Polynomial and rational approximations and Schröder's processes of the first and second kind

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Summary

In [10, 11], Schröder proposed two fixed point processes to find a simple root α of a nonlinear equation $f(x) = 0$. The iteration functions (IF) of arbitrary order $p \geq 2$ associated to the two processes will be noted $E_p(x)$ for the Schröder's process of the first kind, and $S_p(x)$ for the Schröder's process of the second kind.

For the order 2 processes we have

$$E_2(x) = S_2(x) = N_f(x) = x = \frac{f(x)}{f^{(1)}(x)},$$

where $N_f(x)$ is the Newton's IF. In general for $p > 2$, the processes can be expressed as

$$E_p(x) = x - \frac{f(x)}{f^{(1)}(x)} \left. \Lambda_{p-2}(\xi) \right|_{\xi = f(x)/f^{(1)}(x)}, \tag{0.1}$$

and

$$S_p(x) = x - \frac{f(x)}{f^{(1)}(x)} \left. \left[\frac{\Gamma_{p-3}(\xi)}{\Gamma_{p-2}(\xi)} \right] \right|_{\xi = f(x)/f^{(1)}(x)}, \tag{0.2}$$

where $\Lambda_q(\xi)$ and $\Gamma_q(\xi)$ are polynomials of degree $q \geq 0$, such that $\Lambda_q(0) = 1 = \Gamma_q(0)$. We will describe these polynomials in Sections 3 and 4.

A question raised and discussed in [6, 8, 9] is to explain the possible link between the two processes. The main result of this paper is to show that $E_p(x)$ is a polynomial approximation of $S_p(x)$, that $S_p(x)$ is a rational approximation of $E_p(x)$, and explain the relation between the polynomials $\Lambda_{p-2}(\xi)$, $\Gamma_{p-3}(\xi)$ and $\Gamma_{p-2}(\xi)$ in (0.1) and (0.2). In this paper we complete the work done in [8, 9].

In the next section we present some notations and definitions used in this paper. In Sections 3 and 4 we present the two Schröder's processes and their corresponding polynomials $\Lambda_q(\xi)$ and $\Gamma_q(\xi)$. We prove the main result in the last section.

References

[1] T.M. Apostol, Calculating higher derivatives of inverses, *American Mathematical Monthly*, 107 (2000), 738-741.

[2] E. Bodewig, On types of convergence and on the behavior of approximations in the neighborhood of a multiple root of an equation, *Quart. Appl. Math.*, 7 (1949), 325-333.

[3] W. Gander, On Halley's iteration method, *American Mathematical Monthly*, 92 (1985), 131-134

[4] A.S. Househoulder, *Principles of Numerical Analysis*, McGraw Hill, Columbus, New York, 1953.

[5] W.P. Johnson, Combinatorics of higher derivatives of inverses, *American Mathematical Monthly*, 109 (2002), 273-277.

[6] B. Kalantari, I. Kalantari, and R. Zaare-Nahandi, A basic family of iteration functions for polynomial root-finding and its characterizations, *Journal of computational and Applied Mathematics*, 80 (1997), 209-226.

[7] M.S. Petković and D. Herceg, On rediscovered iteration methods for solving equations, *Journal of Computational and Applied Mathematics*, 107 (1999), 275-284.

[8] L.D. Petković and M.S. Petković, The link between Schröder's iteration methods of the first and second kind, *Novi Sad J. Math.*, 38 (2008), 55-63.

[9] M.S. Petković, L.D. Petković, and D. Herceg, On Schröder's families of root-finding methods, *Journal of Computational and Applied Mathematics*, 233 (2010), 1755-1762.

[10] E. Schröder, Uber unendlich viele Algorithmen zur Auflosung der Gleichungen, *Math. Ann.*, 2 (1870), 317-365.

[11] E. Schröder, *On Infinitely Many Algorithms for Solving Equations*, Translated by G.W. Stewart, Institute for advanced Computer Studies, TR-92-121 and TR-2990, University of Maryland, 1992.

[12] M. Shub and S. Smale, Computation complexity: On the geometry of polynomials and the theory of cost I, *Ann. Sci. École Norm. Sup.*, 18 (1985), 107-142.

[13] J.F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.