# Inverse function, Taylor's expansion, and high-order algorithms for $n$-th root computation.

**François Dubeau and Calvin Gnang**

Département de mathématiques
Faculté des sciences, Université de Sherbrooke
2500, boul. de l'Université, Sherbrooke (Qc), Canada, J1K 2R1
francois.dubeau@usherbrooke.ca

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### Summary

The computation of the $n$-th root $r^{1/n}$ of a strictly positive real number $r$ has a long history [1, 14]. In more recent work, for example in [10] and [20], continued fraction expansions are used to derive methods to compute $r^{1/2}$. Also, in [12] methods similar to those presented in [20] are obtained as special case of a determinantal family of root-finding methods [11]. For the computation of $n$-th root third order and fourth order methods are presented in [6]. General high order methods can be derived from the application of Newton's method to an appropriate modified function [2] or using a modified Newton's method applied to the original function [7, 8]. These two approaches have been revisited in [3]. Using combinations of basic functions identified for methods proposed in [2, 7], new high-order methods are derived for the computation of $r^{1/2}$ in [15].

The goal of this paper is twofold. The first contribution is the presentation of a simple way to obtain two known families of high order methods to compute $r^{1/n}$ established by Dubeau [2] (*Computing*, 57 (1996), 365-369), and by Hernandez and Romero [7] (*International Journal of Computer Mathematics*, 81 (2004), 1001-1014). It happens that both methods are related to the Taylor's expansion of an inverse function. Not only we show that the method [7, 8] can be derived in an elementary manner, but we also show that it is a simple application of the Schröder's process dating from 1870 [16, 17], see also [19]. In both case, we also show that the process can be applied to find new higher order methods.

For the second contribution, we point out a general process to increase the rate of convergence of a fixed point method. Linear combinations of two methods of the same order $m$ are considered to increase the order $m$ to $m+1$ by zero in the asymptotic constant.

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