudi et al. (2001), Genest & Rémillard (2004) to construct test of the mutual independence of the components of $\mathbf{X}$ based on the observations $\{X_i = (X_{i,1}, ..., X_{i,d})$, for $i = 1, .., n\}$ and using the statistic

$$Stat_n = \int_{[0,1]^d} n \left\{ C_n(\mathbf{u}) - \prod_{j=1}^{d} u_j \right\}^2 d\mathbf{u},$$

where $C_n(\mathbf{u})$ is the empirical copula, originally proposed by Deheuvels (1979) and defined as

$$C_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} \mathbb{I}\left\{U_{i;j} \leq u_j\right\}, \text{ for } \mathbf{u} = (u_1, ..., u_d) \in [0,1]^d,$$

where $U_{i;j} = F_{j;n}(X_{i,j})$, for $j = 1, ..., d$, with $F_{j;n}$ is the empirical cumulative distribution function of the component X_j. An interesting aspect of the above test statistic is that, under the mutual independence of the components $X_1, ..., X_d$ of $\mathbf{X}$, the empirical process $\mathbb{C}_n(\mathbf{u}) = \sqrt{n}\,(C_n(\mathbf{u}) - C_\pi(\mathbf{u}))$ can be decomposed, using the Möbius transform; see Blum et al. (1961), Rota (1964) and Genest & Rémillard (2004), into $2^d - d - 1$ sub-processes $\sqrt{n}\mathcal{M}_A\left(C_n\right), A \subseteq \{1, ..., d\}, |A| > 1$, that converge jointly to tight centered mutually independent Gaussian processes. One important property of this decomposition is that mutual independence among $X_1, ..., X_d$ is equivalent to having $\mathcal{M}_A\left(C\right)(\mathbf{u}) = 0$, for all $\mathbf{u} \in [0,1]^d$ and all $A \subseteq \{1, ..., d\}$ such that $|A| > 1$. Consequently, instead of the single test statistic $Stat_n$, this suggests considering $2^d - d - 1$ test statistics of the form

$$M_{A,n} = \int_{[0,1]^d} n \left\{\mathcal{M}_A\left(C_n\right)(\mathbf{u})\right\}^2 d\mathbf{u},$$

where $A \subseteq \{1, ..., d\}, |A| > 1$, that are asymptotically mutually independent under the null hypothesis of independence.

The above decomposition has been recently extended by Beran et al. (2007) and Kojadinovic & Holmes (2009) to the situation where one wants to test the mutual independence of several continuous random vectors. As an alternative to the statistic $Stat_n$, Genest & Rémillard (2004); see also Genest et al. (2007), studied several ways to combine the $2^d - d - 1$ statistics $M_{A,n}$ into one global statistic for testing independence. Using some Monte Carlo experiments, the test based on this combination tends to give the best results and was found to frequently outperform the test based on $Stat_n$; see Genest & Rémillard (2004) and Kojadinovic & Holmes (2009). Finally, in addition to the test statistics constructed from the distribution and copula functions, other researchers have considered using empirical characteristic functions; for the review see Feuerverger (1993), Bilodeau & Lafaye de Micheaux (2005), among others.

In this paper, we propose several nonparametric copula-based tests for independence that are more flexible, have better power, and are easy to implement. The first test is a Cramér-von Mises-type test that is constructed using Bernstein empirical copula. Recall that the Bernstein empirical copula was first proposed and investigated by Sancetta & Satchell (2004) for i.i.d. data. The latter show that, under some regularity conditions, any copula function can be approximated by a Bernstein copula. Recently Jansen et al. (2012) have shown that the Bernstein empirical copula outperforms the classical empirical copula estimator $C_n(\mathbf{u})$. This result has motivated us to replace the standard empirical copula in the process $\mathbb{C}_n(\mathbf{u})$ defined above by the Bernstein copula function for the construction of new nonparametric test that can be used to investigate the following null hypothesis H_0:

$$H_0 : C(\mathbf{u}) = C_\pi(\mathbf{u}) \equiv \prod_{j=1}^{d} u_j, \text{ for } \mathbf{u} \in [0,1]^d,$$

where, as before, $C(\mathbf{u})$ is the copula function that corresponds to the joint cumulative distribution of the random vector $\mathbf{X}$. We find that the test based on Bernstein empirical copula outperforms that which is based on the empirical copula $C_n(\mathbf{u})$. However, we also find that the two tests provide a poor power when the null hypothesis is, for example, a Student T copula with zero Kendall's tau. The difficulty of distinguishing between Student T copula distribution and zero Kendall's tau and the independent copula is illustrated in Figure 1 and discussed in Section 3.2. This might explains the poor power of nonparametric copula distribution-based tests.

To overcome the above problem, we introduce two other nonparametric tests based on Bernstein empirical copula *density*. Bouezmarni et al. (2010) have proposed an estimator of Bernstein copula density and derived its asymptotic properties under dependent data. These properties have also recently reinvestigated in Jansen et al. (2014). The motivation for using a Bernstein copula *density* can be found in Figure 1 which shows that working with copula *density,* instead of copula distribution, easily help to distinguish between Student T copula density with zero Kendall's tau and the independent copula density. For this reason and others, our second test is a Cramér-von Mises-type test which uses the Bernstein copula density estimator. Our third test is based on Kullback-Leibler divergence originally defined in terms of probability density functions to measure the divergence between two densities. We first rewrite the Kullback-Leibler divergence in terms of copula density, see Blumentritt & Schmid (2012). We then construct our third test of independence using an estimator of Kullback-Leibler divergence which is defined as a logarithmic function of the Bernstein copula density estimator. Finally, we provide the asymptotic distribution of each of the above three tests and consider a Monte Carlo experiment to investigate the performance of our tests of independence. In particular, we study the power functions of the proposed tests and we compare them with that of the test based on the empirical copula process considered in Deheuvels (1981c), Genest et al. (2006), and Kojadinovic & Holmes (2009).

The remainder of the paper is organized as follows. In section 2, we provide the definition of the Bernstein copula distribution and its properties. Thereafter, we define the process of Bernstein copula $\{\mathbb{B}_{k,n}(\mathbf{u}) : \mathbf{u} \in [0,1]^d\}$ and we construct our first test of independence based on the latter process. In section 3, we define the Bernstein density copula estimator and we propose an alternative class of nonparametric test of independence based on the Bernstein density copula. Section 4 is devoted to our third nonparametric test of independence which is based on the Kullback-Lieber divergence defined in terms of Bernstein copula density. Testing that this measure is equal to zero is equivalent to testing for independence. Section 5 reports the results of a Monte Carlo simulation exercise to illustrate the performance of the proposed tests statistics. Finally, we conclude in Section 6.

2 Test of independence based on Bernstein copula

2.1 Bernstein empirical copula

In this section, we define the estimator of the Bernstein copula distribution and discuss its asymptotic properties. This estimator will be applied later to build our first test of independence. Recall that the Bernstein polynomial estimators have been originally used as smooth estimators to estimate the cumulative distribution function and the corresponding probability density function. For example, Babu et al. (2002) have used the Bernstein polynomial to estimate the univariate probability density and the distribution function. Thereafter, Leblanc (2009, 2010, 2012a,b) has established the asymptotic properties (consistency, efficiency, normality) of the Bernstein estimator of the probability density and the distribution function. He has also shown that these estimators outperform the classical nonparametric estimators such as the empirical distribution function.

Sancetta & Satchell (2004) were the first to introduce the Bernstein polynomial for the estimation of copula distributions for i.i.d. data. Jansen et al. (2012) studied the asymptotic properties of this estimator (almost sure consistency rates, asymptotic normality). They provide explicit expressions for the asymptotic bias and asymptotic variance and show that Benrstien empirical copula outperforms the classical empirical copula in terms of the asymptotic mean squared error.

We next define the Bernstein copula estimator and examine its properties. From Sancetta & Satchell (2004) , we recall that the Bernstein polynomial of order k, for $k > 0$, of a copula function C can be defined as follows

$$B_k(\mathbf{u}) = \sum_{v_1=0}^{k} ... \sum_{v_d=0}^{k} C(\frac{v_1}{k}, ..., \frac{v_d}{k}) \prod_{j=1}^{d} P_{v_j,k}(u_j), \text{ for } \mathbf{u} = (u_1, ..., u_d) \in [0,1]^d, \tag{1}$$

where k plays the role of a bandwidth parameter and $P_{v_j,k}(u_j)$ is the binomial distribution function:

$$P_{v_j,k}(u_j) = \left(\begin{array}{c} k \\ v_j \end{array} \right) u_j^{v_j} \, (1-u_j)^{k-v_j}. \tag{2}$$

Since C is continuous on $[0,1]^d$, we have,

$$\lim_{k\to\infty} B_k(\mathbf{u}) = C(\mathbf{u}), \quad \text{uniformly in } \mathbf{u} \in [0,1]^d$$

In addition, under the conditions specified in their Theorem 1, Sancetta & Satchell (2004) show that $B_k(\mathbf{u})$ in (1) is itself a copula. Thus, to estimate the copula function C, they propose the following estimator (hereafter Bernstein empirical copula),

$$C_{k,n}(\mathbf{u}) = \sum_{v_1=0}^{k} ... \sum_{v_d=0}^{k} C_n(\frac{v_1}{k}, ..., \frac{v_d}{k}) \prod_{j=1}^{d} P_{v_j,k}(u_j), \text{ for } \mathbf{u} = (u_1, ..., u_d) \in [0,1]^d,$$

where C_n is the standard empirical copula estimator defined in the introduction. The order k will depend on the sample size n, with $k \to \infty$ if $n \to \infty$. In the rest of the paper and for the simplicity of exposition we denote by

$$\sum_{v} \equiv \sum_{v_1=0}^{k} ... \sum_{v_d=0}^{k}.$$

We now define the empirical Bernstein copula process under the null hypothesis of independence:

$$\mathbb{B}_{k,n}(\mathbf{u}) = n^{1/2}(C_{k,n}(\mathbf{u}) - C_\pi(\mathbf{u})), \text{ for } \mathbf{u} \in [0,1]^d,$$

where $C_\pi(\mathbf{u})$ is the copula function under independence.

In the following we recall a recent result of Jansen et al. (2012) on the asymptotic behavior of the empirical Bernstein copula process for $d \geq 2$. This will be needed to establish the asymptotic distribution of our first test of independence in Section 2.2.

Lemma 1 (Jansen and al, 2012). *Assume bounded third order partial derivatives for C on $(0,1)^d$. Suppose that k tends to infinity such that $n^{1/2}k^{-1}$ goes to zero when n is large. Then, the process $\mathbb{B}_{k,n}$ converges weakly to the Gaussian process, $\mathbb{C}(\mathbf{u})$, with mean zero and covariance function*

$$\mathrm{E}\Bigg[\Bigg(\mathbb{I}(U_1 \leq u_1, ..., U_d \leq u_d) - C(\mathbf{u}) - \sum_{j=1}^{d} C_{u_j}(\mathbf{u})(\mathbb{I}(U_j \leq u_j) - u_j)\Bigg)$$
$$\cdot \Bigg(\mathbb{I}(U_1 \leq v_1, ..., U_d \leq v_d) - C(\mathbf{v}) - \sum_{j=1}^{d} C_{v_j}(\mathbf{v})(\mathbb{I}(U_j \leq v_j) - v_j)\Bigg)\Bigg],$$

where C is the distribution function of $\mathbf{U} = (U_1, ..., U_d)$ and C_{u_j} denotes the partial derivative of C with respect to u_j.

2.2 *Test of independence*

In this section we use the empirical Bernstein copula process described in the previous section to build our first test of independence. The latter is defined in terms of copula function as follows:

$$\mathbb{H}_0 : C(\mathbf{u}) = C_\pi(\mathbf{u}) = \prod_{j=1}^{d} u_j, \text{ for } \mathbf{u} \in [0,1]^d.$$

To test $\mathbb{H}_0$, one can consider statistics, such as Kolmogorov-Smirnov or Cramér–von Mises statistics, that measure the departure from the null of independence. In other words, we can build tests of $\mathbb{H}_0$ using the Kolmogorov-Smirnov or Cramér–von Mises test statistics that measure the difference between the empirical Bernstein copula $C_{k,n}(\mathbf{u})$ and the independent copula function $C_\pi(\mathbf{u}) = \prod_{j=1}^{d} u_j$. Here, we consider the following Cramér–von Mises-type test statistic:

$$T_n = n \int_{[0,1]^d} \left[C_{k,n}(\mathbf{u}) - \prod_{j=1}^{d} u_j \right]^2 d\mathbf{u} = n \int_{[0,1]^d} \mathbb{B}_{k,n}^2(\mathbf{u}) d\mathbf{u}. \tag{3}$$

We have the following result that follows from Lemma 1 and the continuous mapping theorem.

Proposition 1. *Assume that the conditions in Lemma 1 are satisfied. Then, under the null hypothesis of independence $\mathbb{H}_0$, the statistic T_n in (3) converges in distribution to the following integral of a Gaussian process:*

$$\int_{[0,1]^d} \mathbb{C}^2(\mathbf{u}) d\mathbf{u},$$

where the process $\mathbb{C}(\mathbf{u})$ is defined in Lemma 1.

In practice, particularly for finite samples, our simulation results recommend to use a Monte Carlo-based method, instead of the asymptotic distribution, for the calculation of critical values (p-values) of the test statistic in (3). These simulations show that a Monte Carlo-based approach provides a better approximation for the distribution of the test statistic. Briefly speaking, the Monte Carlo-based method consists in generating several samples of the data under the null hypothesis of independence. For each of these samples, we calculate the test statistic in (3). We next use the empirical distribution of the calculated test statistics to compute the $1-\alpha th$ quantile, where α is a significance level. We then reject the null hypothesis of independence if the observed test statistic (computed using the real data) is greater than the calculated $1-\alpha th$ quantile.

Finally, the following proposition provides an explicit expression for the test statistic T_n.

Proposition 2. *If we note*

$$\sum_{v_1=0}^{k} \dots \sum_{v_d=0}^{k} \sum_{s_1=0}^{k} \dots \sum_{s_d=0}^{k} = \sum_{(v,s)},$$

then we have

$$T_n = n \sum_{(k,s)} C_n(\frac{v_1}{k}, \dots, \frac{v_d}{k}) C_n(\frac{s_1}{k}, \dots, \frac{s_d}{k}) \prod_{j=1}^{d} \binom{k}{v_j}\binom{k}{s_j} \beta(v_j + s_j + 1, 2k - v_j - s_j + 1)$$
$$- 2n \sum_{v_1=0}^{d} \dots \sum_{v_d=0}^{d} C_n(\frac{v_1}{k}, \dots, \frac{v_d}{k}) \prod_{j=1}^{d} \binom{k}{v_j} \beta(v_j + 2, k - v_j + 1) + \frac{n}{3^d},$$

where $\beta(.,.)$ is the beta function defined as $\beta(w_1, w_2) = \int_0^1 t^{w_1-1} (1-t)^{w_2-1} \, dt$, for w_1 and w_2 two positive integers.

Proof of Proposition 2. Observe that:

$$\begin{aligned}
T_n &= n \int_{[0,1]^d} (C_{k,n}(\mathbf{u}) - \prod_{j=1}^{d} u_j)^2 du_1 \dots du_d \\
&= n \int_{[0,1]^d} (C_{k,n}(\mathbf{u}))^2 du_1 \dots du_d - 2n \int_{[0,1]^d} C_{k,n}(\mathbf{u}) (\prod_{j=1}^{d} u_j) du_1 \dots du_d \\
&+ n \int_{[0,1]^d} (\prod_{j=1}^{d} u_j)^2 du_1 \dots du_d \\
&= I_1 - I_2 + I_3.
\end{aligned}$$

Now we calculate I_1

$$
\begin{aligned}
I_1 &= n\int_{[0,1]^d} (C_{k,n}(\mathbf{u}))^2 du_1...du_d \\
&= n\int_{[0,1]^d} \sum_{(v,s)} C_n(\frac{v_1}{k},...,\frac{v_d}{k})C_n(\frac{s_1}{k},...,\frac{s_d}{k}) \\
&\times \prod_{j=1}^{d} P_{v_j,k}(u_j)P_{s_j,k}(u_j)du_1...du_d \\
&= n\sum_{(v,s)} \tbinom{k}{v_1}...\tbinom{k}{v_d}\tbinom{k}{s_1}...\tbinom{k}{s_d} C_n(\frac{v_1}{k},...,\frac{v_d}{k})C_n(\frac{s_1}{k},...,\frac{s_d}{k}) \\
&\times \int_{[0,1]^d} \prod_{j=1}^{d} u^{v_j+s_j}(1-u)^{2k-v_j-s_j}du_1...du_d \\
&= n\sum_{(v,s)} C_n(\frac{v_1}{k},...,\frac{v_d}{k})C_n(\frac{s_1}{k},...,\frac{s_d}{k}) \\
&\times \prod_{j=1}^{d} \tbinom{k}{v_j}\tbinom{k}{s_j}\beta(v_j+s_j+1, 2k-v_j-s_j+1).
\end{aligned}
$$

In a similar way we have:

$$
I_2 = 2n\sum_{v_1=0}^{k}...\sum_{v_d=1}^{k} C_n(\frac{v_1}{k},...,\frac{v_d}{k})\prod_{j=1}^{d}\tbinom{k}{v_j}\beta(v_j+2, k-v_j+1).
$$

Hence the result in Proposition 2.

■

3 Test of independence based on Bernstein copula density

We next propose a second test of independence based on Bernstein copula *density* instead of the empirical Bernstein copula. Before, we need to define the Bernstein copula density estimator.

3.1 Bernstein copula density estimator

If it exists, the copula density, denoted by c, which corresponds to the copula function C is given by:

$$
c(\mathbf{u}) = \partial^d C(\mathbf{u})/\partial u_1...\partial u_d.
$$

Since the Bernstein copula function in (1) is absolutely continuous, it follows that the Bernstein copula density is defined as:

$$c_k(\mathbf{u}) = \sum_{v_1=0}^{k} ... \sum_{v_d=0}^{k} C(\frac{v_1}{k}, ..., \frac{v_d}{k}) \prod_{j=1}^{d} P'_{v_j,k}(u_j),$$

where $P'_{v_j,k}$ is the derivative of $P_{v_j,k}$ with respect to u. Hence, the Bernstein estimator of the copula density is given by

$$\hat{c}_n(\mathbf{u}) = \sum_{v_1=0}^{k} ... \sum_{v_d=0}^{k} C_n(\frac{v_1}{k}, ..., \frac{v_d}{k}) \prod_{j=1}^{d} P'_{v_j,k}(u_j).$$

The above estimator is proposed and investigated in Sancetta & Satchell (2004) for i.i.d. data. Later, Bouezmarni et al. (2010) have used the Bernstein polynomial to estimate the copula density in the presence of dependent data. They provide the asymptotic properties (asymptotic bias, asymptotic variance, uniform a.s. convergence and asymptotic normality) of the Bernstein density copula estimator for α-mixing data. Recently, Jansen et al. (2014) have reinvestigate this estimator by establishing its asymptotic normality under i.i.d. data.

Now to build our second test statistic for testing the null of independence, we rather use the definition of the Bernstein copula density estimator in Bouezmarni et al. (2010) given by

$$\hat{c}_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} K_k(\mathbf{u}, S_i), \text{ for } \mathbf{u} \in [0,1]^d, \tag{4}$$

where

$$K_k(\mathbf{u}, S_i) = k^d \sum_{\nu_1=1}^{k-1} ... \sum_{\nu_d=1}^{k-1} A_{S_i,\nu} \prod_{j=1}^{d} P_{\nu_j,k-1}(u_j),$$

with $S_i = (F_{i;n}(X_{i1}), ..., F_{d;n}(X_{id}))$, where $F_{j;n}(.)$, for $j = 1, ..., d$, is the empirical distribution of the random component X_j, $A_{S_i,\nu} = \mathbf{1}_{\{S_i \in B_\nu\}}$, for $B_\nu = \left[\frac{\nu_1}{k}, \frac{\nu_1+1}{k}\right] \times ... \times \left[\frac{\nu_d}{k}, \frac{\nu_d+1}{k}\right]$, k is an integer playing the role of the bandwidth parameter, and $P_{\nu_j,k-1}(u_j)$ is the binomial distribution function defined as in (2).

3.2 Test of independence

In order to test the null hypothesis of independence, which is equivalent to testing

$$H_0 : c(\mathbf{u}) = 1, \qquad \mathbf{u} \in [0,1]^d,$$

we consider the following Cramér–von Mises-type statistic, which is based on the Bernstein copula density estimator,

$$I_n(\mathbf{u}) = \int_{[0,1]^d} \{\hat{c}_n(\mathbf{u}) - 1\}^2 d\mathbf{u}. \tag{5}$$

It is important to notice that building tests of independence based on Bernstein copula density instead of copula distribution, can be motivated by the fact that the copula density captures better the dependence even when the Kendall's tau coefficient is small or zero. For example, it is quite straightforward to see that when the Kendall's tau is equal to zero, one can not distinguish between the student copula distribution and the independent copula, however it is easier to distinguish between their corresponding densities. In this case, the lower and upper tail dependence of the student copula density are equal to 0.1816901, even when the Kendall's tau is equal to zero. This situation is illustrated in 1 where Kendall's tau is taken equal to zero. From this, we see that it is not possible to distinguish between the sub-figures in the top and bottom of the left-hand side panel of Figure 1, which correspond to the student copula distribution (in the top) and the independent copula distribution (in the bottom). However, as we see in the right-hand side panel of Figure 1, it is very simple to distinguish between the student copula density (in the top) and the independent copula density (in the bottom).

The following proposition provides a practical expression for the test statistic I_n in the bivariate case.

Proposition 3. *Using the notation in Proposition 2, we have for $d = 2$*

$$
\begin{aligned}
I_n &= \int_{[0,1]^2} (\hat{c}_n(u,v) - 1)^2 dudv \qquad (6)\\
&= \int_{[0,1]^2} (\hat{c}_n(u,v)^2 dudv - 1 \\
&= \sum_{(h,l,s,t)} C_n(\frac{h}{k}, \frac{l}{k}) C_n(\frac{s}{k}, \frac{t}{k}) \tbinom{k}{h}\tbinom{k}{l}\tbinom{k}{s}\tbinom{k}{t} \\
&\times \beta(h+s, 2k-h-s-1)\beta(l+t, 2k-l-t-1) \\
&\times \left[\frac{hs(2k-2)}{h+s+1} - k(h+s) + \frac{k^2(h+s-1)}{2k-1}\right] \\
&\times \left[\frac{lt(2k-2)}{t+t+1} - k(l+t) + \frac{k^2(l+t-1)}{2k-1}\right] - 1.
\end{aligned}
$$

Proof of Proposition 3. The proof is similar to that of Proposition 2.

■

To derive the asymptotic distribution of the test statistic I_n, some additional regularity assumptions are needed. We consider the following set of standard assumptions on the process of $\mathbf{X}$ and the bandwidth parameter k of the Bernstein copula density estimator.

Assumption A: The random vector $\mathbf{X}$, with cumulative distribution function $F(X)$, has a copula function C and copula density c. We assume that c is twice continuously differentiable on $(0,1)^d$ and bounded away from 0.

Assumption B: We assume that for $k \to \infty$, $nk^{-(d/2)-2} \to 0$ and $n^{-1/2}k^{d/4} \to 0$.

We now state the asymptotic distribution of our second test statistic under the null hypothesis.

Theorem 1. *Under Assumptions **A** and **B** and under H_0, we have*

$$
\frac{nk^{-d/2}}{(\sqrt{2}\pi/4)^{d/2}} \left(I_n - 2^{-d} n^{-1} (\pi k)^{d/2}\right) \quad \xrightarrow{d} \quad N(0,1), \;\; \textit{as } n \to \infty,
$$

where I_n is defined in (5).

Proof of Theorem 1. For simplicity of exposition, hereafter we provide the proof for the case $d = 2$. For the more general case $d > 2$, the proof can be obtained in a similar way. First, we denote by, for $\mathbf{u} = (u_1, u_2)$,

$$r_n(\mathbf{u}) = \hat{c}_n(\mathbf{u}) - 1,$$

$$K_k^*(\mathbf{u}, S_i) = K_k(\mathbf{u}, S_i) - \mathbb{E}(K_k(\mathbf{u}, S_i)), \text{ and}$$

$$H_n(a, b) = \textstyle\int K_k^*(\mathbf{u}, a) K_k^*(\mathbf{u}, b) du.$$

Using the above notations, we have

$$\begin{aligned}
I_n - \mathbb{E}(I_n) &= 2 \int \left[r_n(\mathbf{u}) - \mathbb{E}(r_n(\mathbf{u}))\right] \mathbb{E}(r_n(\mathbf{u})) d\mathbf{u} \\
&+ \int \left\{ \left[r_n(\mathbf{u}) - \mathbb{E}(r_n(\mathbf{u}))\right]^2 - \mathbb{E}\left[r_n(\mathbf{u}) - \mathbb{E}(r_n(\mathbf{u}))\right]^2 \right\} d\mathbf{u} \\
&= 2 \int \left[r_n(\mathbf{u}) - \mathbb{E}(r_n(\mathbf{u}))\right] \mathbb{E}(r_n(\mathbf{u})) d\mathbf{u} \\
&+ \frac{2}{n^2} \sum_{i<j} \left\{ H_n(S_i, S_j) - \mathbb{E}(H_n(S_i, S_j)) \right\} \\
&+ \frac{1}{n^2} \sum_{i=1}^{n} \left\{ H_n(S_i, S_i) - \mathbb{E}(H_n(S_i, S_i)) \right\} \\
&= I_{1n} + I_{2n} + I_{3n}.
\end{aligned}$$

Now, we first prove that the terms nkI_{1n} and nkI_{3n} are negligible. We next use the central limit theorem of the U-statistics, see Hall (1984), to show that nkI_{2n} is asymptotically normally distributed with mean zero and variance σ^2. From Bouezmarni et al. (2010) and Jansen et al. (2014), we have

$$\mathbb{E}(r_n(\mathbf{u})) = O(k^{-1})$$

and

$$\hat{c}_n(\mathbf{u}) - \mathbb{E}(\hat{c}_n(\mathbf{u})) = O(k^{-1}) + o_p(n^{-1/2} k^{1/2}).$$

Hence,

$$\begin{aligned}
nk^{-1} I_{1n} &= O_p(nk^{-1}k^{-2} + nk^{-1}n^{-(1/2)-2}) \\
&= O_p(nk^{-3} + nk^{-(1/2)-2}).
\end{aligned}$$

and under Assumption **B**, we get

$$nk^{-1}I_{1n} = o_p(1).$$

For the term I_{3n}, we have

$$\begin{aligned}
\mathbb{V}ar(H_n(S_i, S_i) &\approx \mathbb{V}ar\left[\int K_k^2(\mathbf{u}, S_i)du_1du_2\right] \\
&= k^4\mathbb{V}ar\left[\int \sum_{\nu_1}\sum_{\nu_2} A_{S_i,\nu}P^2_{\nu_1,k-1}(u_1)P^2_{\nu_2,k-1}(u_2)du_1du_2\right] \\
&\approx k^4 \int \sum_{\nu_1}\sum_{\nu_2} P_nP^4_{\nu_1,k-1}(u_1)P^4_{\nu_2,k-1}(u_2)du_1du_2,
\end{aligned}$$

where

$$\begin{aligned}
P_n &= \int_{\frac{\nu_1}{k}}^{\frac{\nu_1+1}{k}} \int_{\frac{\nu_2}{k}}^{\frac{\nu_2+1}{k}} c(u_1, u_2)du_1du_2 \\
&= \frac{1}{k^2} \qquad \text{under the null hypothesis.}
\end{aligned}$$

Consequently,

$$\mathbb{V}ar(H_n(S_i, S_i) = O(k^{1/2}).$$

Hence

$$\begin{aligned}
nk^{-1}I_{3n} &= O_p(n^{-1/2}k^{-1/2}) \\
&= o_p(1) \qquad \text{under Assumption B.}
\end{aligned}$$

For the third term I_{2n}, we can follow similar arguments as in proof of Theorem 1 in the appendix of Bouezmarni et al. (2012) to show that

$$\mathbb{E}\left(H_n^*(S_i, S_j)\right)^2 = \left(\tfrac{\pi}{4}\right)^2,$$

$$\left\|H_n^*(S_i, S_1)H_n^*(S_i, S_2)\right\|_4 = o(1), \text{ and}$$

$$\left\|H_n^*(S_i, S_j)\right\|_4 = o(1), \quad \text{with} \quad H_n^*(S_i, S_j) = k^{-1}H_n(S_i, S_j).$$

Consequently, from Hall (1984) we obtain

$$\frac{\sqrt{2}}{n\pi/4}\sum_{i<j}\left(H_n^*(S_i, S_j) - \mathbb{E}(H_n^*(S_i, S_j))\right) \xrightarrow{d} N(0, 1).$$

Hence

$$\frac{2\sqrt{2}}{\pi} n k^{-1} I_{2n} \xrightarrow{d} N(0,1),$$

Therefore

$$\frac{2\sqrt{2}}{\pi} n k^{-1} \left(I_n - \mathbb{E}(I_n)\right) \to N(0,1).$$

Finally, the expression of the the bias $\mathbb{E}(I_n)$ can be obtained using Bouezmarni et al. (2010) and Jansen et al. (2014):

$$\mathbb{E}(I_n) = \mathbb{E}\left(\int \left[\hat{c}_n(\mathbf{u}) - 1\right]^2 d\mathbf{u}\right) \approx \frac{k}{n}\frac{\pi}{4},$$

which concludes the proof of Theorem 1.

■

As for the test statistic T_n in (3), in practice we recommend to use a Monte Carlo-based method, instead of the asymptotic distribution, for the calculation of critical values (p-values) of the test statistic in (5). A brief description of the Monte Carlo-based approach can be found in the paragraph after Proposition 1.

4 Test of independence based on Kullback-Leibler divergence

4.1 Measure of dependence based on Kullback-Leibler divergence

Relative entropy, known as Kullback-Leibler divergence, is a measure of multivariate association which is originally defined in terms of probability density functions. Following Blumentritt & Schmid (2012), in this section we redefine the Kullback-Leibler measure in terms of copula density to disentangle the dependence structure from the marginal distributions. Blumentritt & Schmid (2012) use this copula density-based definition to propose an estimator of the measure of dependence based on Bernstein copula density estimator. The latter is guaranteed to be non-negative. The non-negativity of the Bernstein estimators avoids having negative values inside the logarithmic function of the Kullback distance. Furthermore, there is no boundary bias problem when we use the Bernstein estimator, because by smoothing with beta densities the Bernstein copula density does not assign weights outside its support.

We now review the theoretical aspects of the above measure of dependence. Joe (1987), Joe (1989a), and Joe (1989b) have introduced relative entropy as a measure of multivariate association in the random vector $\mathbf{X}$. This relative entropy is defined as

$$\delta(\mathbf{X}) = \int_{\mathbb{R}^d} \log \left[\frac{f(x)}{\prod_{i=1}^{d} f_i(x_i)} \right] f(x) dx, \tag{7}$$

where f is the probability density of the random vector $\mathbf{X}$ and f_i is the marginal probability density of its component X_i, for $i = 1, ..., d$. According to Sklar (1959), the density function of the joint process X can be expressed as

$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d)) \prod_{i=1}^{d} f_i(x_i), \tag{8}$$

where c is the density copula function. Using Equation (8), we can show that the relative entropy in (7) can be rewritten in terms of copula density as

$$\delta(\mathbf{X}) = \delta(c) = \int_{[0,1]^d} \log [c(u)] c(u) du. \tag{9}$$

The measure of dependence $\delta(c)$ does not depend on the marginal distributions of $\mathbf{X}$, but only on its copula C via the copula density c. Thus, saying that the null hypothesis of independence is satisfied, which corresponds to $c(u) \equiv 1$, is equivalent to say that the measure of dependence $\delta = 0$. We next define a nonparametric estimate of δ that we use to build our third test of independence and establish its asymptotic normality.

4.2 Test of independence based on Bernstein estimator of δ

We have shown that the measure of dependence δ can be rewritten in terms of copula density function c. Thus, following Blumentritt & Schmid (2012), an estimator of δ can be obtained by replacing the unknown copula density by its Bernstein copula density estimator. As we saw in Section 3.1, the latter estimator is defined as

$$\hat{c}_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} K_k(\mathbf{u}, S_i), \text{ for } \mathbf{u} \in [0,1]^d, \tag{10}$$

where

$$K_k(\mathbf{u}, S_i) = k^d \sum_{\nu_1=1}^{k-1} ... \sum_{\nu_d=1}^{k-1} A_{S_i,\nu} \prod_{j=1}^{d} P_{\nu_j,k-1}(u_j),$$

and the terms S_i, $F_{j;n}(.)$, $A_{S_i,\nu}$, k, and $P_{\nu_j,k-1}(u_j)$ are defined in Section 3.1. Hence, an estimator of the measure δ is given by

$$\hat{\delta}_n(c) = \int_{[0,1]^d} \log\left[\hat{c}_n(u)\right] dC_n(u) = \frac{1}{n}\sum_{i=1}^{n} \log(\hat{c}_n(S_i)). \tag{11}$$

Our third test of independence is based on $\hat{\delta}_n(c)$. As we said previously, saying that the null of independence is true is equivalent to say that the measure of dependence δ is equal to zero. Thus, the statistic in (11) can be naturally used as a test statistic to test the null hypothesis of independence H_0. The following theorem provides the asymptotic normality of the test statistic $\hat{\delta}_n(c)$.

Theorem 2. *Under Assumptions* ***A*** *and* ***B*** *and H_0, we have*

$$\frac{nk^{-(d/2)}}{\sqrt{2}(\pi/4)^{d/2}}\left(2\hat{\delta}_n - 2^{-d}n^{-1}(\pi k)^{d/2}\right) \to N(0,1), \ \ \text{as } n \to \infty,$$

where $\hat{\delta}_n$ is defined in (11).

Proof of Theorem (2): Using a Taylor series expansion around point $x^* = 1$ for the function $g(x) = x\log(x)$, this leads to

$$\delta(c) = \int (g(c(u)))du = \sum_{r=2}^{\infty} \frac{(-1)^r}{r(r-1)} \int (c(u)-1)^r du.$$

Consequently, we can write

$$\hat{\delta}_n(c) \approx \frac{1}{2}\int (\hat{c}_n(u)-1)^2 du = \frac{1}{2} I_n.$$

Hence the proof of the result in Theorem 2.

■

As for the test statistics T_n and I_n in (3) and (5), respectively, in practice we recommend to use a Monte Carlo-based method, instead of the asymptotic distribution, for the calculation of critical values (p-values) of the test statistic in (11). A brief description of the Monte Carlo-based approach can be found in the paragraph after Proposition 1.

5 Simulation studies

We run a Monte Carlo experiment to investigate the performance of nonparametric tests of independence proposed in the previous sections. In particular, we study the power of the test statistics T_n, I_n and δ_n using different samples sizes: $n = 100$, 200, 400, 500. To calculate the critical values, at the significance level $\alpha = 5\%$, of the test statistics under the null of independence, we simulate independent data using the independent copula, which is available on the R package Copula. Subsequently, to evaluate the power of our nonparametric tests, we use different copula functions to generate data under different degrees of dependence, which correspond to the following values of the Kendall's tau coefficient $\tau = 0, 0.25, 0.5$. The copulas under consideration are Normal, Student, Clayton and Gumbel copulas. Moreover, the power functions of our test statistics T_n, I_n and δ_n are compared with that of the following natural competitor test, which is based on the empirical copula process considered in Deheuvels (1981c), Genest, Quessy, and Rémillard (2006), and Kojadinovic and Holmes (2009):

$$S_n = n \int_{[0,1]^2} \{C_n(u,v) - C_\pi(u,v)\}^2 du dv. \tag{12}$$

The test statistics T_n, I_n and δ_n depend on the bandwidth parameter k, which is needed to estimate the copula density (distribution). We take k equal to the integer part of $cn^{1/2}$, for different values of $c = 0.5, 1, 1.5, 2$ that satisfy Assumption **B**. This is common practice in nonparametric testing where no optimal bandwidth is available. In fact, this would typically involve an Edgeworth expansion of the asymptotic distribution of the test statistics, as proposed in Omelka et al. (2009) for kernel estimator of copula, which is complex and left for future research. However, in our simulations we considered various values of c and we investigate the sensitivity of the power functions of the test statistics T_n, I_n and δ_n to the bandwidth parameter k. Finally, the number of replications used to compute the critical values and the empirical power functions is equal to 1000.

The simulation results for the empirical power of the test statistics T_n, I_n, δ_n, and S_n are reported in Tables 1-3. Table 1 compares the power of the test statistics (3) and (12) which are based on copula distributions. From this, we see that the power functions of T_n and S_n are quite compared, except for the case of Student T copula distribution where we can clearly see that the power of our test T_n is slightly better than the competitor test S_n, especially when $\tau = 0$ and the sample size is large. Note that for the Student T case, $\tau = 0$ does not mean independence, and and this case illustrates the capacity of our non-parametric test to detect some nonlinear dependence. Table 2 compares the power of the test statistics I_n and S_n which are based on copula density and copula distribution, respectively. From this table and for Student T copula and $\tau = 0$ cases, we see that there is a power improvement, compared to the results in Table 1, when one uses the copula density-based test statistic in (5) instead of the copula distribution-based test statistic in (3). This improvement becomes very important when the sample size is large. For example, for $n = 400, 500$ the power of the test statistic I_n can be approximately 7 or 8 times bigger than the power of the test statistic S_n, and this is for different values of the bandwidth k. A similar pattern is observed when we compare the test statistics (11) and (12); see Table 3. The good performance of the test statistics I_n and δ_n can be explained by the results in Figure 1; see also the discussion in the second paragraph of Section 3.2.

Finally, Figures 2-4 illustrate the sensitivity of the power functions of our tests of independence to the bandwidth parameter k. We focus on the Student T copula, because it is the most relevant case according to the results in Tables 1-3. To evaluate the sensitivity of our tests we consider many values of k. Figures 2-4 show that the power functions of all tests are generally insensitive to the bandwidth k, except in the presence of low dependence (when the Kendall's tau is equal to zero) and for large samples. For $\tau = 0$, we find that the power of our three tests is an increasing function of the bandwidth k, with more sensitivity in the case of test statistics I_n and δ_n. For the other values of τ, the power functions are insensitive to k, except for $\tau = 0.25$. Overall, when the sample size is bigger than 200, we recommend to use a bandwidth k which is bigger than or equal to 35.

6 Conclusion

We provided three different nonparametric tests of independence between continuous random variables based on Bernstein empirical copula and Bernstein empirical copula density. The first two tests were constructed based on functional of Cramér-von Mises of the Bernstein empirical copula process and the Bernstein density copula, respectively. The third test is based on Kullback-Leibler divergence originally defined in terms of probability density functions to measure the divergence between two densities. We first rewrote the Kullback-Leibler divergence in terms of copula density, see Blumentritt & Schmid (2012). We then constructed our third test of independence using an estimator of Kullback-Leibler divergence which is defined as a logarithmic function of the Bernstein copula density estimator. Finally, we provided the asymptotic distribution of each of the above three tests and considered a Monte Carlo experiment to investigate the performance of our tests of independence. In particular, we studied the power functions of the proposed tests and we compared them with that of the test based on the empirical copula process considered in Deheuvels (1981c), Genest et al. (2006), and Kojadinovic & Holmes (2009).

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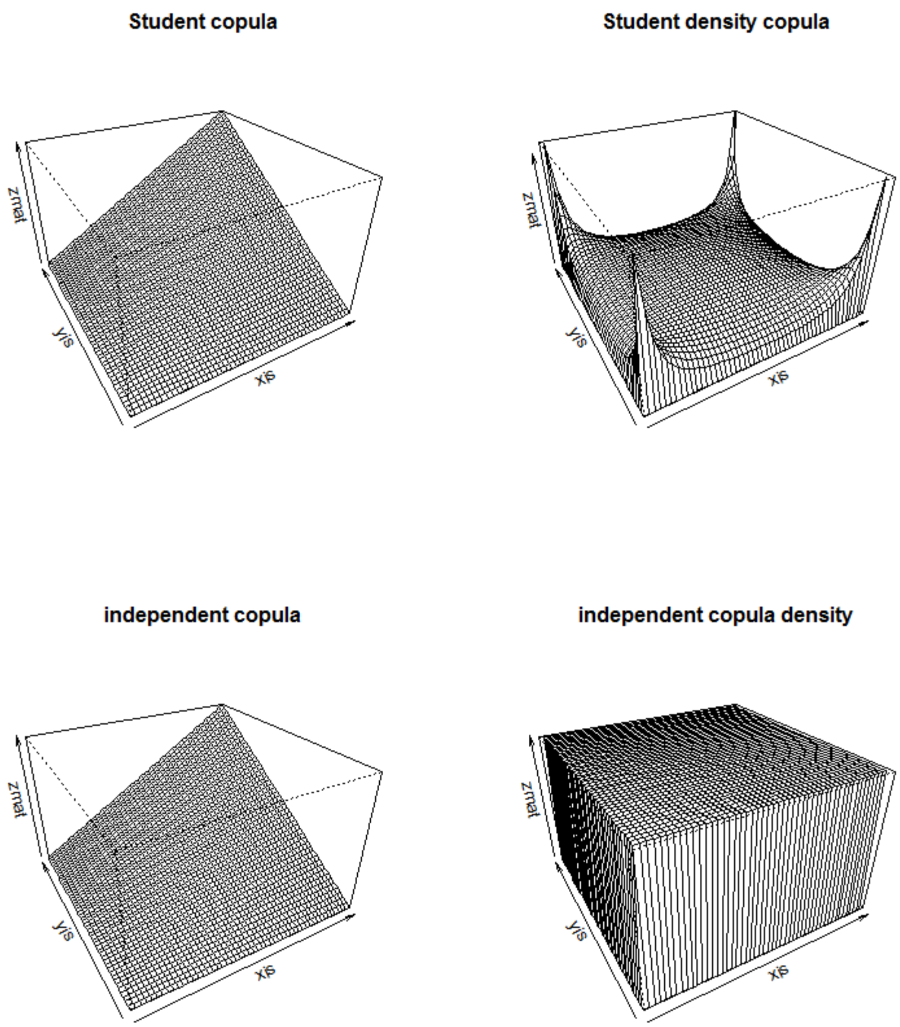

Figure 1: *This figure compares the student copula distribution (in the top of the left-hand side panel) and the independent copula distribution (in the bottom of the left-hand side panel) and between the student copula density (in the top of the right-hand side panel) and the independent copula density (in the bottom of the right-hand side panel). The independence here corresponds to the case where the Kendall's tau is taken equal to zero.*

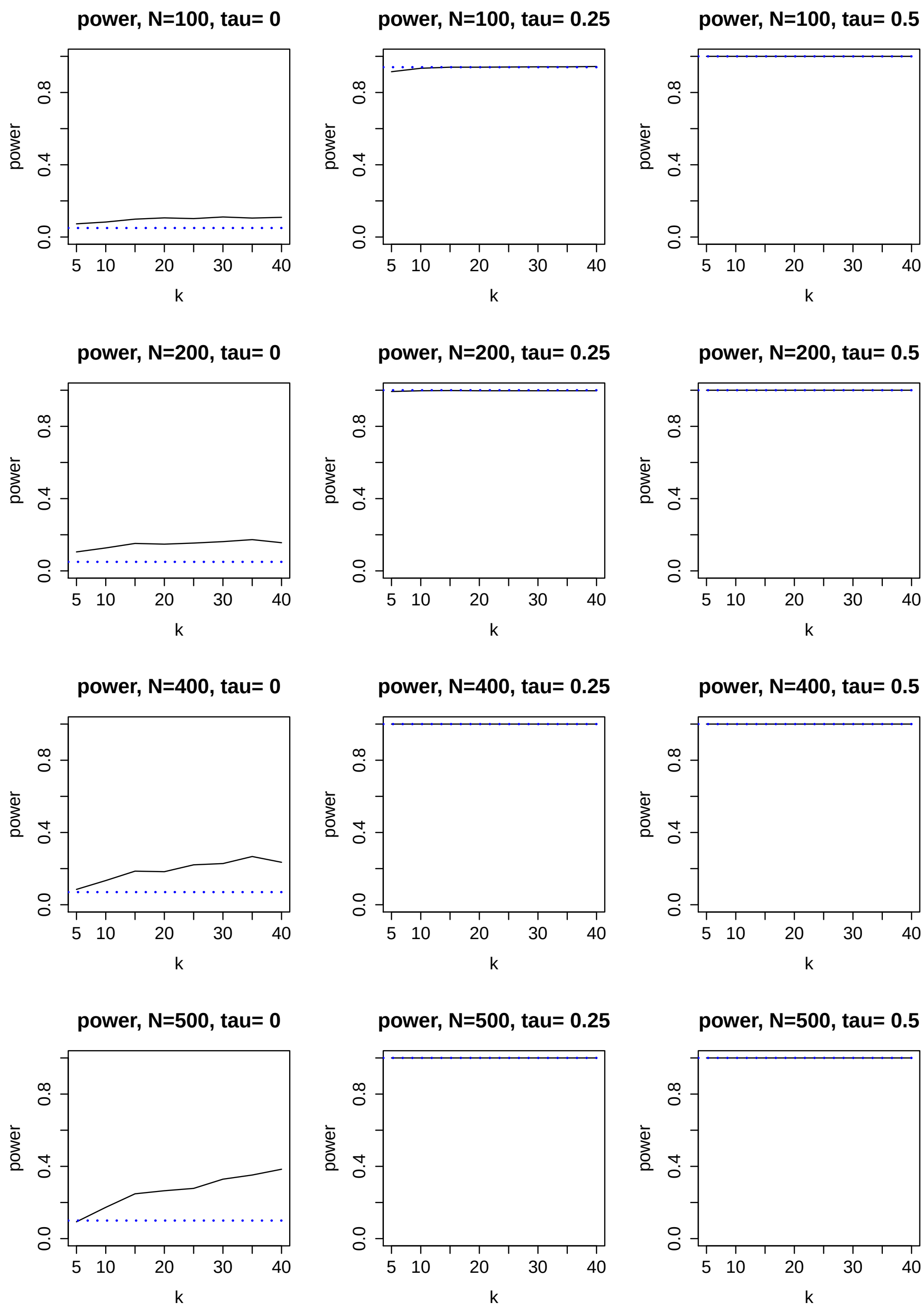

Figure 2: *This figure plots the power functions of the test statistics T_n (solid line) and S_n (dash line) as functions of the bandwidth parameter k for student copula distribution.*

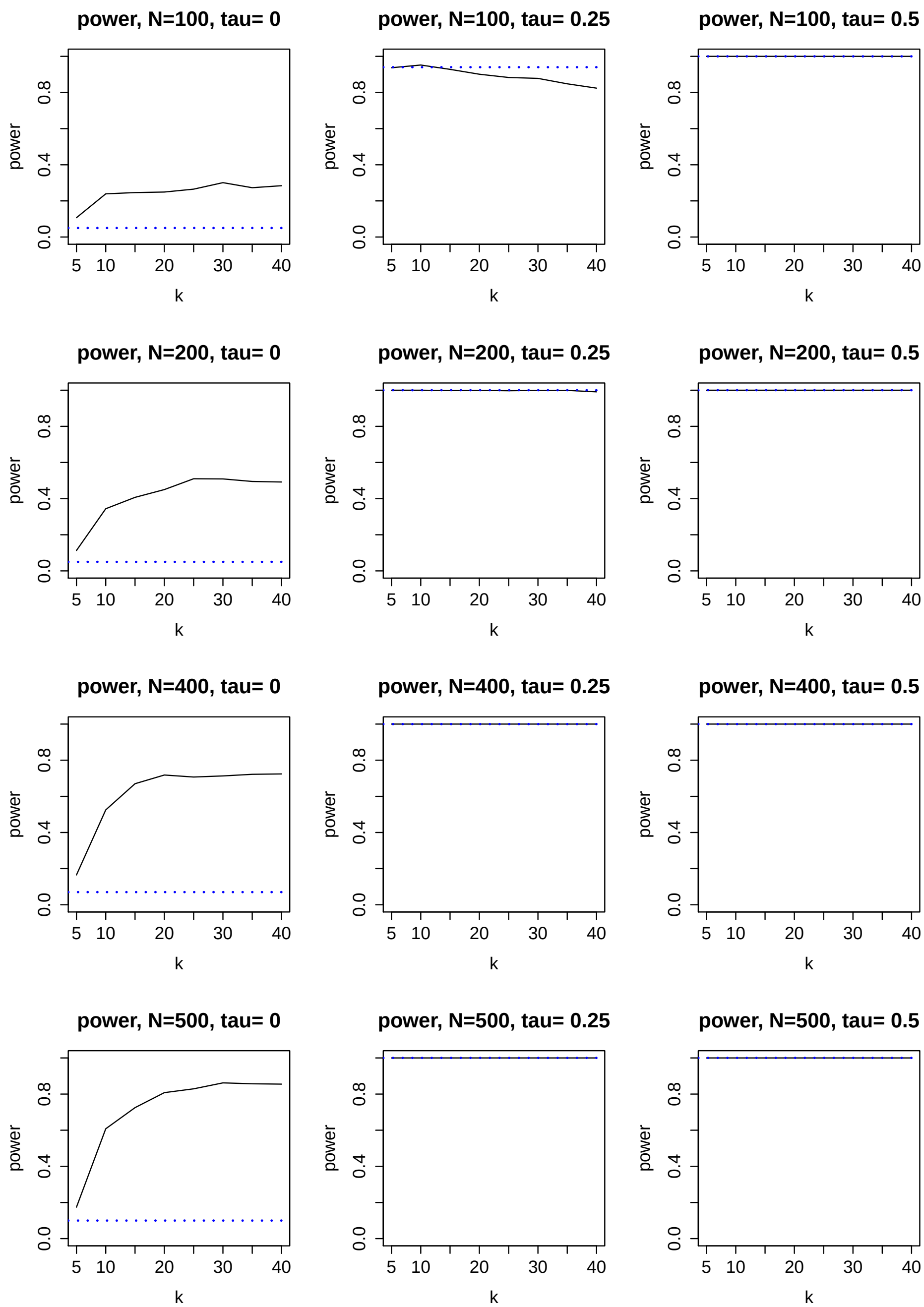

Figure 3: *This figure plots the power functions of the test statistics* I_n *(solid line) and* S_n *(dash line) as functions of the bandwidth parameter* k *for student copula density.*

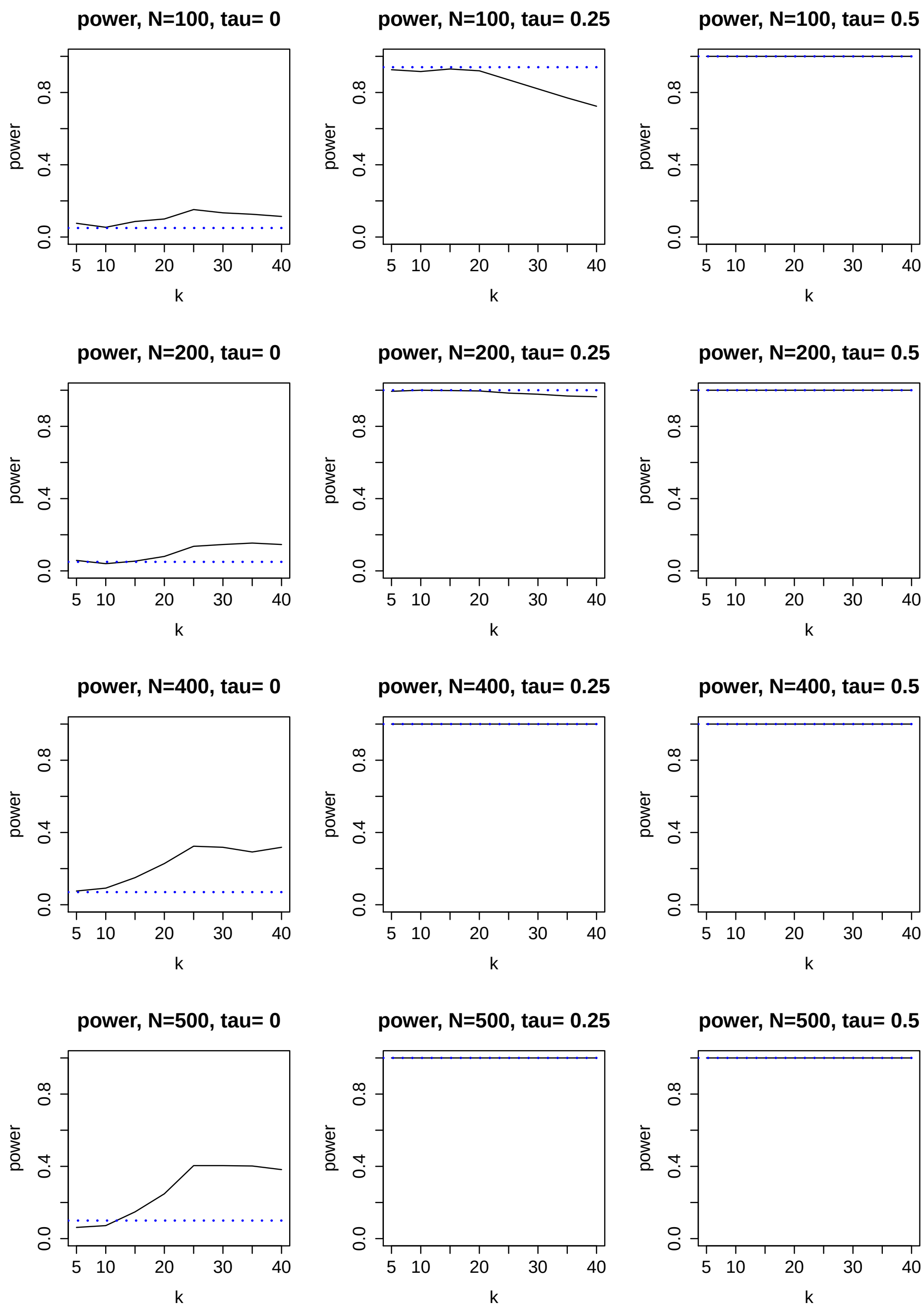

Figure 4: *This figure plots the power functions of the test statistics δ_n (solid line) and S_n (dash line) as functions of the bandwidth parameter k for student copula density.*

Normal copula												
Test statistic T_n												
	N = 100			N = 200			N = 400			N = 500		
c	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$
0.5	0.04	0.92	1.00	0.07	1.00	1.00	0.06	1.00	1.00	0.04	1.00	1.00
1	0.04	0.94	1.00	0.06	1.00	1.00	0.06	1.00	1.00	0.04	1.00	1.00
1.5	0.05	0.94	1.00	0.06	1.00	1.00	0.06	1.00	1.00	0.04	1.00	1.00
2	0.04	0.94	1.00	0.06	1.00	1.00	0.06	1.00	1.00	0.04	1.00	1.00
Test statistic S_n												
	0.04	0.94	1.00	0.04	0.99	1.00	0.05	1.00	1.00	0.05	1.00	1.00
Student copula												
Test statistic T_n												
0.5	0.07	0.92	1.00	0.10	0.99	1.00	0.13	1.00	1.00	0.17	1.00	1.00
1	0.10	0.94	1.00	0.15	1.00	1.00	0.18	1.00	1.00	0.26	1.00	1.00
1.5	0.11	0.94	1.00	0.15	1.00	1.00	0.23	1.00	1.00	0.33	1.00	1.00
2	0.11	0.94	1.00	0.16	1.00	1.00	0.23	1.00	1.00	0.38	1.00	1.00
Test statistic S_n												
	0.04	0.92	1.00	0.05	1.00	1.00	0.07	1.00	1.00	0.10	1.00	1.00
Clayton copula												
Test statistic T_n												
0.5	0.05	0.93	1.00	0.05	1.00	1.00	0.06	1.00	1.00	0.06	1.00	1.00
1	0.05	0.94	1.00	0.06	1.00	1.00	0.06	1.00	1.00	0.07	1.00	1.00
1.5	0.06	0.96	1.00	0.06	1.00	1.00	0.06	1.00	1.00	0.07	1.00	1.00
2	0.05	0.94	1.00	0.06	1.00	1.00	0.06	1.00	1.00	0.07	1.00	1.00
Test statistic S_n												
	0.05	0.94	1.00	0.04	1.00	1.00	0.05	1.00	1.00	0.05	1.00	1.00
Gumbel copula												
Test statistic T_n												
0.5	0.05	0.94	1.00	0.06	1.00	1.00	0.06	1.00	1.00	0.06	1.00	1.00
1	0.05	0.96	1.00	0.05	1.00	1.00	0.06	1.00	1.00	0.06	1.00	1.00
1.5	0.05	0.96	1.00	0.06	1.00	1.00	0.05	1.00	1.00	0.07	1.00	1.00
2	0.06	0.97	1.00	0.05	1.00	1.00	0.05	1.00	1.00	0.06	1.00	1.00
Test statistic S_n												
	0.03	0.93	1.00	0.05	1.00	1.00	0.05	1.00	1.00	0.05	1.00	1.00

Table 1: *This table compares the empirical size and power of the test statistics T_n and S_n for different copulas (Normal, Student, Clayton and Gumbel copulas), different values of Kendall's tau coefficient τ ($\tau = 0, 0.25, 0.5$), different sample sizes n ($n = 100, 200, 400, 500$), and different values for the bandwidth $k = cn^{1/2}$ ($c = 0.5, 1, 1.5, 2$).*

Normal copula												
Test statistic I_n												
	N = 100			N = 200			N = 400			N = 500		
c	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$
0.5	0.05	0.94	1.00	0.04	1.00	1.00	0.06	1.00	1.00	0.06	1.00	1.00
1	0.06	0.90	1.00	0.03	1.00	1.00	0.04	1.00	1.00	0.06	1.00	1.00
1.5	0.04	0.82	1.00	0.03	0.99	1.00	0.04	1.00	1.00	0.07	1.00	1.00
2	0.06	0.77	1.00	0.04	0.99	1.00	0.05	1.00	1.00	0.07	1.00	1.00
Test statistic S_n												
	0.04	0.94	1.00	0.04	0.99	1.00	0.05	1.00	1.00	0.05	1.00	1.00
Student copula												
Test statistic I_n												
0.5	0.11	0.94	1.00	0.11	1.00	1.00	0.52	1.00	1.00	0.61	1.00	1.00
1	0.25	0.93	1.00	0.41	1.00	1.00	0.72	1.00	1.00	0.81	1.00	1.00
1.5	0.25	0.90	1.00	0.45	1.00	1.00	0.71	1.00	1.00	0.86	1.00	1.00
2	0.30	0.88	1.00	0.51	1.00	1.00	0.72	1.00	1.00	0.85	1.00	1.00
Test statistic S_n												
	0.04	0.92	1.00	0.05	1.00	1.00	0.07	1.00	1.00	0.10	1.00	1.00
Clayton copula												
Test statistic I_n												
0.5	0.06	0.96	1.00	0.06	1.00	1.00	0.05	1.00	1.00	0.06	1.00	1.00
1	0.05	0.96	1.00	0.04	1.00	1.00	0.05	1.00	1.00	0.05	1.00	1.00
1.5	0.04	0.93	1.00	0.04	1.00	1.00	0.05	1.00	1.00	0.06	1.00	1.00
2	0.06	0.92	1.00	0.06	1.00	1.00	0.06	1.00	1.00	0.06	1.00	1.00
Test statistic S_n												
	0.05	0.94	1.00	0.04	1.00	1.00	0.05	1.00	1.00	0.05	1.00	1.00
Gumbel copula												
Test statistic I_n												
0.5	0.05	0.95	1.00	0.04	1.00	1.00	0.05	1.00	1.00	0.07	1.00	1.00
1	0.06	0.93	1.00	0.05	1.00	1.00	0.06	1.00	1.00	0.06	1.00	1.00
1.5	0.04	0.90	1.00	0.04	1.00	1.00	0.05	1.00	1.00	0.07	1.00	1.00
2	0.06	0.87	1.00	0.06	1.00	1.00	0.05	1.00	1.00	0.06	1.00	1.00
Test statistic S_n												
	0.03	0.93	1.00	0.05	1.00	1.00	0.05	1.00	1.00	0.05	1.00	1.00

Table 2: *This table compares the empirical size and power of the test statistics I_n and S_n for different copulas (Normal, Student, Clayton and Gumbel copulas), different values of Kendall's tau coefficient τ ($\tau = 0, 0.25, 0.5$), different sample sizes n ($n = 100, 200, 400, 500$), and different values for the bandwidth $k = cn^{1/2}$ ($c = 0.5, 1, 1.5, 2$).*

Normal copula												
Test statistic δ_n												
	N = 100			N = 200			N = 400			N = 500		
c	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0$	$\tau = 0.25$	$\tau = 0.5$
0.5	0.07	0.91	1.00	0.05	1.00	1.00	0.05	1.00	1.00	0.06	1.00	1.00
1	0.03	0.93	1.00	0.03	1.00	1.00	0.06	1.00	1.00	0.06	1.00	1.00
1.5	0.04	0.93	1.00	0.03	1.00	1.00	0.06	1.00	1.00	0.05	1.00	1.00
2	0.04	0.82	1.00	0.04	0.99	1.00	0.07	1.00	1.00	0.05	1.00	1.00
Test statistic S_n												
	0.04	0.94	1.00	0.04	0.99	1.00	0.05	1.00	1.00	0.05	1.00	1.00
Student copula												
Test statistic δ_n												
0.5	0.08	0.93	1.00	0.06	0.99	1.00	0.09	1.00	1.00	0.07	1.00	1.00
1	0.09	0.93	1.00	0.05	1.00	1.00	0.23	1.00	1.00	0.25	1.00	1.00
1.5	0.10	0.92	1.00	0.08	1.00	1.00	0.32	1.00	1.00	0.40	1.00	1.00
2	0.13	0.82	1.00	0.15	0.98	1.00	0.32	1.00	1.00	0.38	1.00	1.00
Test statistic S_n												
	0.04	0.92	1.00	0.05	1.00	1.00	0.07	1.00	1.00	0.10	1.00	1.00
Clayton copula												
Test statistic δ_n												
0.5	0.06	0.94	1.00	0.05	1.00	1.00	0.05	1.00	1.00	0.04	1.00	1.00
1	0.03	0.96	1.00	0.03	1.00	1.00	0.05	1.00	1.00	0.05	1.00	1.00
1.5	0.04	0.94	1.00	0.04	1.00	1.00	0.04	1.00	1.00	0.06	1.00	1.00
2	0.05	0.88	1.00	0.04	0.99	1.00	0.04	1.00	1.00	0.06	1.00	1.00
Test statistic S_n												
	0.05	0.94	1.00	0.04	1.00	1.00	0.05	1.00	1.00	0.05	1.00	1.00
Gumbel copula												
Test statistic δ_n												
0.5	0.06	0.93	1.00	0.07	1.00	1.00	0.05	1.00	1.00	0.06	1.00	1.00
1	0.04	0.93	1.00	0.04	1.00	1.00	0.05	1.00	1.00	0.07	1.00	1.00
1.5	0.04	0.93	1.00	0.03	1.00	1.00	0.05	1.00	1.00	0.06	1.00	1.00
2	0.05	0.83	1.00	0.04	0.99	1.00	0.06	1.00	1.00	0.06	1.00	1.00
Test statistic S_n												
	0.03	0.93	1.00	0.05	1.00	1.00	0.05	1.00	1.00	0.05	1.00	1.00

Table 3: *This table compares the empirical size and power of the test statistics δ_n and S_n for different copulas (Normal, Student, Clayton and Gumbel copulas), different values of Kendall's tau coefficient τ ($\tau = 0, 0.25, 0.5$), different sample sizes n ($n = 100, 200, 400, 500$), and different values for the bandwidth $k = cn^{1/2}$ ($c = 0.5, 1, 1.5, 2$).*