ON DIFFERENTIAL SYSTEMS WITH STRONGLY INDEFINITE VARIATIONAL STRUCTURE

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ABSTRACT. We prove some multiplicity results for a class of first-order superquadratic Hamiltonian systems and a class of indefinite superquadratic elliptic systems which lead to the study of strongly indefinite functionals. The nonlinear terms are not assumed to satisfy the Ambrosetti-Rabinowitz superquadratic condition. To establish the existence of solutions we apply a version of the symmetric mountain pass theorem for strongly indefinite functionals, also established in this paper, which covers the situations where all the Palais-Smale sequences of the energy functional may be unbounded.

1. INTRODUCTION

This paper is concerned with the existence and the multiplicity of critical points of strongly indefinite even functionals which appear in the study of periodic solutions of Hamiltonian systems and of solutions of certain class of elliptic systems. We recall that a functional $\Phi : X \to \mathbb{R}$ defined on a Banach space is said to be strongly indefinite if it is neither bounded from above nor from below, even modulo subspaces of finite dimension or codimension. Its study then gives rise to an interesting and challenging variational problem, because the usual powerful critical point theorems in [1, 3, 8, 33] cannot be applied directly.

Critical Point Theory for strongly indefinite even functionals was studied in [4, 10, 12, 6, 13, 18, 21]. In his seminal paper [13], Benci introduced various index and pseudo-index theories whose definitions, however, depended on the topology of the sublevel sets of the functional, and he required a certain dimension property which severely restricts the class of problems for which his results apply. In [4, 18, 21], the Galerkin type approximations were used to reduce the study of strongly indefinite functionals to a semidefinite situation where the basic idea of Lusternik-Schnirelmann theory applies. However, in order to control the critical points of the reduced functional it is required in theses papers that the original functional satisfies a strong version of the usual compactness condition in Critical Point Theory. In the recent papers [10, 12], the first two authors of this paper generalized the well known fountain theorems of Bartsch and Willem to the case of strongly indefinite functionals. In contrast with [4, 18, 21], they used no reduction method and carried out the proofs directly in the infinite-dimensional setting.

The first goal of this paper is to extend the results in [10, 12] to more general situations for which the Palais-Smale sequences of the functionals may be unbounded. This will have a crucial importance in applications, since we can free ourselves from

1

the Palais-Smale type assumptions such as the Ambrosetti-Rabinowitz superlinear condition and its variations, which are extensively used in literature in the study of superlinear problems. We stress that in contrast with the common feature in the study of symmetric strongly indefinite functionals, our approach does not use any reduction method. The main ingredient is a weak-strong topology introduced by Kryszewski and Szulkin in [20]. Our critical point theorems are stated in section 2.

In section 3, we consider the applications to the problem of finding infinitely many large energy periodic solutions of the following first-order Hamiltonian system

$$\begin{cases} \partial_t u - \Delta_x u + V(x)u = H_v(t, u, v) & \text{in } \mathbb{R} \times \Omega, \\ -\partial_t v - \Delta_x v + V(x)v = H_u(t, u, v) & \text{in } \mathbb{R} \times \Omega, \end{cases} \tag{HS}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geqslant 1$) is a bounded smooth domain, and H is a superquadratic C^1-function which is T-periodic ($T > 0$) with respect to the t-variable. By periodic solution, we mean a solution $z = (u, v) : \mathbb{R} \times \Omega \to \mathbb{R}^{2M}$ of (HS) satisfying the conditions

$$\begin{aligned} z(t, x) &= z(t + T, x) & \forall (t, x) \in \mathbb{R} \times \Omega \\ z(t, x) &= 0 & \forall (t, x) \in \mathbb{R} \times \Omega. \end{aligned}$$

Setting

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \mathcal{J}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad z = (u, v),$$

$$A = \mathcal{J}_0(-\Delta_x + V), \text{ and } \mathcal{H} = -\frac{1}{2}(Az, z) + H(t, x, z),$$

system (HS) simply reads

$$\mathcal{J} \partial_t z = \mathcal{H}_z, \quad (t, x) \in \mathbb{R} \times \Omega,$$

which is the form of unbounded Hamiltonian systems or infinite-dimensional Hamiltonian systems in $L^2(\Omega, \mathbb{R}^{2M})$. This kind of systems were studied under various assumptions by Brezis and Nirenberg [14], and by Clément et al. [16].

Usually, in the study of superquadratic Hamiltonian systems, the nonlinear terms satisfy the following condition introduced by Ambrosetti and Rabinowitz [1]

$$\exists \mu > 2, R > 0 \quad ; \quad 0 < \mu H(t, z) \leqslant H_z(t, z)z \text{ for } |z| \geqslant R, \tag{AR}$$

see, for instance, [2, 5, 7, 29] and references therein. It is well known that this condition is mainly used to verify the boundedness of the energy functional and without it the problem then becomes very complicated. Moreover, without (AR) it may happen that all the Palais-Smale sequences of the functional are unbounded.

Let

$$\widetilde{H}(t, u, v) := \frac{1}{2}H_z(t, x, z) \cdot z - H(t, x, z), \quad S^1 := \mathbb{R}/(T\mathbb{Z}), \text{ and } \Theta = S^1 \times \Omega.$$

Our assumptions on the potential V are the following:

(V_1) $V \in C(\overline{\Omega}, \mathbb{R})$.
(V_2) $0 \notin \sigma(S)$, $S = -\Delta_x + V$.

For the Hamiltonian H we make the following assumptions:

(H_1) $H \in C^1(\Theta \times \mathbb{R}^{2M}, \mathbb{R})$ is T-periodic with respect to t.
(H_2) $H(t, x, 0) = 0$ and $|H_z(t, x, z)| = \circ(|z|)$ as $z \to 0$, uniformly in (t, x).
(H_3) $H(t, x, z) > 0$ for $z \neq 0$ and $\frac{H(t, x, z)}{|z|^2} \to \infty$ as $|z| \to \infty$, uniformly in (t, x).
(H_4) $\exists R > 0$ such that

(1) $\widetilde{H}(t,x,z) \geqslant a_1|z|^2$ if $|z| \geqslant R$,

(2) $\left(\frac{|H_z(t,x,z)|}{|z|}\right)^\sigma \leqslant a_2\widetilde{H}(t,x,z)$ if $|z| \geqslant R$,

where $a_1, a_2 > 0$, $\sigma > 1$ if $N = 1$ and $\sigma > \frac{N}{2} + 1$ if $N \geqslant 2$.

Under these assumptions, Mao et al. [26] obtained a nontrivial periodic solution for (HS) by using a local linking theorem. We will prove here that in case where the Hamiltonian is even with respect to z, the number of periodic solutions is in fact infinite. This fact has already been observed by Bartsch and Ding in [5] where they studied (HS) in both the cases where Ω is bounded, or $\Omega = \mathbb{R}^N$. However, they used condition (AR) to verify the boundedness of the Palais-Smale sequences of the energy functional associated to (HS), which was crucial for their argument. Moreover, one does not know whether the solutions they obtained there are large energy solutions. We would like to emphasize that in our situation we do not know whether the Palais-Smale sequences of that energy functional are bounded.

Our result reads as follows:

Theorem 1.1. *Assume that* (V_1), (V_2), *and* (H_1)-(H_4) *are satisfied. If in addition* H *in even in* z, *then* (HS) *has infinitely many pairs* $\pm z_k$ *of* T-*periodic solutions such that* $\|z_k\|_\infty \to \infty$ *as* $k \to \infty$.

Finally, as another application of our abstract result, we consider in section 4 the following elliptic system of Hamiltonian type

$$\begin{cases} -\Delta u = g(x,v) \text{ in } \Omega, \\ -\Delta v = f(x,u) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{ES}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$. The solutions of this problem describe steady states of some reaction-diffusion systems which are derived from several applications, such as mathematical biology or chemical reactions.

We study this problem under the following assumptions:

(E_1) $f,g \in \mathcal{C}(\Omega \times \mathbb{R})$ and there is a constant $C > 0$ such that

$$|f(x,u)| \leqslant C(1 + |u|^{p-1}) \text{ and } |g(x,u)| \leqslant C(1 + |u|^{q-1}) \quad \forall(x,u),$$

where $p, q > 2$ satisfy $\dfrac{1}{p} + \dfrac{1}{q} > 1 - \dfrac{2}{N}$.

Furthermore, in case $N \geqslant 5$ we impose

$$\frac{1}{p} > \frac{1}{2} - \frac{2}{N} \quad \text{and} \quad \frac{1}{q} > \frac{1}{2} - \frac{2}{N}.$$

(E_2) $f(x,u) = \circ(u)$ and $g(x,u) = \circ(u)$ as $u \to 0$, uniformly in x.

(E_3) $F(x,u)/u^2 \to \infty$ and $G(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$.

(E_4) $u \mapsto f(x,u)/|u|$ and $u \mapsto g(x,u)/|u|$ are increasing in $(-\infty,0) \cup (0,+\infty)$.

(E_5) $f(x,-u) = -f(x,u)$ and $g(x,-u) = -g(x,u)$ for all (x,u).

Before we state our result on this problem, we recall the following definition.

Definition 1.2. *We say that (u, v) is a strong solution of* (ES) *if* $u \in W^{2,p/(p-1)}(\Omega) \cap W_0^{1,p/(p-1)}(\Omega)$, $v \in W^{2,q/(q-1)}(\Omega) \cap W_0^{1,q/(q-1)}(\Omega)$ *and* (u, v) *satisfies*

$$\begin{cases} -\Delta u = g(x, v) & \text{a.e. in } \Omega, \\ -\Delta v = f(x, u) & \text{a.e. in } \Omega. \end{cases}$$

We will also prove the following

Theorem 1.3. *Under assumptions* (E_1)-(E_5), (ES) *has infinitely many pairs of strong solutions* $\pm(u_k, v_k)$ *such that* $\|(u_k, v_k)\| \to \infty$ *as* $k \to \infty$, *where* $\|\cdot\|$ *represents the norm in the space*

$$W^{2,p/(p-1)}(\Omega) \cap W_0^{1,p/(p-1)}(\Omega) \times W^{2,q/(q-1)}(\Omega) \cap W_0^{1,q/(q-1)}(\Omega).$$

System (ES) has been already studied from the variational point of view by many authors. Hulshof and van der Vorst [19], and de Figueiredo and Felmer [17] obtained positive solutions by requiring among others that the Hamiltonian $H(u, v) = F(u) + G(v)$ satisfies (AR). In [18], Felmer and Wang considered the case where the full superquadratic range is not reached, and by using a variation of (AR) they obtained infinitely many solutions in case where H is even in (u, v) by using the Galerkin approximation. Recently, Szulkin and Weth [31] improved the above results by reducing the energy functional to the Nehari-Pankov manifold. To circumvent the difficulty that the Nehari-Pankov manifold is not necessary of class C^1, they required the maps in assumption (E_4) to be strictly increasing. In [9], the first author of this paper showed that it is sufficient that these maps are only increasing. He applied the generalized fountain theorem obtained in [11] to a family of perturbed functionals. However, working with a family of modified functionals makes things unnecessary complicated. We considerably simplify this approach in the present paper.

Throughout the paper we denote by $|\cdot|_r$ the norm of the Lebesgue space L^r.

2. CRITICAL POINT THEOREMS FOR STRONGLY INDEFINITE FUNCTIONALS

Let X be a separable Hilbert space which admits an orthogonal decomposition $X = X^- \oplus X^+$, where X^- is closed and $X^+ = (X^-)^\perp$. We denote $\langle , \rangle$ the inner product of X.

Let $(a_j)_{j \geqslant 0}$ be an orthonormal basis of X^-. We define on X a new norm by setting

$$\|u\| := \max \left(\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |\langle P^- u, a_j \rangle|, \|P^+ u\| \right), \ u \in X,$$

where $P^\pm : X \to X^\pm$ are orthogonal projections, and we denote by τ the topology generated by this norm. This topology was introduced by Kryszewski and Szulkin in [20] and is related to the topology on X which is strong on X^+ and weak on X^-. More precisely, if (u_n) is a bounded sequence in X then

$$u_n \xrightarrow{\tau} u \iff P^- u_n \rightharpoonup P^- u \text{ and } P^+ u_n \to P^+ u.$$

Now we recall some standard notations in Critical Point Theory:
Let $\Phi : X \to \mathbb{R}$, $a, b \in \mathbb{R}$, $S \subset X$, and $\alpha > 0$. We denote

$$dist(u, S) := \|u - S\|, \quad dist_\tau(u, S) := \|u - S\|, \quad S_\alpha := \{u \in X \ ; \ dist(u, S) \leqslant \alpha\},$$

$$\Phi^b := \{u \in X \ ; \ \Phi(u) \leqslant a\}, \quad \Phi_a := \{u \in X \ ; \Phi(u) \geqslant a\}, \text{ and } \Phi_a^b := \Phi_a \cap \Phi^b.$$

We say that a functional $\Phi \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$ if the following holds:

$(PS)_c$ Any sequence $(u_n) \subset X$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ $\big($Palais-Smale sequence at level $c\big)$ possesses a convergent subsequence.

This is the famous compactness condition in Critical Point Theory introduced by Palais and Smale [27]. Since we would like to consider some situations where this condition does not hold, we will use its following generalization due to Cerami [15]:

$(Ce)_c$ Any sequence $(u_n) \subset X$ such that $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ $\big($Cerami sequence at level $c\big)$ possesses a convergent subsequence.

If $(Ce)_c$ holds we say that the functional Φ satisfies the Cerami condition at the level c $\big((Ce)_c$-condition for short$\big)$.

We now consider the class of C^1 functionals $\Phi : X \to \mathbb{R}$ which satisfy:

(KS) $\Phi(u) = \frac{1}{2}\|P^+ u\|^2 - \frac{1}{2}\|P^- u\|^2 - \Psi(u)$, where $\Psi \in C^1(X, \mathbb{R})$ is bounded below, weakly sequentially lower semicontinuous, and Ψ' is weakly sequentially continuous.

The following deformation lemma will play a key role in the proof of our abstract results.

Lemma 2.1 (Deformation lemma). *Assume that Φ satisfies (KS). Let $S \subset X$ $\big($with $-S = S$ if Φ is even$\big)$, $d \geqslant b$ and $\varepsilon, \theta > 0$ such that*

$$\forall u \in \Phi^{-1}([b - 2\varepsilon, d + 2\varepsilon]) \cap S_{2\theta}, \ \big(1 + \|u\|\big)\|\Phi'(u)\| \geqslant \frac{8\varepsilon}{\theta}. \tag{2.1}$$

Then there exists $\eta \in C([0, 1] \times \Phi^{d+2\epsilon}, X)$ such that:

(i) $\eta(t, u) = u$ *if $t = 0$ or if $u \notin \Phi^{-1}([b - 2\varepsilon, d + 2\varepsilon]) \cap S_{2\theta}$,*
(ii) $\eta(1, \Phi^{d+\varepsilon} \cap S) \subset \Phi^{b-\varepsilon}$,
(iii) $\Phi(\eta(\cdot, u))$ *is non increasing, $\forall u \in \Phi^{d+2\varepsilon}$,*
(iv) *each point $(t, u) \in [0, 1] \times \Phi^{d+2\varepsilon}$ has a τ-neighborhood $N_{(t,u)}$ such that $\big\{v - \eta(s, v) \,\big|\, (s, v) \in N_{(t,u)} \cap ([0, 1] \times \Phi^{d+2\varepsilon})\big\}$ is contained in a finite-dimensional subspace of X,*
(v) η *is τ-continuous,*
(vi) *if Φ is even then $\eta(t, \cdot)$ is odd $\forall t \in [0, 1]$.*

Proof. We define the vector field

$$w(u) := 2\nabla\Phi(u)/\|\Phi'(u)\|^2, \quad u \in \Phi^{-1}\big([b - 2\varepsilon, d + 2\varepsilon]\big).$$

By assumption (KS) we know that Φ' is weakly sequentially semicontinuous, and this implies that the function

$$v \in \Phi_{b-2\varepsilon}^{d+2\varepsilon} \mapsto \big\langle \Phi'(v), w(u)\big\rangle \in \mathbb{R}$$

is τ-continuous. Hence every $u \in \Phi_{b-2\varepsilon}^{d+2\varepsilon}$ has a τ-neighborhood N_u such that

$$\big\langle \Phi'(v), w(u)\big\rangle > 1 \quad \forall v \in N_u, \tag{2.2}$$

$$\|u\| < 2\|v\| \quad \forall v \in N_u, \tag{2.3}$$

where (2.3) holds because the set $\big\{z \in X \,;\, \|z\| \leqslant \alpha\big\}$ is τ-closed for any $\alpha \geqslant 0$. Now since (KS) implies that Φ is τ-upper semicontinuous, the set $\tilde{N} := \Phi^{-1}(\,] - \infty, b - 2\varepsilon[)$ is τ-open. It follows that $\mathcal{N} := \tilde{N} \cup \big\{N_v \,;\, b - 2\varepsilon \leqslant \Phi(v) \leqslant d + 2\varepsilon\big\}$ is a τ-open covering of the metric space $\big(\Phi^{d+2\varepsilon}, \tau\big)$. We can extract a τ-locally finite

τ-open covering $\mathcal{M} := \{M_i\,;\, i \in I\}$ of $\Phi^{c+2\epsilon}$ finer than $\mathcal{N}$.
Let
$$V := \bigcup_{i \in I} M_i.$$

For every $i \in I$ we have either $M_i \subset N_v$ for some v or $M_i \subset \tilde{N}$. In the first case we define $v_i := w(v)$ and in the second case $v_i := 0$. Consider a τ-Lipschitz continuous partition of unity $\{\lambda_i\,;\, i \in I\}$ subordinated to $\mathcal{M}$ and define on V the vector field
$$h(u) := \sum_{i \in I} \lambda_i(u)v_i.$$

Let us define
$$A := \Phi^{-1}\big([b - 2\epsilon, d + 2\epsilon]\big) \cap S_{2\delta}, \quad B =: \Phi^{-1}\big([b - \epsilon, d + \epsilon]\big),$$

$$\psi(u) := dist_\tau\big(u, V\backslash A\big)\Big[dist_\tau\big(u, V\backslash A\big) + dist_\tau\big(u, B\big)\Big]^{-1} \quad \text{on} \quad V,$$

and
$$f(u) := -\psi(u)h(u), \quad u \in V.$$

Clearly, the vector field f is locally Lipschitz continuous and τ-locally Lipschitz continuous. By using (2.1), (2.3) and the definition of the vector field h we see that
$$\|f(u)\| \leqslant \frac{\theta}{4\varepsilon}\big(1 + 2\|u\|\big) \quad \forall u \in V.$$

It follows from Corollary 7.6 in [30] that the problem
$$\begin{cases} \frac{d}{dt}\sigma(t, u) = f(\sigma(t, u)) \\ \sigma(0, u) = u \in \Phi^{d+2\varepsilon} \end{cases}$$

has a unique solution $\sigma(\cdot, u)$ defined on $\mathbb{R}^+$.
We define $\eta : [0, 1] \times \Phi^{d+2\varepsilon}$ by setting
$$\eta(t, u) := \sigma\big((d - b + 2\varepsilon)t, u\big).$$

An argument similar to that in the proof of Lemma 8 in [10] shows that η satisfies (i)-(v). If Φ is even, then we replace $h(u)$ by $\frac{1}{2}\big(h(u) - h(-u)\big)$ in such a way that f is odd. (vi) is then a consequence of the existence and the uniqueness of the solution of the above Cauchy problem. $\qquad\square$

Now we can state our first critical point theorem, which extends the generalized saddle point theorem in [24] to the case where the Palais-Smale sequences of Φ may be unbounded.

Theorem 2.2 (Saddle point theorem). *Assume that Φ satisfies (KS). If there exists $R > 0$ such that*

$(A_0) \qquad b := \inf_{u \in X^+} \Phi(u) > \sup_{u \in X^-, \|u\| = R} \Phi(u) \quad and \quad d := \sup_{u \in X^-, \|u\| \leqslant R} \Phi(u) < \infty,$

then for some $c \in [b, d]$, there is a sequence $(u_n) \subset X$ such that
$$\Phi(u_n) \to c \quad and \quad \big(1 + \|u_n\|\big)\Phi'(u_n) \to 0 \text{ as } n \to \infty.$$

Moreover, if Φ satisfies the $(Ce)_\alpha$ condition for all $\alpha \in [b, d]$, then Φ has a critical value in $[b, d]$.

The proof follows the lines of [24].

Proof. We assume by contradiction that for every $c \in [b, d]$ there is no $(Ce)_c$ sequence for Φ. Then there exists $\varepsilon > 0$ such that

$$u \in \Phi^{-1}\big([b - 2\varepsilon, d + 2\varepsilon]\big) \Rightarrow \big(1 + \|u\|\big)\|\Phi'(u)\| \geqslant \varepsilon.$$

Let

$$M := \big\{u \in X^- \, ; \, \|u\| = R\big\}.$$

We apply Lemma 2.1 with $S = X$ and $\theta = 8$, and we define $\mu : [0, 1] \times M \to X$ by setting $\mu(t, u) := P^- \eta(t, u)$, where η is given by Lemma 2.1. Clearly, (iv) and (v) of Lemma 2.1 imply that μ is τ-continuous and each $(t, u) \in [0, 1] \times \Phi^{d+2\varepsilon}$ has a τ-neighborhood $N_{(t,u)}$ such that $\big\{v - \mu(s, v) \, \big| \, (s, v) \in N_{(t,u)} \cap ([0, 1] \times \Phi^{d+2\varepsilon})\big\}$ is contained in a finite-dimensional subspace of X.

We claim that $0 \notin \mu\big([0, 1] \times \partial M\big)$. Indeed, if there exists $(t_0, u_0) \in [0, 1] \times \partial M$ such that $\mu(t_0, u_0) = 0$, then $\eta(t_0, u_0) \in X^+$ and by (iii) of Lemma 2.1 and assumption (A_0) we have

$$\Phi(u_0) = \Phi\big(\eta(0, u_0)\big) \geqslant \Phi\big(\eta(t_0, u_0)\big) \geqslant b > \sup_{\partial M} \Phi,$$

which contradicts the fact that $u_0 \in \partial M$.

μ is then an admissible homotopy (in the sense of Kryszewski and Szulkin [20]) such that $0 \notin \mu\big([0, 1] \times \partial M\big)$. It follows from Theorem 2.4-(iii) of [20] that the Kryszewski-Szulkin's degree (see [20]) $deg_{KS}\big(\mu(t, \cdot), int(M), 0\big)$ is well defined and does not depend on $t \in [0, 1]$. Hence

$$deg_{KS}\big(\mu(1, \cdot), int(M), 0\big) = deg_{KS}\big(\mu(0, \cdot), int(M), 0\big) = deg_{KS}\big(id, int(M), 0\big).$$

It follows from Theorem 2.4-(i) of [20] that there is $u \in int(M)$ such that $\mu(1, u) = 0$, which implies that $\eta(1, u) \in X^+$. By the definition of b we have $b \leqslant \Phi\big(\eta(1, u)\big)$. But (ii) of Lemma 2.1 implies, since $M \subset \Phi^{d+\varepsilon}$, that $\Phi\big(\eta(1, u)\big) \leqslant b - \varepsilon$. This gives a contradiction. $\qquad\square$

In order to obtain a multiplicity result we need to introduce some notations. We consider an orthonormal basis $(e_j)_{j \geqslant 0}$ of X^+ and we set

$$X_k^- := X^- \oplus \big(\oplus_{j=0}^k \mathbb{R}e_j\big), \quad X_k^+ := \overline{\oplus_{j=k}^\infty \mathbb{R}e_j}, \text{ and } B_k^-(\rho) := \big\{u \in X_k^- \mid \|u\| \leqslant \rho\big\}.$$

Theorem 2.3 (Fountain theorem). *Assume that Φ satisfies (KS), that Φ is even, and that there exist $\rho_k > r_k > 0$ such that:*

(A_1)
$$a_k := \sup_{u \in X_k^-; \|u\| = \rho_k} \Phi(u) \leqslant 0, \quad d_k := \sup_{u \in X_k^-; \|u\| \leqslant \rho_k} \Phi(u) < \infty.$$

(A_2)
$$b_k := \inf_{u \in X_k^+; \|u\| = r_k} \Phi(u) \to \infty, \quad k \to \infty.$$

Then, there exist a sequence $(u_k^n)_n \subset X$ and a number $c_k \geqslant b_k$ such that

$$\Phi(u_k^n) \to c_k \quad \text{and} \quad \big(1 + \|u_k^n\|\big)\Phi'(u_k^n) \to 0 \text{ as } n \to \infty.$$

Moreover, if Φ satisfies the $(Ce)_c$ condition for any $c > 0$, then Φ has an unbounded sequence of critical values.

Theorem 2.3 generalized Theorem 12 in [10] where the sequence (u_k^n) was a Palais-Smale sequence.

Proof. Let Γ_k be the set of maps $\gamma : B_k^-(\rho_k) \to X$ such that:

(a) γ is odd, τ-continuous and $\gamma|_{\partial B_k^-(\rho_k)} = id$,

(b) each $u \in int(B^- k(\rho_k))$ has a τ-neighborhood $\mathcal{N}_u$ in X_k^- such that $(id - \gamma)\big(\mathcal{N}_u \cap int(B_k^-(\rho_k))\big)$ is contained in a finite dimensional subspace of X,

(c) $\Phi(\gamma(u)) \leqslant \Phi(u) \ \forall u \in B_k^-(\rho_k)$.

(A_1) and (A_2) imply that there is k_0 big enough such that $b_k > a_k$ for every $k \geqslant k_0$. We define for $k \geqslant k_0$

$$c_k := \inf_{\gamma \in \Gamma_k} \sup_{u \in B_k^-(\rho_k)} \Phi\big(\gamma(u)\big).$$

It follows from Lemma 10 in [10] that if $\gamma \in \Gamma_k$ then

$$\gamma\big(B_k^-(\rho_k)\big) \cap \big\{u \in X_k^+ \,;\, \|u\| = r_k\big\} \neq \varnothing,$$

which implies that $c_k \geqslant b_k$.

We assume by contradiction that there is no Cerami sequence of Φ at level c_k. Then there exist $\varepsilon \in (0, \frac{c_k - a_k}{2})$ and $\theta > 0$ such that

$$u \in \Phi^{-1}([c_k - 2\varepsilon, c_k + 2\varepsilon]) \cap \gamma\big(B_k^-(\rho_k)\big)_{2\theta} \quad \Rightarrow \quad \big(1 + \|u\|\big)\|\Phi'(u)\| \geqslant \frac{8\varepsilon}{\theta},$$

where $\gamma \in \Gamma_k$ is such that

$$\sup_{B_k^-(\rho_k)} \Phi \circ \gamma \leqslant c_k + \varepsilon. \tag{2.4}$$

We apply Lemma 2.1 with $b = d = c_k$ and $S = \gamma\big(B_k^-(\rho_k)\big)$. We assume that

$$c_k - 2\epsilon > a_k. \tag{2.5}$$

We define on $B_k^-(\rho_k)$

$$\beta(u) := \eta(1, \gamma(u)).$$

It follows from $(i), (ii), (iv), (v), (vi)$ of Lemma 2.1 and (2.5) that $\beta \in \Gamma_k$.
Now by using (2.4) and (ii) of Lemma 2.1 we obtain

$$\sup_{\substack{u \in X_k^- \\ \|u\| = \rho_k}} \Phi(\beta(u)) = \sup_{\substack{u \in X_k^- \\ \|u\| = \rho_k}} \Phi(\eta(1, \gamma(u))) \leqslant c_k - \varepsilon$$

giving a contradiction with the definition of c_k. It then follows that Φ has a Cerami sequence at level c_k. $\qquad \square$

Using the same argument as in the proof of Theorem 6 in [12], we can prove the following dual version of Theorem 2.3.

Theorem 2.4 (Dual fountain theorem). *Let $\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \Psi(u)$, where $\Psi \in \mathcal{C}^1(X, \mathbb{R})$ is even, bounded below, and weakly sequentially lower semicontinuous, and Ψ' is weakly sequentially continuous.*
If $\forall k \geqslant k_0, \exists \rho_k > r_k > 0$ such that

$(B_1) \quad a^k := \inf\limits_{u \in X_k^+, \|u\| = \rho_k} \Phi(u) \geqslant 0, \quad b^k := \sup\limits_{u \in X_k^-, \|u\| = r_k} \Phi(u) < 0,$

$(B_2) \quad d^k := \inf\limits_{u \in X_k^+, \|u\| \leqslant \rho_k} \Phi(u) \to 0, \ k \to \infty,$

Then there exist $c^k \in [d^k, 0[$ and a sequence $(u_k^n) \subset X$ such that

$$\Phi(u_k^n) \to c_k \quad and \quad \big(1 + \|u_n\|\big)\Phi'(u_k^n) \to 0, \quad as \ n \to \infty.$$

Moreover, if Φ satisfies the $(Ce)_c$ condition for any $c \in [d^{k_0}, 0[$, then Φ has a sequence (u_k) of critical points such that $\Phi(u_k) \to 0^-$ as $k \to \infty$.

3. Periodic solutions of superquadratic Hamiltonian systems

In this section, we apply our multiplicity result above to the Hamiltonian system

$$\begin{cases} \partial_t u - \Delta_x u = H_v(t, u, v) & \text{in } \mathbb{R} \times \Omega, \\ -\partial_t v - \Delta_x v = H_u(t, u, v) & \text{in } \mathbb{R} \times \Omega. \end{cases} \tag{HS}$$

We assume here that assumptions (V_1), (V_2), and (H_1)-(H_4) are satisfied, and that H is even in z.

By (V_1) the operator $S = -\Delta_x + V$ acting on $L^2(\Omega, \mathbb{R}^{2M})$ is self-adjoint with domain $D(S) = H^2(\Omega, \mathbb{R}^{2M}) \cap H_0^1(\Omega, \mathbb{R}^{2M})$.

Let $L := \mathcal{J}\partial_t + A$. Observing that he operators A and $\mathcal{J}A$ acting on $L^2(\Omega, \mathbb{R}^{2M})$ with domains $D(A) = D(\mathcal{J}A) = H^2(\Omega, \mathbb{R}^{2M}) \cap H_0^1(\Omega, \mathbb{R}^{2M})$ are self-adjoint, then the operator L acting on $L^2(S^1, L^2(\Omega, \mathbb{R}^{2M}))$ is also self-adjoint. We know by [5] that

$$L^2(S^1, L^2(\Omega, \mathbb{R}^{2M})) \cong L^2(S^1 \times \Omega, \mathbb{R}^{2M}), \quad 0 \notin \sigma(A) \cup \sigma(\mathcal{J}A), \quad 0 \notin \sigma(L),$$

so there is an orthogonal decomposition

$$L^2(\Omega, \mathbb{R}^{2M}) = E^+ \oplus E^-,$$

such that L is negative on E^- and positive on E^+.

The space $X = D(|L|^{\frac{1}{2}})$ is a Hilbert space with the inner product

$$\langle w, w \rangle = \langle |L|^{\frac{1}{2}} w, |L|^{\frac{1}{2}} w \rangle_{L^2}$$

and norm $\|z\|^2 = \langle z, z \rangle$.

Moreover, we have an orthogonal decomposition

$$X = X^- \oplus X^+ \text{ with } X^\pm = X \cap E^\pm.$$

We have the following

Lemma 3.1 ([5], Lemma 4.6). *The embedding of X in $L^r(S^1 \times \Omega, \mathbb{R}^{2M})$ is compact for $r \geqslant 2$ if $N = 1$, and for $r \in [2, 2(N+2)/2)$ if $N \geqslant 2$.*

Define the functional $\Phi : X \to \mathbb{R}$,

$$\Phi(z) = \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \Psi(z), \text{ where } \Psi(z) = \int_\Theta H(t, x, z). \tag{3.1}$$

Critical points of Φ are weak solutions of (HS).

Lemma 3.2. (1) $\Phi \in \mathcal{C}^1(X, \mathbb{R})$.
(2) Ψ *is bounded below and is weakly sequentially lower semicontinuous.*
(3) Ψ' *is weakly sequentially continuous.*

Proof. (1) By (H_4)-(2) we have for $|z| \geqslant R$ and for $(t, x) \in \Theta$

$$|H_z(t, x, z)|^\sigma \leqslant a_2 \widetilde{H}(t, x, z)|z|^\sigma$$

$$\leqslant a_2 \left(\frac{1}{2} H_z(t, x, z)z - H(t, x, z) \right) |z|^\sigma$$

$$\leqslant \frac{a_2}{2}|H_z(t, x, z)||z|^{\sigma+1}.$$

Hence

$$|H_z(t, x, z)| \leqslant a_3|z|^{\frac{\sigma+1}{\sigma-1}} \quad \forall |z| \geqslant R, \quad \forall (t, x) \in \Theta,$$

where $a_3 > 0$ is constant.

On the other hand, since H_z is continuous, we have $|H_z(t, x, z)| \leqslant a_4$ for $|z| \leqslant R$ and $(t, x) \in \Theta$. We deduce that

$$|H_z(t, x, z)| \leqslant a_4 + a_3 |z|^{\frac{\sigma+1}{\sigma-1}} \quad \forall z \in \mathbb{R}^{2M}, \quad \forall (t, x) \in \Theta. \tag{3.2}$$

Let $\varepsilon > 0$. (H_2) implies that there is $\delta > 0$ such that

$$|H_z(t, x, z)| \leqslant \varepsilon |z| \quad \text{pour tout } |z| < \delta.$$

If $|z| \geqslant \delta$, then we have in view of (3.2)

$$|H_z(t, x, z)| \leqslant a_4 + a_3 |z|^{\frac{\sigma+1}{\sigma-1}} \leqslant a_4 \frac{|z|^p}{\delta^p} + a_3 |z|^p = \left(\frac{a_4}{\delta^p} + a_3\right)|z|^p \quad \left(\text{with } p = \frac{\sigma+1}{\sigma-1}\right).$$

Hence

$$\forall \varepsilon > 0, \ \exists C(\varepsilon) > 0 \ ; \ |H_z(t, x, z)| \leqslant \varepsilon |z| + C(\varepsilon)|z|^p \quad \forall z \in \mathbb{R}^{2M}. \tag{3.3}$$

Now one can easily verify that

$$\left\{ \begin{array}{l} \sigma > 1 \text{ if } N = 1 \\ \sigma > 1 + \frac{N}{2} \text{ if } N \geqslant 2 \end{array} \right. \implies \left\{ \begin{array}{l} p + 1 > 2 \text{ si } N = 1 \\ 2 < p + 1 < 2(N + 2)/N) \text{ if } N \geqslant 2. \end{array} \right.$$

It then follows from Lemma 3.1 that Ψ is well defined on X. We can now use a standard argument to show that Φ is of class C^1 on X and

$$\langle \Phi'(z), w \rangle = \langle z^+ - z^-, w \rangle - \int_\Theta w H_z(t, x, z), \quad \forall z, w \in X. \tag{3.4}$$

(2) Since $H \geqslant 0$, the functional Ψ is bounded below. Let $z_n \rightharpoonup z$ and $c \in \mathbb{R}$ such that $\Psi(z_n) \leqslant c$. By Lemma 3.1 we have $z_n \to z$ in L^2 and, up to a subsequence, $z_n(t, x) \to z(x, t)$ a.e. in Θ. Using Fatou's lemma we obtain $\Psi(z) \leqslant c$, which shows that ψ is weakly lower semicontinuous.

(3) Let $z_n \rightharpoonup z$. By Lemma 3.1 we have $z_n \to z_n$ in L^{p+1} $\left(\text{where } p = \frac{\sigma+1}{\sigma-1}\right)$ which implies, using (3.3) and Theorem $A.2$ in [32], that $|H(t, x, z_n) - H(t, x, z)|_{\frac{p+1}{p}} \to 0$. It is then easy to deduce, using Fatou's lemma, that Ψ' is weakly sequentially continuous.

$\square$

We now prove that the functional Φ satisfies the Cerami condition.

Lemma 3.3. *Let $(z_n) \subset X$ be such that*

$$d = \sup \Phi(z_n) < \infty \quad and \quad \left(1 + \|z_n\|\right)\Phi'(z_n) \to 0.$$

Then (z_n) possesses a convergent subsequence.

Proof. Let us first show that the sequence (z_n) is bounded.

Arguing by contradiction we suppose that (z_n) is unbounded. Then, up to a subsequence, we have $\|z_n\| \to \infty$.

Note that

$$\langle \Phi'(z_n), z_n^+ - z_n^- \rangle = \|z_n^+\|^2 + \|z_n^-\|^2 - \int_\Theta (z_n^+ - z_n^-) \cdot H_z(t, x, z_n)$$

$$= \|z_n\|^2 \left(1 - \int_\Theta \frac{(z_n^+ - z_n^-) \cdot H_z(t, x, z_n)}{\|z_n\|^2}\right).$$

Since

$$|\langle \Phi'(z_n), z_n^+ - z_n^- \rangle| \leqslant \|\Phi'(z_n)\| \|z_n^+ - z_n^-\| \leqslant 2\|z_n\| \|\Phi'(z_n)\| \to 0,$$

one necessary has

$$\int_\Theta \frac{(z_n^+ - z_n^-) \cdot H_z(t, x, z_n)}{\|z_n\|^2} \to 1. \tag{3.5}$$

Set $w_n = z_n/\|z_n\|$.

Assumption (H_4)-(1) implies that $\widetilde{H}(t, x, z) \geqslant a_1|z|^2 - c_1$ for all (t, x, z). We deduce that for n big enough one has

$$1 + d \geqslant \Phi(z_n) - \frac{1}{2}\langle \Phi'(z_n), z_n \rangle = \int_\Theta \widetilde{H}(t, x, z_n) \tag{3.6}$$

$$\geqslant a_2|z_n|^2 - c_1|\Theta|, \tag{3.7}$$

where $|\Theta|$ denotes the Lebesgue measure of Θ. (3.7) shows that $(|z_n|_2)$ is bounded. Hence

$$|w_n|_2 = \frac{|z_n|_2}{\|z_n\|} \to 0. \tag{3.8}$$

On the other hand we have

$$\left| \int_\Theta \frac{(z_n^+ - z_n^-) \cdot H_z(t, x, z_n)}{\|z_n\|^2} \right| \leqslant \int_\Theta \frac{|z_n^+ - z_n^-|}{\|z_n\|} \frac{|H_z(t, x, z_n|}{\|z_n\|}$$

$$= \int_\Theta |w_n^+ - w_n^-||w_n| \frac{|H_z(t, x, z_n|}{|z_n|}$$

$$\leqslant \left(\int_\Theta \left(\frac{|H_z(t, x, z_n|}{|z_n|}\right)^\sigma \right)^{1/\sigma} \left(\int_\Theta \left(|w_n^+ - w_n^-||w_n|\right)^{\sigma'} \right)^{1/\sigma'},$$

with $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$. Since

$$|w_n^+ - w_n^-||w_n| = |w_n^+ - w_n^-||w_n^+ + w_n^-| \leqslant (|w_n^+| + |w_n^-|)^2 \leqslant 2(|w_n^+|^2 + |w_n^-|^2),$$

we have

$$\int_\Theta \left(|w_n^+ - w_n^-||w_n|\right)^{\sigma'} \leqslant c_2\left(|w_n^+|_{2\sigma'}^{2\sigma'} + |w_n^-|_{2\sigma'}^{2\sigma'}\right).$$

By using the fact that $2\sigma' = p + 1 \in [2, 2(N + 2)/N]$, we have

$$|w_n^\pm|_{2\sigma'} \leqslant |w_n^\pm|_2^\alpha |w_n^\pm|_{2(N+2)/N}^{1-\alpha} \quad (\text{ interpolation inequality}),$$

where $\alpha \in [0, 1]$ is such that $\frac{\alpha}{2} + \frac{1-\alpha}{2(N+2)/N} = 1$. Since X embeds continuously in $L^{2(N+2)/N}$, we have $|w_n^\pm|_{2(N+2)/N} \leqslant c_3\|w_n^\pm\| \leqslant c_3$ (because $\|w_n^\pm\| \leqslant \|w_n\| = 1$).

Now, since the decomposition $X = X^- \oplus X^+$ is also orthogonal with respect to the L^2-norm, we have $|w_n^\pm|_2 \leqslant |w_n|_2$. Hence $|w_n^\pm|_{2\sigma'} \leqslant c_3|w_n|_2^\alpha$, which implies

$$\left| \int_\Theta \frac{(z_n^+ - z_n^-) \cdot H_z(t, x, z_n)}{\|z_n\|^2} \right| \leqslant c_4\left(\int_\Theta \left(\frac{|H_z(t, x, z_n|}{|z_n|}\right)^\sigma \right)^{1/\sigma} |w_n|_2^{2\alpha}. \tag{3.9}$$

Let $\varepsilon > 0$. By (H_2) there exists $\delta > 0$ such that $|H_z(t, x, z)| \leqslant \varepsilon|z|$ for $|z| < \delta$. Then

$$\int_\Theta \left(\frac{|H_z(t, x, z_n|}{|z_n|}\right)^\sigma = \int_{\Theta \cap \{|z_n| < \delta\}} \left(\frac{|H_z(t, x, z_n|}{|z_n|}\right)^\sigma + \int_{\Theta \cap \{\delta \leqslant |z_n| \leqslant R\}} \left(\frac{|H_z(t, x, z_n|}{|z_n|}\right)^\sigma$$

$$+ \int_{\Theta \cap \{|z_n| \geqslant R\}} \left(\frac{|H_z(t, x, z_n|}{|z_n|}\right)^\sigma$$

$$\leqslant |\Theta|\varepsilon^\sigma + \sup_{\Theta \cap \{\delta \leqslant |z_n| \leqslant R\}} \left(\frac{|H_z(t, x, z_n|}{|z_n|}\right)^\sigma |\Theta| + a_2 \int_{\Theta \cap \{|z_n| \geqslant R\}} \widetilde{H}(t, x, z_n),$$

where we have used (H_4)-(2) for the last term in the r.h.s. of the equality. We deduce from (3.6) that $\left(\int_{\Theta \cap \{|z_n| \geqslant R\}} \widetilde{H}(t,x,z_n)\right)$ is bounded. Hence $\left(\int_{\Theta} \left(\frac{|H_z(t,x,z_n)|}{|z_n|}\right)^\sigma\right)$ is bounded. It then follows from (3.9) and (3.8) that

$$\int_\Theta \frac{(z_n^+ - z_n^-) \cdot H_z(t,x,z_n)}{\|z_n\|^2} \to 0,$$

which contradicts (3.5). With this contradiction, we conclude that (z_n) is bounded.

We may now suppose that $z_n \rightharpoonup z$ in X. Since

$$\|z_n^\pm - z^\pm\|^2 = \pm\langle \Phi'(z_n) - \Phi'(z), z_n^\pm - z^\pm \rangle \pm \int_\Theta (z_n^\pm - z^\pm) \cdot H_z(t,x,z_n),$$

a standard argument using Lemma 3.1 shows that $z_n \to z$. □

Proof of Theorem 1.1. Let (e_i) be an orthonormal basis of X^+. We recall that

$$X_k^- = X^- \oplus \left(\oplus_{j=0}^k \mathbb{R}e_j\right) \text{ and } X_k^+ = \overline{\oplus_{j=k}^\infty \mathbb{R}e_j}.$$

Let $z \in X_k^-$. By (H_3) we have

$$\forall \delta > 0 \quad \exists c_\delta > 0 ; \ H(t,x,z) \geqslant \delta|z|^2 - c_\delta.$$

Then

$$\Phi(z) = \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \int_\Theta H(t,x,z) \leqslant \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \delta|z|_2^2 + c_\delta|\Theta|.$$

Since the decomposition $X = X^- \oplus X^+$ is orthogonal with respect to L^2-norm, we have $|z^+|_2 \leqslant |z|_2$. Since all norms are equivalent on $\oplus_{j=0}^k \mathbb{R}e_j$, there is $c_1 > 0$ such that $c_1\|z^+\|^2 \leqslant |z|_2^2$. Hence

$$\Phi(z) \leqslant \left(\frac{1}{2} - c_1\delta\right)\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 + c_1|\Theta|.$$

Let us choose δ such that $c_1\delta \geqslant 1$. Then we have

$$\Phi(z) \leqslant -\frac{1}{2}\|z\|^2 + c_\delta|\Theta| \to -\infty \text{ as } \|z\| \to \infty. \tag{3.10}$$

Let $z \in X_k^+$. Then for every $\varepsilon > 0$ we have

$$\Phi(z) = \frac{1}{2}\|z\|^2 - \int_\Theta H(t,x,z)$$

$$\geqslant \frac{1}{2}\|z\|^2 - \frac{\varepsilon}{2}|z|_2^2 - \frac{c(\varepsilon)}{p+1}|z|_{p+1}^{p+1} \text{ (by (3.3))}$$

$$\geqslant \frac{1}{2}\|z\|^2 - c_2\varepsilon|z|_2^2 - \frac{c(\varepsilon)}{p+1}|z|_{p+1}^{p+1} \text{ (because } X \hookrightarrow L^2)$$

$$\geqslant \left(\frac{1}{2} - c_2\varepsilon\right)\|z\|^2 - \frac{c(\varepsilon)}{p+1}\beta_k^{p+1}\|z\|^{p+1},$$

where $\beta_k = \sup_{\substack{w \in X_k^+ \\ \|w\|=1}} |w|_{p+1}$. By choosing $\varepsilon \leqslant \frac{1}{2c_2}$ we obtain

$$\Phi(z) \geqslant \frac{1}{2}\left(\frac{1}{2}\|u\|^2 - c_3\beta_k^{p+1}\|z\|^{p+1}\right).$$

Hence, since $\beta_k \to 0$ as $k \to \infty$ (see [32], Lemma 3.8), we have

$$\|z\| = r_k := \left(c_3(p+1)\beta_k^{p+1}\right)^{1/(1-p)} \implies \Phi(z) \geqslant \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p+1}\right)r_k^2 \to 0, \ k \to \infty. \tag{3.11}$$

(3.10) and (3.11) show that the conditions (A_1) and (A_2) of Theorem 2.3 are satisfied. In view of Lemmas 3.2 and 3.3, we can apply Theorem 2.3 and get the result. $\qquad\square$

4. SOLUTIONS OF SUPERQUADRATIC ELLIPTIC SYSTEMS

Now we consider the problem

$$\begin{cases} -\Delta u = g(x, v) \text{ in } \Omega, \\ -\Delta v = f(x, u) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{ES}$$

and we assume in the sequel that the assumptions (E_1)-(E_5) are satisfied.

Consider the Laplacian as the operator

$$-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega),$$

and let $(\varphi_j)_{j\geqslant 1}$ a corresponding system of orthogonal and $L^2(\Omega)$-normalized eigenfunctions, with eigenvalues $(\lambda_j)_{j\geqslant 1}$. Then, writing

$$u = \sum_{j=1}^{\infty} a_j \varphi_j, \quad \text{with } a_j = \int_\Omega u\varphi_j,$$

we set, for $0 \leqslant s \leqslant 2$

$$E^s := \left\{ u \in L^2(\Omega) \,\Big|\, \sum_{j=1}^{\infty} \lambda_j^s |a_j|^2 < \infty \right\}$$

and

$$A^s(u) := \sum_{j=1}^{\infty} \lambda_j^{s/2} a_j \varphi_j, \quad \forall u \in D(A^s) = E^s.$$

One can verify easily that A^s is an isomorphism onto $L^2(\Omega)$. We denote $A^{-s} := (A^s)^{-1}$. It is well known (see [22]) that the space E^s is a fractional Sobolev space with the inner product

$$\langle u, v \rangle_s = \int_\Omega A^s u A^s v.$$

Lemma 4.1 ([28])**.** E^s embeds continuously in $L^r(\Omega)$ for $s > 0$ and $r \geqslant 1$ satisfying $\frac{1}{r} \geqslant \frac{1}{2} - \frac{s}{N}$. Moreover, the embedding is compact in the case of strict inequality.

By assumption (E_1), there exist $s, t > 0$ such that $s + t = 2$ and

$$\frac{1}{p} > \frac{1}{2} - \frac{s}{N} \quad \text{and} \quad \frac{1}{q} > \frac{1}{2} - \frac{t}{N}. \tag{4.1}$$

We consider the functional

$$\Phi(u, v) := \int_\Omega A^s u A^t v - \Psi(u, v), \quad (u, v) \in E^s \times E^t.$$

where

$$\Psi(u, v) = \int_\Omega \Big(F(x, u) + G(x, v) \Big).$$

It follows from Lemma 4.1 that the inclusions $E^s \hookrightarrow L^p(\Omega)$ and $E^t \hookrightarrow L^q(\Omega)$ are continuous. This, together with the estimate

$$\left| \int_\Omega A^s u A^t v \right| \leqslant |A^s u|_2 |A^t u|_2 = \|u\|_s \|v\|_t,$$

imply that the functional Φ above is well defined on $E := E^s \times E^t$.
Now a standard argument shows that if assumption (E_1) holds, then the functional Φ is of class $\mathcal{C}^1$ on E and its critical points are weak solutions of (ES).
We recall that $(u, v) \in E^s \times E^t$ is a weak solution of (ES) if

$$\int_\Omega \left(A^s u A^t k + A^s h A^t v \right) - \int_\Omega \left(h f(x, u) + k g(x, v) \right) = 0, \quad \forall (h, k) \in E^s \times E^t.$$

The following regularity result is due to de Figueiredo and Felmer [17].

Lemma 4.2. *If $(u, v) \in E^s \times E^t$ is a weak solution of* (ES)*, then (u, v) is a strong solution of* (ES)*.*

We endow $E = E^s \times E^t$ with the inner product

$$\left\langle (u, v), (\phi, \varphi) \right\rangle_{s \times t} = \left\langle u, \phi \right\rangle_s + \left\langle v, \varphi \right\rangle_t, \quad (u, v), (\phi, \varphi) \in E,$$

and the associated norm $\| (u, v) \|_{s \times t}^2 = \left\langle (u, v), (u, v) \right\rangle_{s \times t}$.
In the following we assume without loss of generality that $s \geqslant t$.
One can easily verify that E has the orthogonal decomposition $\big($with respect to $\left\langle \cdot, \cdot \right\rangle_{s \times t}\big)$ $E = E^+ \oplus E^-$, where

$$E^+ := \left\{ (u, A^{s-t} u) \mid u \in E^s \right\} \quad \text{and} \quad E^- := \left\{ (u, -A^{s-t} u) \mid u \in E^s \right\}. \tag{4.2}$$

If we denote by $P^\pm : E \to E^\pm$ the orthogonal projections, then a direct calculation yields

$$P^\pm(u, v) = \frac{1}{2} \big(u \pm A^{t-s} v, v \pm A^{s-t} u \big), \quad \forall (u, v) \in E, \tag{4.3}$$

and

$$\Phi(u, v) = \frac{1}{2} \| P^+(u, v) \|_{s \times t}^2 - \frac{1}{2} \| P^-(u, v) \|_{s \times t}^2 - \int_\Omega \left(F(x, u) + G(x, v) \right). \tag{4.4}$$

Let us recall the following technical result, which will play a crucial role in the verification of the Cerami condition.

Lemma 4.3 ([23], Lemma 3.2)**.** *Let $x \in \mathbb{R}^N$, $u, v, s \in \mathbb{R}$ such that $s \geqslant -1$ and $su + v \neq 0$. If (E_2)-(E_4) are satisfied then*

$$f(x, u) \left[s(\frac{s}{2} u + (1 + s) v) \right] + F(x, u) - F(x, u + v) \leqslant 0,$$
$$g(x, u) \left[s(\frac{s}{2} u + (1 + s) v) \right] + G(x, u) - G(x, u + v) \leqslant 0.$$

Lemma 4.4. *Let $(z_n = (u_n, v_n)) \subset E$ and $c > 0$ such that*

$$\Phi(z_n) \to c \quad \text{and} \quad \big(1 + \| z_n \|_{s \times t} \big) \Phi'(z_n) \to 0.$$

Then (z_n) has a convergent subsequence.

Proof. Let us first show that (z_n) is bounded.
If (z_n) is unbounded, then we can assume that $\| z_n \|_{s \times t} \to \infty$. Set $w_n = z_n / \| z_n \|_{s \times t}$.
Then, up to a subsequence, we have $w_n \rightharpoonup w$ in E.
Since $F, G \geqslant 0$, we have for n big enough

$$c - 1 \leqslant \Phi(z_n) \leqslant \frac{1}{2} \| z_n^+ \|_{s \times t}^2 - \frac{1}{2} \| z_n^- \|_{s \times t}^2 = \frac{1}{2} \| z_n \|_{s \times t}^2 \big(\| w_n^+ \|_{s \times t}^2 - \| w_n^- \|_{s \times t}^2 \big).$$

Recall that $\|w_n^+\|_{s\times t}^2 + \|w_n^-\|_{s\times t}^2 = 1$. Hence $\|w_n^+\|_{s\times t}^2 \geqslant \frac{1}{2} + \frac{c-1}{\|z_n\|_{s\times t}^2}$. But since $\|z_n\|_{s\times t}^2 \to \infty$, we have for n big enough $\frac{|c-1|}{\|z_n\|_{s\times t}^2} \leqslant \frac{1}{6}$. It follows that $\|w_n^+\|_{s\times t}^2 \geqslant \frac{1}{3}$. Now fix $r > 0$ and set $s_n = \frac{r}{\|z_n\|_{s\times t}} - 1$ and $y_n = -\frac{r}{\|z_n\|_{s\times t}} z_n^- = -rw_n^-$. Then $s_n \geqslant -1$, $y_n \in E^-$ and the sequences (s_n) and (y_n) are bounded.
By setting $y_n = (a_n, b_n)$, where $a_n \in E^s$ and $b_n \in E^t$, we have

$$\left\langle \Phi'(z_n), s_n(\frac{s_n}{2}+1)z_n + (1+s_n)y_n \right\rangle = \left\langle z_n^+ - z_n^-, s_n(\frac{s_n}{2}+1)z_n + (1+s_n)y_n \right\rangle$$
$$- \int_\Omega \left(s_n(\frac{s_n}{2}+1)u_n + (1+s_n)a_n \right) f(x, u_n)$$
$$- \int_\Omega \left(s_n(\frac{s_n}{2}+1)v_n + (1+s_n)b_n \right) g(x, v_n).$$

Since (s_n) and (y_n) are bounded, we obtain

$$\left| \left\langle \Phi'(z_n), s_n(\frac{s_n}{2}+1)z_n + (1+s_n)y_n \right\rangle \right| \leqslant \|\Phi'(z_n)\| \| s_n(\frac{s_n}{2}+1)z_n + (1+s_n)y_n \|$$
$$\leqslant C_1 \|\Phi'(z_n)\| \|z_n\|_{s\times t} + C_2 \|\Phi'(z_n)\| \to 0.$$

We deduce that for n big enough

$$\left\langle z_n^+ - z_n^-, s_n(\frac{s_n}{2}+1)z_n + (1+s_n)y_n \right\rangle \leqslant 1 + \int_\Omega \left(s_n(\frac{s_n}{2}+1)u_n + (1+s_n)a_n \right) f(x, u_n)$$
$$+ \int_\Omega \left(s_n(\frac{s_n}{2}+1)v_n + (1+s_n)b_n \right) g(x, v_n). \tag{$\star$}$$

Noting that $rw_n^+ = z_n + (s_n z_n + y_n)$ and

$$\left\langle z_n^+ - z_n^-, s_n(\frac{s_n}{2}+1)z_n + (1+s_n)y_n \right\rangle = -\frac{1}{2}\|z_n^+\|_{s\times t}^2 + \frac{1}{2}\|z_n^-\|_{s\times t}^2 + \frac{1}{2}r^2\|w_n^+\|_{s\times t}^2 + \frac{1}{2}r^2\|w_n^-\|_{s\times t}^2,$$

we have

$$\Phi(rw_n^+) - \Phi(z_n) = -\frac{1}{2}\|w_n^-\|_{s\times t}^2 + \left\langle z_n^+ - z_n^-, s_n(\frac{s_n}{2}+1)z_n + (1+s_n)y_n \right\rangle$$
$$+ \int_\Omega \left(F(x, u_n) - F(x, (1+s_n)u_n + a_n) \right)$$
$$+ \int_\Omega \left(G(x, u_n) - G(x, (1+s_n)u_n + a_n) \right).$$

We deduce by using $(\star)$ that for n sufficiently large

$$\Phi(rw_n^+) - \Phi(z_n) \leqslant 1$$
$$+ \int_\Omega \left[\left(s_n(\frac{1}{2}s_n+1)u_n + (1+s_n)a_n \right) f(x, u_n) + F(x, u_n) - F\left(x, (1+s_n)u_n + a_n \right) \right]$$
$$+ \int_\Omega \left[\left(s_n(\frac{1}{2}s_n+1)v_n + (1+s_n)b_n \right) g(x, u_n) + G(x, u_n) - G\left(x, (1+s_n)v_n + b_n \right) \right].$$

We conclude by using Lemma 4.3 that for n big enough, $\Phi(rw_n^+) - 1 \leqslant \Phi(z_n)$.
If $w^+ = 0$, then $w_n^+ \to 0$ in $L^p(\Omega) \times L^q(\Omega)$ by Lemma 4.1. By (E_1) and Theorem A.2 in [32] we have $\int_\Omega F\left(x, (1+s_n)u_n + a_n\right) \to 0$ and $\int_\Omega G\left(x, (1+s_n)v_n + b_n\right) \to 0$.

Since $\Phi(z_n) \to c$, we then obtain for n sufficiently large

$$
\begin{aligned}
c + 1 \geqslant \Phi(z_n) &\geqslant \Phi(rw_n^+) - 1 \\
&= \frac{1}{2} r^2 \|w_n\|_{s \times t}^2 - \int_\Omega F\big(x, (1 + s_n)u_n + a_n\big) - \int_\Omega G\big(x, (1 + s_n)v_n + b_n\big) - 1 \\
&\geqslant \frac{1}{2} r^2 \|w_n\|_{s \times t}^2 - 2 \\
&\geqslant \frac{r^2}{6} - 2 \quad \Big(\text{because } \|w_n^+\|_{s \times t}^2 \geqslant \frac{1}{3}\Big).
\end{aligned}
$$

This leads to a contradiction if we take r arbitrary large. Hence $w^+ \neq 0$ and then $w \neq 0$.

Setting $w_n = (w_n^1, w_n^2) \in E^s \times E^t$, we have

$$
\begin{aligned}
\frac{\Phi(z_n)}{\|z_n\|_{s \times t}^2} &= \frac{1}{2} \|w_n^+\|_{s \times t}^2 - \frac{1}{2} \|w_n^-\|_{s \times t}^2 - \int_\Omega \frac{F(x, u_n)}{\|z_n\|_{s \times t}^2} - \int_\Omega \frac{G(x, u_n)}{\|z_n\|_{s \times t}^2} \\
&= \frac{1}{2} \|w_n^+\|_{s \times t}^2 - \frac{1}{2} \|w_n^-\|_{s \times t}^2 - \int_{w_n^1 \neq 0} \frac{F(x, w_n^1 \|z_n\|_{s \times t})}{\big|w_n^1 \|z_n\|_{s \times t}\big|^2} - \int_{w_n^2 \neq 0} \frac{G(x, w_n^2 \|z_n\|_{s \times t})}{\big|w_n^1 \|z_n\|_{s \times t}\big|^2} \\
&\leqslant C - \int_{w_n^1 \neq 0} \frac{F(x, w_n^1 \|z_n\|_{s \times t})}{\big|w_n^1 \|z_n\|_{s \times t}\big|^2} - \int_{w_n^2 \neq 0} \frac{G(x, w_n^2 \|z_n\|_{s \times t})}{\big|w_n^1 \|z_n\|_{s \times t}\big|^2}.
\end{aligned}
$$

Since $(\Phi(z_n))$ is bounded and $\|z_n\|_{s \times t} \to \infty$, the l.h.s. tends to 0. By applying Fatou's lemma and using (E_3), we see that the r.h.s. goes to $-\infty$, which is a contradiction. Consequently the sequence (z_n) is bounded.

Now a standard argument shows that the bounded sequence (z_n) has a convergent subsequence. $\qquad\square$

Proof of Theorem 1.3. Let $(a_j)_{j \geqslant 0}$ be an orthonormal basis of E^s. We define an orthonormal basis $(e_j)_{j \geqslant 0}$ of E^+ by setting

$$
e_j := \frac{1}{\sqrt{2}} \big(a_j, A^{s-t} a_j\big).
$$

Let

$$
E_k^- = E^- \oplus \big(\oplus_{j=0}^k \mathbb{R} e_j\big) \quad \text{and} \quad E_k^+ = \overline{\oplus_{j=k}^\infty \mathbb{R} e_j}.
$$

(E_3) implies that for every $\delta > 0$ there is $C_\delta > 0$ such that

$$
F(x, u) \geqslant \delta |u|^2 - C_\delta \quad \text{and} \quad G(x, u) \geqslant \delta |u|^2 - C_\delta, \quad \forall(x, u). \tag{4.5}
$$

Let $z \in E_k^-$. Then $z = \big(u, A^{s-t} u\big) + \big(v, -A^{s-t} v\big)$, where $v \in E^s$ and $u \in E_k^s := \oplus_{j=0}^k \mathbb{R} a_j$. We obtain by using (4.5), the fact that E^{s-t} embeds continuously in $L^2(\Omega)$, and the parallelogram identity

$$
\begin{aligned}
\Phi(z) &= \|u\|_s^2 - \|v\|_s^2 - \int_\Omega \Big(F(x, u + v) + G(x, A^{s-t}(u - v))\Big) \\
&\leqslant \|u\|_s^2 - \|v\|_s^2 - 2C\delta\big(|u|_2^2 + |v|_2^2\big) + 2C_\delta |\Omega|.
\end{aligned}
$$

Since all the norms are equivalent on the finite-dimensional subspace E_k^s, there is a constant $c_1 > 0$ such that $c_1 \|u\|_s \leqslant |u|_2$. Hence

$$
\Phi(z) \leqslant (1 - c_2 \delta) \|u\|_s^2 - \|v\|_s^2 - 2C_\delta |\Omega|.
$$

Choose $\delta > \frac{1}{c_2}$. Hence $\Phi(z) \to -\infty$ as $\|z\|_{s \times t} \to \infty$, and consequently assumption (A_1) of Theorem 2.3 is satisfied.

Let $z \in E_k^+$. Then $z = (u, A^{s-t}u)$ with $u \in \overline{\oplus_{j=k}^{\infty} \mathbb{R}a_j}$, and

$$\Phi(z) = \frac{1}{2}\|z\|_{s\times t}^2 - \int_{\Omega} \Big(F(x, u) + G(x, A^{s-t}u) \Big)$$

$$\geqslant \frac{1}{2}\|z\|_{s\times t}^2 - 2\int_{\Omega} \Big(F(x, u) + G(x, A^{s-t}u) \Big).$$

By (E_1) there is a constant $C_1 > 0$ such that

$$|F(x, u)| \leqslant C_1(1 + |u|^p) \quad \text{and} \quad |G(x, A^{s-t}u)| \leqslant C_1(1 + |A^{s-t}u|^q).$$

Hence

$$\Phi(z) \geqslant \|u\|_s^2 - 2C_1|u|_p^p - 2C_1|A^{s-t}u|_q^q - 4C_1|\Omega|.$$

We define

$$\beta_{1,k} := \sup_{\substack{u\in\oplus_{j=k}^{\infty}\mathbb{R}a_j \\ \|u\|_s=1}} |u|_p, \quad \beta_{2,k} := \sup_{\substack{v\in\oplus_{j=k}^{\infty}\mathbb{R}(A^{s-t}a_j) \\ \|v\|_t=1}} |v|_q,$$

and $\beta_k = \max\{\beta_{1,k}; \beta_{2,k}\}$.
Then

$$\Phi(z) \geqslant \|u\|_s^2 - 2C_1\beta_k^p\|u\|_s^p - 2C_1\beta_k^q\|u\|_s^q - 4C_1|\Omega|.$$

We assume without loss of generality that $q \leqslant p$ and we set

$$r_k := \big(C_1 p \beta_k^p\big)^{\frac{1}{2-p}}.$$

Then for $\|z\|_{s\times t} = \sqrt{2}\|u\|_s = r_k$ we have

$$\Phi(z) \geqslant \widetilde{b}_k := K\beta_k^{\frac{2p}{2-p}}\Big[\Big(\frac{1}{4} - \frac{1}{p(\sqrt{2})^p}\Big) - A\beta_k^{\frac{2(q-p)}{2-p}}\Big] - 4C_1|\Omega|, \tag{4.6}$$

where $K, A > 0$ are constant.
We know by Lemma 3.8 in [32] that $\beta_{1,k} \to 0$ and $\beta_{2,k} \to 0$ as $k \to \infty$. This implies that $\widetilde{b}_k \to \infty$ as $k \to \infty$, and then that assumption (A_2) of Theorem 2.3 is satisfied.

An argument similar to that in the proof of Lemma 3.2 shows that Ψ is weakly sequentially lower semicontinuous and that Ψ' is weakly sequentially continuous. Since we have shown above that the functional Φ satisfies the Cerami condition at every critical level, we can apply Theorem 2.3 and get the desired result. $\qquad\square$

References

[1] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* 14 (1973) 349-381.

[2] V. Barbu, Periodic solutions to unbounded Hamiltonian system, *Discrete Contin. Dyn. Syst.* 1 (1995) 277-283.

[3] T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, *Nonlinear Anal.* **20** (1993) 1205-1216.

[4] T. Bartsch, M. Clapp, Critical Point Theory for Indefinite Functionals with Symmetries, *J. Funct. Anal.* 138 (1996) 107-136.

[5] T. Bartsch, Y.H. Ding, Homoclinic solutions of an infinite-dimensional Hamiltonian system, *Math. Z.* 240 (2002) 289-310.

[6] T. Bartsch, Y. H. Ding, On a nonlinear Schrödinger equation with periodic potential, *Math. Ann.* 313 (1999) 15-37.

[7] T. Bartsch, A. Szulkin, *Hamiltonian systems: periodic and homoclinic solutions by variational methods.* Handbook of differential equations: ordinary differential equations. Vol. II, 77-146, Elsevier B. V., Amsterdam, 2005.

[8] T. Bartsch, M. Willem, On an elliptic equation with concave and convex nonlinearities, *Proc. Amer. Math. Soc.* **123** (1995) 3555-3561.

[9] C. J. Batkam, On a superquadratic elliptic system with strongly indefinite structure, arXiv:1305.5780.

[10] C. J. Batkam, F. Colin, Generalized Fountain Theorem and applications to strongly indefinite semilinear problems, *J. Math. Anal. Appl.* 405 (2013) 438-452.

[11] C. J. Batkam, F. Colin, *On multiple solutions of a semilinear Schrödinger equation with periodic potential*, Nonlinear Anal. **84** (2013) 39-49.

[12] C. J. Batkam, F. Colin, The effects of concave and convex nonlinearities in some noncooperative elliptic systems, *Annali di Matematica*, in press, DOI 10.1007/s10231-013-0343-9.

[13] V. Benci, On critical point theory for indefinite functionals in the presence of symmetries, *Trans. Amer. Math. Soc.* **274** (1982) 533-572.

[14] H. Brezis, L. Nirenberg, Characterization of the ranges of some nonlinear operators and applications to boundary value problems, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (4) 5 (1978) 225-326.

[15] G. Cerami, An existence criterion for the critical points on unbounded manifolds (Italian), *Istit. Lombardo Accad. Sci. Lett. Rend. A* 112 (1978) 332-336 (1979).

[16] P. Clément, P. Felmer, and E. Mitidieri, Homoclinic orbits for a class of infinite dimensional Hamiltonial systems, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 24 (1997) 367-393.

[17] D. G. de Figueiredo, P. Felmer, On superquadratic elliptic systems, *Trans. Amer. Math. Soc.* 343 (1994) 99-116.

[18] P. Felmer, Z.-Q. Wang, Multiplicity for symmetric indefinite functionals: application to Hamiltonian and elliptic systems. *Topol. Methods Nonlinear Anal.* 12 (1998) 207-226.

[19] J. Hulshof, R. VanDer Vorst, Differential systems with strongly indefinite variational structure, *J. Funct. Anal.* 114 (1993) 32-58.

[20] W. Kryszewski, A. Szulkin, Generalized linking theorem with an application to a semilinear Schrödinger equation, *Adv. Differential Equation* 3 (1998) 441-472.

[21] Y.Q. Li, A limit index theory and its application, *Nonlinear Anal. TMA* 25 (1995) 1371-1389.

[22] J. L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. I, Dunod, Paris, 1968.

[23] S. Liu, On superlinear Schrödinger equations with periodic potential, *Calc. Var. Partial Differential Equations* 45 (2012) 1-9.

[24] S. Liu, Z. Shen, Generalized saddle point theorem and asymptotically linear problems with periodic potential, *Nonlinear Anal.* 86 (2013) 52-57.

[25] Y. Long, Periodic solutions for a class of nonautonomous Hamiltonian system, *Nonlinear Anal.* 41 (2000) 455-463.

[26] A. Mao, S. Luan, and Y. Ding, Periodic solutions for a class of first order super-quadratic Hamiltonian system, *J. Math. Anal. Appl.* 330 (2007) 584-596.

[27] R. S. Palais, S. Smale, A generalized Morse theory, *Bull. Amer. Math. Soc.* 70 1964 165-172.

[28] A. Persson, Compact linear mappings between interpolation spaces, *Ark. Mat.* 5 (1964) 215-219.

[29] P.H. Rabinowitz, Periodic solutions of Hamiltonian system, *Comm. Pure Appl. Math.* 31 (1978) 157-184.

[30] M. Schechter, The Use of Cerami Sequences in Critical Point Theory, *Abstract and Applied Analysis*, vol. 2007, Article ID 58948, 28 pages, 2007. doi:10.1155/2007/58948

[31] A. Szulkin, T. Weth, *The method of Nehari manifold*, Handbook of nonconvex analysis and applications, 597-632, Int. Press, Somerville, MA, 2010.

[32] M. Willem, *Minimax Theorems*, Birkhauser, Boston (1996).

[33] W. Zou, Variant fountain theorems and their applications, *Manuscripta Math.* 104 (2001) 343-358.

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