THE EFFECTS OF A CONCAVE-CONVEX NONLINEARITY ON A NONCOOPERATIVE ELLIPTIC SYSTEM

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Abstract. We consider a noncooperative elliptic system with the combined effects of concave and convex nonlinearities in bounded domains. Using variational methods we prove the existence of infinitely many large and small energy solutions. The proof relies on new critical point theorem which guarantees the existence of infinitely many critical values of a wide class of even strongly indefinite functionals.

1. Introduction

This paper is concerned with the multiplicity of solutions of the following noncooperative elliptic system:

$$(S_{\lambda,\mu,\delta}) \qquad \begin{cases} -\Delta u = \lambda|u|^{p-2}u + \frac{\alpha}{\alpha+\beta}\delta|u|^{\alpha-2}u|v|^{\beta} \text{ in } \Omega, \\ \\ \Delta v = \mu|v|^{q-2}v + \frac{\beta}{\alpha+\beta}\delta|u|^{\alpha}|v|^{\beta-2}v \text{ in } \Omega, \\ \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, λ, μ and δ are real parameters, $\alpha, \beta > 1$ and $1 < p, q < 2 < \alpha + \beta < 2^*$; with $2^* := \infty$ if $N = 1, 2$ and $2^* := 2N/N - 2$ if $N \geqslant 3$. It is well known that the solutions of $(S_{\lambda,\mu,\delta})$ can be found as critical points of the following functional

$$J(u,v) := \int_\Omega \Big[\frac{1}{2}\big(|\nabla u|^2 - |\nabla v|^2\big) - \frac{\lambda}{p}|u|^p - \frac{\mu}{q}|v|^q - \frac{\delta}{\alpha+\beta}|u|^\alpha|v|^\beta\Big]dx,$$

defined on the Hilbert space $H_0^1(\Omega)\times H_0^1(\Omega)$. The functional J is strongly indefinite, in the sense that its quadratic part has infinitely many positive and infinitely many negative eigenvalues. Therefore, in addition to the combined effects of concave and convex nonlinearities, this strongly indefiniteness is also a difficulty we must overcome in this paper.

The starting point on the study of $(S_{\lambda,\mu,\delta})$ is the following semilinear scalar elliptic equation:

$$(S) \quad \begin{cases} -\Delta u = \gamma|u|^{r-2}u + \nu|u|^{s-2}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\gamma, \nu \in \mathbb{R}$ and $1 < r < 2 < s < 2^*$. In the celebrated paper [1] Ambrosetti, Brézis and Cerami showed that for $\nu = 1$ and $\gamma > 0$ small, (S) has infinitely many

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solutions with positive energy

$$\Phi(u) = \int_{\Omega} \Big(\frac{1}{2} |\nabla u|^2 - \frac{\gamma}{r} |u|^r - \frac{\nu}{s} |u|^s \Big) dx,$$

and also infinitely many solutions with negative energy. Their result was generalized later by Bartsch and Willem [2] who obtained, by using the fountain theorem and its dual version, the existence of infinitely many positive large energy solutions and infinitely many negative small energy solutions of (S) provided $\nu > 0$ and $\gamma > 0$ respectively. Their approach also allowed them to give a positive answer to problem (c) of [1].

Inspired by these previous papers, we investigate the existence of infinitely many solutions of $(S_{\lambda,\mu,\delta})$. Our main results are stated as follows:

Theorem 1. *Assume that $\delta > 0$, $\lambda, \mu \in \mathbb{R}$. Then $(S_{\lambda,\mu,\delta})$ has a sequence of solutions (u_k, v_k) such that $J(u_k, v_k) \to \infty$ as $k \to \infty$.*

Theorem 2. *Assume that $\delta \leqslant 0$, $\lambda > 0$ and $\mu \geqslant 0$. Then $(S_{\lambda,\mu,\delta})$ has a sequence of solutions (y_k, z_k) such that $J(y_k, z_k) < 0$ and $J(y_k, z_k) \to 0$ as $k \to \infty$.*

While Theorem 1 is a consequence of the generalized fountain theorem established by the authors in [3], Theorem 2 will be proved by using a new critical point theorem for even strongly indefinite functionals, established in this paper, which can be viewed as a generalization of the dual fountain theorem of Bartsch and Willem [2].

Semilinear and quasilinear elliptic systems of cooperative type, with the combination of concave and convex nonlinearities have been widely studied (see for instance [4, 5, 6, 7] and references therein). However this paper seems to be the first one considering the noncooperative case.

The paper is organized as follows: In section 2 we state and prove the abstract critical point theorems, and in section 3 we apply them in order to prove the existence of solutions.

2. Critical point theorems

Let $X = Y \oplus Z$ be a Hilbert space, with inner product $(\cdot)$ and norm $\|\cdot\|$, where

$$Y = \overline{\bigoplus_{j=0}^{\infty} \mathbb{R} e_j} \quad \text{and} \quad Z = \overline{\bigoplus_{j=0}^{\infty} \mathbb{R} f_j},$$

with $\|e_j\| = \|f_j\|$, $\forall j \geqslant 0$.
Denote by $P : X \to Y$ and $Q : X \to Z$ the orthogonal projections, and consider on X the τ−topology of Kryszewski and Szulkin, that is the topology associated to the following norm

$$|||u||| := \max\Big(\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} |(Pu, e_j)|, \|Qu\| \Big), \ u \in X.$$

τ has the following interesting property (see [8], Chapter 6): If $(u_n) \subset X$ is a bounded sequence, then

$$u_n \xrightarrow{\tau} u \iff Pu_n \rightharpoonup Pu \ \text{and} \ Qu_n \to Qu.$$

We refer to [3] for the proof of the following lemma:

Lemma 3 (Deformation lemma)**.** *Let $\varphi \in \mathcal{C}^1(X,\mathbb{R})$ be an even functional which is τ-upper semicontinuous and such that $\nabla\varphi$ is weakly sequentially continuous. Let $S \subset X$ such that $-S = S$, and $c \in \mathbb{R}$, $\epsilon, \theta > 0$ such that*

$$\forall u \in \varphi^{-1}([c-2\epsilon, c+2\epsilon]) \cap S_{2\theta},\ \|\varphi'(u)\| \geqslant \frac{8\epsilon}{\theta}. \tag{1}$$

Then there exists $\eta \in \mathcal{C}([0,1]\times\varphi^{c+2\epsilon}, X)$ such that:

(i) $\eta(t,u) = u$ *if* $t = 0$ *or if* $u \notin \varphi^{-1}([c-2\epsilon, c+2\epsilon]) \cap S_{2\theta}$,
(ii) $\eta(1, \varphi^{c+\epsilon} \cap S) \subset \varphi^{c-\epsilon}$,
(iii) $\|\eta(t,u) - u\| \leqslant \frac{\theta}{2}\ \forall u \in \varphi^{c+2\epsilon},\ \forall t \in [0,1]$,
(iv) $\varphi(\eta(\cdot,u))$ *is non increasing,* $\forall u \in \varphi^{c+2\epsilon}$,
(v) *Each point $(t,u) \in [0,1]\times\varphi^{c+2\epsilon}$ has a τ-neighborhood $N_{(t,u)}$ such that $\{v - \eta(s,v) \,|\, (s,v) \in N_{(t,u)} \cap ([0,1]\times\varphi^{c+2\epsilon})\}$ is contained in a finite-dimensional subspace of X,*
(vi) η *is τ-continuous,*
(vii) $\eta(t,\cdot)$ *is odd* $\forall t \in [0,1]$,

where $S_\alpha := \{u \in X \,|\, dist(u,S) \leqslant \alpha\}$ $\forall \alpha > 0$ and $\varphi^a := \{u \in X \,|\, \varphi(u) \leqslant a\}$ $\forall a \in \mathbb{R}$.

We adopt the following notations:

$$Y_k := Y \oplus (\oplus_{j=0}^{k}\mathbb{R}f_j), \qquad Z_k := \overline{\oplus_{j=k}^{\infty}\mathbb{R}f_j},$$

$B_k := \{u \in Y_k \,|\, \|u\| \leqslant \rho_k\}$, $N_k := \{u \in Z_k \,|\, \|u\| = r_k\}$ where $0 < r_k < \rho_k$, $k \geqslant 1$.

The following theorem was proved by Batkam and Colin in [3]. It will be very helpful for finding large energy solutions.

Theorem 4 (Generalized fountain theorem)**.** *Let $\varphi \in \mathcal{C}^1(X,\mathbb{R})$ be an even functional which is τ-upper semicontinuous and such that $\nabla\varphi$ is weakly sequentially continuous. If, for every $k \geqslant k_0$, there exists $\rho_k > r_k > 0$ such that:*

(A_1) $a_k := \sup_{u\in Y_k,\ \|u\|=\rho_k} \varphi(u) \leqslant 0$ and $\sup_{u\in Y_k,\ \|u\|\leqslant\rho_k} \varphi(u) < \infty$.

(A_2) $b_k := \inf_{u\in Z_k,\ \|u\|=r_k} \varphi(u) \to \infty,\ k \to \infty$.

(A_3) *φ satisfies the $(PS)_c$ condition,* $\forall c > 0$.

Then φ has an unbounded sequence of critical values.

In order to find small energy solutions, we need a new critical point theorem. We introduce the following notations:

$$Y^k := \overline{\oplus_{j=k}^{\infty}\mathbb{R}e_j}, \qquad Z^k := \left(\oplus_{j=0}^{k}\mathbb{R}e_j\right) \oplus Z,$$

$B^k := \{u \in Y^k \,|\, \|u\| \leqslant \sigma_k\}$, $N^k := \{u \in Z^k \,|\, \|u\| = s_k\}$ where $0 < s_k < \sigma_k$, $k \geqslant 2$.

Theorem 5. *Let $\Phi \in \mathcal{C}^1(X,\mathbb{R})$ be an even functional which is τ-lower semicontinuous and such that $\nabla\Phi$ is weakly sequentially continuous. If, for every $k \geqslant k_0$, there exists $\sigma_k > s_k > 0$ such that:*

(B_1) $a^k := \inf_{u\in Y^k,\ \|u\|=\sigma_k} \Phi(u) \geqslant 0$,

(B_2) $b^k := \sup_{\substack{u\in Z^k\\ \|u\|=s_k}} \Phi(u) < 0,$

(B_3) $d^k := \inf_{\substack{u\in Y^k\\ \|u\|\leqslant\sigma_k}} \Phi(u) \to 0,\ k\to\infty,$

(B_4) Φ *satisfies the* $(PS)_c$ *condition* $\forall c \in [d_{k_0}, 0[$.

Then Φ *has a sequence of critical points* (u_k) *such that* $\Phi(u_k) < 0$ *and* $\Phi(u_k) \to 0$ *as* $k\to\infty$.

Proof. Let Γ^k be the set of maps $\gamma : B^k \to X$ such that:

(a) γ is odd, $\tau-$continuous and $\gamma|_{\partial B^k} = id$,
(b) each $u \in int(B^k)$ has a $\tau-$neighborhood $\mathcal{N}_u$ in Y^k such that $(id-\gamma)\big(\mathcal{N}_u \cap int(B^k)\big)$ is contained in a finite dimensional subspace of X,
(c) $\Phi(\gamma(u)) \geqslant \Phi(u)\ \forall u \in B^k$.

Define

$$c^k := \sup_{\gamma\in\Gamma^k} \inf_{u\in B^k} \Phi(\gamma(u)).$$

It is clear that $c^k \leqslant b^k$. Following the proof of Lemma 10 in [3], one can verify easily that if γ satisfies (a) and (b), then $\gamma(B^k) \cap N^k \neq \varnothing$, and this implies that $d^k \leqslant c^k$.
Let $\epsilon \in]0, \frac{a^k-c^k}{2}[$, $\theta > 0$ and let $\gamma \in \Gamma^k$ such that

$$c^k - \epsilon \leqslant \inf_{B^k} \Phi(\gamma(u)). \tag{2}$$

We claim that

$$\exists u \in \Phi^{-1}\big([c^k-2\epsilon, c^k+2\epsilon]\big) \cap \big(\gamma(B^k)\big)_{2\theta} \text{ such that } \|\Phi'(u)\| \leqslant \frac{8\epsilon}{\theta}. \tag{3}$$

In fact assume by negation that (3) is not satisfied and apply Lemma 3 with $\varphi = -\Phi$ and $S = \gamma(B^k)$. Following the proof of Theorem 11 in [3], one can verify easily that the map $\chi : B^k \ni u \mapsto \eta(1, \gamma(u))$ belongs to Γ^k. By using (2) and (ii) of Lemma 3 we deduce that

$$c^k \geqslant \inf_{\substack{u\in Y^k\\ \|u\|=\sigma_k}} \Phi(\chi(u)) \geqslant c^k + \epsilon,$$

which gives the contradiction.
Now we deduce from (3) the existence of a sequence $(u_n^k) \subset X$ such that

$$\Phi(u_n^k) \to c^k \text{ and } \Phi'(u_n^k) \to 0,\ \ n\to\infty.$$

We then conclude by using (B_4) and (B_3). □

In the sequel $|u|_p$ is the usual $L^p(\Omega)$ norm.

3. Proof of the main results

We choose on the Sobolev space $H_0^1(\Omega)$ the norm $\|u\| = |\nabla u|_2$ and on $X := H_0^1(\Omega) \times H_0^1(\Omega)$ the product norm $\|(u,v)\| = \big(\|u\|^2 + \|v\|^2\big)^{\frac{1}{2}}$.
Consider the functional J defined on X by

$$J(u,v) := \int_\Omega \Big[\frac{1}{2}\big(|\nabla u|^2 - |\nabla v|^2\big) - \frac{\lambda}{p}|u|^p - \frac{\mu}{q}|v|^q - \frac{\delta}{\alpha+\beta}|u|^\alpha|v|^\beta\Big]dx. \tag{4}$$

One can verify easily that J is of class $\mathcal{C}^1$ on X and

$$\big\langle J'(u,v),(\phi,\varphi)\big\rangle = \int_\Omega \nabla u\nabla\phi dx - \int_\Omega \nabla v\nabla\varphi dx - \int_\Omega \lambda|u|^{p-2}u\phi dx - \mu\int_\Omega |v|^{q-2}v\varphi dx$$
$$- \frac{\delta}{\alpha+\beta}\int_\Omega \big(\alpha|u|^{\alpha-2}|v|^\beta u\phi + \beta|u|^\alpha|v|^{\beta-2}v\varphi\big)dx \quad (5)$$

We need the following lemma.

Lemma 6. *Assume that* $1 \leqslant p, r < \infty$ *and* $G \in \mathcal{C}(\overline{\Omega}\times\mathbb{R}\times\mathbb{R})$ *such that*

$$|G(x,u,v)| \leqslant c\big(1 + |u|^{\frac{p}{r}} + |v|^{\frac{p}{r}}\big).$$

Then $\forall u, v \in L^p(\Omega)$, $G(\cdot,u,v) \in L^r(\Omega)$ *and the operator* $A : L^p(\Omega)\times L^p(\Omega) \to L^r(\Omega), (u,v) \mapsto G(x,u,v)$ *is continuous.*

The proof is similar to the proof of Theorem $A.2$ in [8] and is omitted.

Lemma 7. ∇J *is weakly sequentially continuous,* $\forall\lambda,\mu,\delta \in \mathbb{R}$.

Proof. Let $(u_n,v_n) \subset X$ such that $(u_n,v_n) \rightharpoonup (u,v)$ in X. We want to show that $\nabla J(u_n,v_n) \rightharpoonup \nabla J(u,v)$.
Using (5) and the Hölder inequality we have

$$|\big\langle J'(u_n,v_n) - J'(u,v),(\phi,\varphi)\big\rangle| \leqslant \Big|\int_\Omega \nabla(u_n-u)\nabla\phi dx - \int_\Omega \nabla(v_n-v)\nabla\varphi dx\Big|$$
$$+ |\lambda|\Big||u_n|^{p-2}u_n - |u|^{p-2}u\Big|_{\frac{p}{p-1}}|\phi|_p + |\mu|\Big||v_n|^{q-2}v_n - |v|^{q-2}v\Big|_{\frac{q}{q-1}}|\varphi|_q$$
$$+ \frac{|\delta\alpha|}{\alpha+\beta}\Big||u_n|^{\alpha-2}u_n|v_n|^\beta - |u|^{\alpha-2}u|v|^\beta\Big|_{\frac{\alpha+\beta}{\alpha+\beta-1}}|\phi|_{\alpha+\beta-1}$$
$$+ \frac{|\delta\beta|}{\alpha+\beta}\Big||u_n|^\alpha v_n|^{\beta-2}v_n - |u|^\alpha v|^{\beta-2}v\Big|_{\frac{\alpha+\beta}{\alpha+\beta-1}}|\varphi|_{\alpha+\beta-1}.$$

Since the Sobolev space $H_0^1(\Omega)$ continuously embeds into $L^p(\Omega)\cap L^q(\Omega)\cap L^{\alpha+\beta-1}(\Omega)$, we infer that

$$|\big\langle J'(u_n,v_n) - J'(u,v),(\phi,\varphi)\big\rangle| \leqslant \Big|\int_\Omega \nabla(u_n-u)\nabla\phi dx - \int_\Omega \nabla(v_n-v)\nabla\varphi dx\Big|$$
$$+ C\Big[|\lambda|\Big||u_n|^{p-2}u_n - |u|^{p-2}u\Big|_{\frac{p}{p-1}} + |\mu|\Big||v_n|^{q-2}v_n - |v|^{q-2}v\Big|_{\frac{q}{q-1}}$$
$$+ \frac{|\delta\alpha|}{\alpha+\beta}\Big||u_n|^{\alpha-2}u_n|v_n|^\beta - |u|^{\alpha-2}u|v|^\beta\Big|_{\frac{\alpha+\beta}{\alpha+\beta-1}}$$
$$+ \frac{|\delta\beta|}{\alpha+\beta}\Big||u_n|^\alpha v_n|^{\beta-2}v_n - |u|^\alpha v|^{\beta-2}v\Big|_{\frac{\alpha+\beta}{\alpha+\beta-1}}\Big]\|(\phi,\varphi)\|.$$

Now by Rellich theorem, $u_n \to u$ in $L^p(\Omega)\cap L^{\alpha+\beta-1}(\Omega)$ and $v_n \to v$ in $L^q(\Omega)\cap L^{\alpha+\beta-1}(\Omega)$. We then conclude by using Theorem A_2 in [8] and Lemma 6. $\square$

Lemma 8. *The functional* J *satisfies the Palais-Smale condition,* $\forall\lambda,\mu,\delta \in \mathbb{R}$. *That is, every sequence* $(u_n,v_n) \subset X$ *such that* $\big(J(u_n,v_n)\big)$ *is bounded and* $J'(u_n,v_n) \to 0$ *as* $n\to\infty$, *has a convergent subsequence.*

Proof. Let $(u_n,v_n)\subset X$ such that

$$d:=\sup_n|J(u_n,v_n)|<\infty \quad \text{and} \quad J'(u_n,v_n)\to 0,\ n\to\infty.$$

First we show that (u_n,v_n) is bounded. We will distinguish two cases, depending on the sign of δ.

Case 1: $\delta\leqslant 0$.
We deduce from (5) that

$$\begin{aligned}\big\langle J'(u_n,v_n),(u_n,0)\big\rangle &= \|u_n\|^2-\lambda|u_n|_p^p-\frac{\delta\alpha}{\alpha+\beta}\int_\Omega|u_n|^\alpha|v|^\beta dx\\ &\geqslant \|u_n\|^2-\lambda|u_n|_p^p.\end{aligned}$$

Hence for n big enough we have

$$\|u_n\|^2-\lambda|u_n|_p^p\leqslant\|u_n\|,$$

which implies, since $p<2$, that (u_n) is bounded.
We easily deduce from (4) and (5) that

$$\begin{aligned}-J(u_n,v_n)+\frac{1}{\alpha+\beta}\big\langle J'(u_n,v_n),(u_n,v_n)\big\rangle &= \Big(\frac{1}{\alpha+\beta}-\frac{1}{2}\Big)\|u_n\|^2+\Big(\frac{1}{2}-\frac{1}{\alpha+\beta}\Big)\|v_n\|^2\\ &\quad+\lambda\Big(\frac{1}{p}-\frac{1}{\alpha+\beta}\Big)|u_n|_p^p+\mu\Big(\frac{1}{q}-\frac{1}{\alpha+\beta}\Big)|v_n|_q^q.\end{aligned}$$

Since (u_n) is bounded, $q<2<\alpha+\beta$ and

$$-J(u_n,v_n)+\frac{1}{\alpha+\beta}\big\langle J'(u_n,v_n),(u_n,v_n)\big\rangle\leqslant d+\|(u_n,v_n)\|$$

for n big enough, we easily deduce that (v_n) is bounded.

Case 2: $\delta>0$.
(5) implies

$$\begin{aligned}\big\langle -J'(u_n,v_n),(0,v_n)\big\rangle &= \|v_n\|^2+\mu|v_n|_q^q+\frac{\delta\beta}{\alpha+\beta}\int_\Omega|u_n|^\alpha|v|^\beta\\ &\geqslant \|v_n\|^2+\mu|v_n|_q^q.\end{aligned}$$

So for n big enough $\|v_n\|^2+\mu|v_n|_q^q\leqslant\|v_n\|$, and (v_n) is bounded.
Again from (4) and (5) we have

$$\begin{aligned}J(u_n,v_n)-\frac{1}{\alpha+\beta}\big\langle J'(u_n,v_n),(u_n,v_n)\big\rangle &= \Big(\frac{1}{2}-\frac{1}{\alpha+\beta}\Big)\|u_n\|^2+\Big(\frac{1}{\alpha+\beta}-\frac{1}{2}\Big)\|v_n\|^2\\ &\quad+\lambda\Big(\frac{1}{\alpha+\beta}-\frac{1}{p}\Big)|u_n|_p^p+\mu\Big(\frac{1}{\alpha+\beta}-\frac{1}{q}\Big)|v_n|_q^q.\end{aligned}$$

Hence for n big enough

$$\begin{aligned}\Big(\frac{1}{2}-\frac{1}{\alpha+\beta}\Big)\|u_n\|^2+\Big(\frac{1}{\alpha+\beta}-\frac{1}{2}\Big)\|v_n\|^2+\lambda\Big(\frac{1}{\alpha+\beta}-\frac{1}{p}\Big)|u_n|_p^p&\\ +\mu\Big(\frac{1}{\alpha+\beta}-\frac{1}{q}\Big)|v_n|_q^q\leqslant d+\|(u_n,v_n)\|.&\end{aligned}$$

And this also implies, since (v_n) is bounded, that (u_n) is bounded.

Now we have, up to a subsequence, $u_n\rightharpoonup u$ and $v_n\rightharpoonup v$ in $H_0^1(\Omega)$, and by Rellich theorem $u_n \to u$ in $L^p(\Omega) \cap L^{\alpha+\beta}(\Omega)$ and $v_n \to v$ in $L^q(\Omega) \cap L^{\alpha+\beta}(\Omega)$. By using (5) one more time, we have

$$\|u_n-u\|^2 = \big\langle J'(u_n,v_n)-J'(u,v),(u_n-u,0)\big\rangle+\lambda\int_\Omega\big(|u_n|^{p-2}u_n-|u|^{p-2}u\big)(u_n-u)dx$$
$$+\frac{\delta\alpha}{\alpha+\beta}\int_\Omega\Big(|u_n|^{\alpha-2}u_n|v_n|^\beta-|u|^{\alpha-2}u|v|^\beta\Big)\big(u_n-u\big)dx$$

and

$$\|v_n-v\|^2 = -\big\langle J'(u_n,v_n)-J'(u,v),(0,v_n-v)\big\rangle-\mu\int_\Omega\big(|v_n|^{q-2}v_n-|v|^{q-2}v\big)(v_n-v)dx$$
$$-\frac{\delta\beta}{\alpha+\beta}\int_\Omega\Big(|u_n|^{\alpha}|v_n|^{\beta-2}v_n-|u|^{\alpha}|v|^{\beta-2}v\Big)\big(v_n-v\big)dx.$$

It is clear that

$$\big\langle J'(u_n,v_n)-J'(u,v),(u_n-u,0)\big\rangle\to 0,\quad n\to\infty,$$
$$\big\langle J'(u_n,v_n)-J'(u,v),(0,v_n-v)\big\rangle\to 0,\quad n\to\infty.$$

Hence by using the Hölder inequality, Lemma 6 and Theorem $A2$ in [8], we finally deduce that $(u_n,v_n)\to(u,v)$ in X. □

In the following we consider an orthonormal basis (e_j) of $H_0^1(\Omega)$.

3.1. **Finding large energy solutions.** We consider the $\tau-$topology on $X = Y \oplus Z$, where Y and Z are defined by:

$$Y := \{0\}\times H_0^1(\Omega) \;\text{ and }\; Z := H_0^1(\Omega)\times\{0\}.$$

Lemma 9. *For every $\delta \geqslant 0$, $\lambda,\mu\in\mathbb{R}$, J is $\tau-$upper semicontinuous..*

Proof. Let $(u_n,v_n)\subset X$ and $C\in\mathbb{R}$ such that $(u_n,v_n)\xrightarrow{\tau}(u,v)$ in X and $J(u_n,v_n)\geqslant C$. By the definition of the $\tau-$topology, $u_n\to u$ in $H_0^1(\Omega)$.
$J(u_n,v_n)\geqslant c$ implies, since $\delta\geqslant 0$, that

$$\frac{1}{2}\|v_n\|^2+C\leqslant\frac{1}{2}\|u_n\|^2-\frac{\lambda}{p}|u_n|_p^p-\frac{\mu}{q}|v_n|_q^q.$$

One can then easily verify, since (u_n) is bounded and $q<2$, that (v_n) is bounded. Up to a subsequence, we may assume that $u_n\to u$ in $L^p(\Omega)$ and $v_n\to v$ in $L^q(\Omega)$, $u_n(x)\to u(x)$ and $u_n(x)\to u(x)$ for almost every $x\in\Omega$. By using Fatou's Lemma and the weak lower semicontinuity of the norm $\|\cdot\|$, we deduce from the inequality

$$-C\geqslant -J(u_n,v_n) = -\frac{1}{2}\|u_n\|^2+\frac{1}{2}\|v_n\|^2+\frac{\lambda}{p}|u_n|_p^p+\frac{\mu}{q}|v_n|_q^q+\frac{\delta}{\alpha+\beta}\int_\Omega|u_n|^\alpha|v_n|^\beta dx,$$

that $-C\geqslant -J(u,v)$. □

Proof of Theorem 1. We recall that

$$Y_k = Y\oplus\Big(\bigoplus_{j=0}^{k}\mathbb{R}e_j\times\{0\}\Big)\;\text{ and }\; Z_k=\overline{\bigoplus_{j=k}^{\infty}\mathbb{R}e_j}\times\{0\}.$$

We first remark that the function $(u,v)\mapsto |u|^{\alpha+\beta}|v|^{\alpha+\beta}$ satisfies the so-called Ambrosetti-Rabinowitz superquadracity condition. Hence it is well known that there are constants $a_1,a_2>0$ such that

$$|u|^{\alpha+\beta}|v|^{\alpha+\beta}\geqslant a_1\big(|u|^{\alpha+\beta}+|v|^{\alpha+\beta}\big)-a_2.$$

Let $(u,v)\in Y_k$. Since on the space $\bigoplus_{j=0}^{k}\mathbb{R}e_j$ all norms are equivalent, we have

$$J(u,v)\leqslant \frac{1}{2}\|u\|^2-\frac{1}{2}\|v\|^2-\frac{\lambda c_1}{p}\|u\|^p-\frac{\mu}{q}|v|_q^q-\frac{\delta a_1 c_2}{\alpha+\beta}\|u\|^{\alpha+\beta}-\frac{\delta a_1}{\alpha+\beta}|v|_{\alpha+\beta}^{\alpha+\beta}+\frac{\delta a_2}{\alpha+\beta}|\Omega|,$$

where $c_1>0$ and $c_2>0$ are constant. We can easily deduce, since $\delta>0$ and $p,q<2<\alpha+\beta$, that $J(u,v)\to-\infty$ as $\|(u,v)\|\to\infty$. Thus condition (A_1) of Theorem 4 is satisfied for ρ_k big enough.

Let $(u,0)\in Z_k$. Then we have

$$J(u,0)=\frac{1}{2}\|u\|^2-\frac{\lambda}{p}|u|_p^p.$$

If $\lambda\leqslant 0$, then $J(u,0)\geqslant\frac{1}{2}\|u\|^2$ and condition (A_2) of Theorem 4 is clearly satisfied. Assume that $\lambda>0$, then

$$J(u,0)\geqslant\frac{1}{2}\|u\|^2-\frac{\lambda}{p}\theta_k^p\|u\|^p,$$

where

$$\theta_k:=\sup_{\substack{w\in\overline{\bigoplus_{j=k}^{\infty}\mathbb{R}e_j}\\ \|w\|=1}}|w|_p. \tag{6}$$

Since by [8] Lemma 3.8, $\theta_k\to 0$ as $k\to\infty$, one can easily verify that $J(u,0)\to\infty$ as $k\to\infty$ if $\|u\|=r_k:=\left(\lambda\theta_k^p\right)^{\frac{1}{p-2}}$. Hence condition (A_2) of Theorem 4 is satisfied. We then conclude by using Lemmas 9, 8 and Theorem 4. □

3.2. **Finding small energy solutions.** In this subsection we define

$$Y:=H_0^1(\Omega)\times\{0\}\quad\text{and}\quad Z:=\{0\}\times H_0^1(\Omega),$$

and we consider the $\tau-$topology on $X=Y\oplus Z$.

Lemma 10. *For every $\delta\leqslant 0$, $\lambda,\mu\in\mathbb{R}$, J is $\tau-$lower semicontinuous.*

Proof. Let $(u_n,v_n)\subset X$ and $C\in\mathbb{R}$ such that $(u_n,v_n)\xrightarrow{\tau}(u,v)$ in X and $J(u_n,v_n)\leqslant C$. It is clear that $v_n\to v$ in $H_0^1(\Omega)$. Since $\delta\leqslant 0$ we have

$$C\geqslant J(u_n,v_n)\geqslant\frac{1}{2}\|u_n\|^2-\frac{1}{2}\|v_n\|^2-\frac{\lambda}{p}|u_n|_p^p-\frac{\mu}{q}|v_n|_q^q,$$

which implies, since (v_n) is bounded and $p<2$, that (u_n) is bounded. We then get the conclusion, as in the proof of Lemma 9, by using one more time Fatou's Lemma and the weak lower semicontinuity of the norm $\|\cdot\|$. □

Proof of Theorem 2. We recall that

$$Y^k:=\overline{\bigoplus_{j=k}^{\infty}\mathbb{R}e_j}\times\{0\}\quad\text{and}\quad Z^k:=\Big(\bigoplus_{j=0}^{k}\mathbb{R}e_j\times\{0\}\Big)\oplus Z.$$

Let $(u,0)\in Y^k$. Then, since $\lambda>0$, we have

$$\begin{aligned}J(u,0)&=\frac{1}{2}\|u\|^2-\frac{\lambda}{p}|u|_p^p\\&\geqslant\frac{1}{2}\|u\|^2-\frac{\lambda}{p}\theta_k^p\|u\|^p,\end{aligned}$$

where θ_k is defined by (6). We choose

$$\sigma_k := 2\Big(\frac{2\lambda}{p}\theta_k^p\Big)^{\frac{1}{2-p}}.$$

Therefore for every $(u,0) \in Y^k$ such that $\|u\| = \sigma_k$, we have

$$J(u,0) \geqslant \Big(\frac{\lambda}{p}\theta_k^p\Big)^{\frac{2}{2-p}} 2^{\frac{p}{2-p}}\big(2^2 - 2^p\big) > 0 \text{ (we recall that } p < 2).$$

On the other hand, it is clear that

$$\frac{1}{2}\|u\|^2 \geqslant J(u,0) \geqslant -\frac{\lambda}{p}\theta_k^p\|u\|^p.$$

Hence, for every $(u,0) \in Y^k$ such that $\|u\| \leqslant \sigma_k$ we have

$$\frac{1}{2}\sigma_k^2 \geqslant J(u,0) \geqslant -\frac{\lambda}{p}\theta_k^p\sigma_k^p,$$

which implies, since $\theta_k \to 0$ that $J(u,0) \to 0$ as $k \to \infty$.
We have then proved that conditions (B_1) and (B_3) of Theorem 5 are satisfied.

Let $(u,v) \in Z^k$. Since the norms $\|\cdot\|$ and $|\cdot|_p$ are equivalent on $\bigoplus_{j=0}^{k} \mathbb{R}e_j$, $\delta \leqslant 0$, $\mu \geqslant 0$ and $H_0^1(\Omega)$ continuously embeds into $L^{\alpha+\beta}(\Omega)$, we have

$$J(u,v) \leqslant \frac{1}{2}\|u\|^2 - \frac{\lambda c_1}{p}\|u\|^p - \frac{\delta c}{\alpha+\beta}\|u\|^{\alpha+\beta} - \frac{1}{2}\|v\|^2 - \frac{\delta c}{\alpha+\beta}\|v\|^{\alpha+\beta},$$

where $c_1 > 0$ and $c > 0$ are constants. It is then easy to verify, since $\lambda > 0$, that condition (B_2) of Theorem 5 is satisfied for s_k small enough.
By using Lemmas 7, 8, 10 we can apply Theorem 5 and get the result. □

Remark 11. *Assume that (u,v) is a solution of $(S_{\lambda,\mu,\delta})$. Then*

$$\big\langle J'(u,v),(u,v)\big\rangle = \|u\|^2 - \|v\|^2 - \lambda|u|_p^p - \mu|v|_q^q - \delta\int_\Omega |u|^\alpha|v|^\beta dx = 0.$$

And this implies that

$$J(u,v) = \lambda\Big(\frac{1}{2}-\frac{1}{p}\Big)|u|_p^p + \mu\Big(\frac{1}{2}-\frac{1}{q}\Big)|v|_q^q + \delta\Big(\frac{1}{2}-\frac{1}{\alpha+\beta}\Big)\int_\Omega |u|^\alpha|v|^\beta dx.$$

Since $p,q < 2 < \alpha+\beta$, it follows that:

(1) *There is no solution of $(S_{\lambda,\mu,\delta})$ with positive energy if $\delta \leqslant 0$, $\lambda \geqslant 0$, $\mu \geqslant 0$.*
(2) *There is no solution of $(S_{\lambda,\mu,\delta})$ with negative energy if $\delta > 0$, $\lambda \leqslant 0$, $\mu \leqslant 0$.*

References

[1] A. Ambrosetti, H. Brézis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Anal.* 122 (1994), no. 2, 519-543.
[2] T. Barstch and M. Willem, On an elliptic equation with concave and convex nonlinearities. *Proc. Amer. Math. Soc.* 123 (1995), no. 11, 3555-3561.
[3] C. J. Batkam and F. Colin, Generalized fountain theorem and application to strongly indefinite semilinear problems, submitted
[4] T.-S. Hsu and H.-L. Lin, Multiple positive solutions for a critical elliptic system with concave-convex nonlinearities, *Proc. Roy. Soc. Edinburgh Sect.* A 139 (2009), no. 6, 1163-1177.
[5] T.-F. Wu, The Nehari manifold for a semilinear elliptic system involving sign-changing weight functions, *Nonlinear Anal.* 68 (2008) 1733-1745.
[6] K. Adriouch and A. EL Hamidi, The Nehari manifold for systems of nonlinear elliptic equations, *Nonlinear Anal.* 64 (2006) 21492164.

[7] Y. Bozhkov and E. Mitidieri, Existence of multiple solutions for quasilinear systems via fibering method, J. *Differential Equations* 190 (2003) 239-267.
[8] Willem M., *Minimax Theorems*, Birkhauser, Boston (1996).

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