ON MAXIMAL GREEN SEQUENCES

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Abstract. Maximal green sequences are particular sequences of quiver mutations which were introduced by Keller in the context of quantum dilogarithm identities and independently by Cecotti-Córdova-Vafa in the context of supersymmetric gauge theory. Our aim is to initiate a systematic study of these sequences from a combinatorial point of view.

Interpreting maximal green sequences as paths in various natural posets arising in representation theory, we prove the finiteness of the number of maximal green sequences for cluster finite quivers, affine quivers and acyclic quivers with at most three vertices. We also give results concerning the possible numbers and lengths of these maximal green sequences.

Finally we describe an algorithm for computing maximal green sequences for arbitrary valued quivers which we used to obtain numerous explicit examples that we present.

Contents

Introduction

Maximal green sequences are maximal chains in a partially ordered set that arises from a cluster exchange graph once an initial seed is fixed. The name "*maximal green sequence*" appears in [Kel11b] where these sequences are used to obtain quantum dilogarithm identities. Moreover, the same sequences appear in theoretical physics where they yield the complete spectrum of a BPS particle, see [CCV11, §4.2].

The partial order relation has been studied by Happel and Unger on a subgraph of the cluster exchange graph [Ung96a, Ung96b, HU05], and recently a number of

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representation-theoretic interpretations of the poset structure of the whole cluster exchange graph has been given [KQ12, KY12, IR12].

The theory of cluster algebras is related to various other fields, and thus the cluster exchange graph can be interpreted in many ways. For instance one can view it as a generalised associahedron, which is known to carry a poset structure (the Tamari poset). While there are certainly a lot more interesting connections, we will focus in this paper mainly on the combinatorial description of the poset structure by quiver mutations as given in [Kel11b], and we intend to initiate a systematic study of maximal green sequences applying representation-theoretic techniques.

The main questions addressed in this paper are: to find sufficient criteria for the existence of maximal green sequences (true for acyclic quivers but not in general) and to study the finiteness of the number of maximal green sequences and their possible lengths.

Organisation of the article. In Section 1, we introduce the notion of maximal green sequences in elementary terms and present some general results. When the proofs do not require any further background we present them in this section. When they do require some additional background, they are postponed to Section 7.

The short Section 2 makes the appearance of maximal green sequences explicit in the context of theoretical physics.

In Section 3, we study maximal green sequences for quivers of finite cluster type. As before, the proofs requiring additional background are postponed to Section 8.

Section 4 presents an analysis of the maximal green sequences for acyclic quivers of infinite representation types; the corresponding proofs are found in Section 9.

The representation-theoretical background underlying the proofs and (part of) the motivations of this article can be found in Section 5 where we recall the various connections between maximal green sequences and some classical posets in representation theory.

In this spirit, we present in Section 6 additional results on the connections between maximal green sequences and the classical Happel-Unger's poset of tilting modules over an algebra, see [HU05].

Sections 7–9 contain the missing proofs.

Finally, Appendix A presents an algorithm that we used for computing numerous explicit examples which can be found in Appendix B. This latter appendix also contain results concerning maximal green sequences for valued quivers; except in this very last part of the article, valued quivers were not considered since the theoretical context for their study is still conjectural under several aspects.

1. Green sequences

Without further specification, *quivers* will always be finite connected oriented graphs and *cluster quivers* will be quivers without loops nor oriented 2-cycles. A quiver is called *acyclic* if it has no oriented cycles. Given a quiver Q, we denote by Q_0 its set of vertices and by Q_1 its set of arrows.

1.1. **Cluster algebras.**

Introduced in [FZ02], cluster algebras are commutative rings equipped with a distinguished set of generators, the *cluster variables*, gathered into possibly overlapping subsets of pairwise compatible variables, the *clusters*, defined recursively with a combinatorial process, the *mutation*. The dynamics of this mutation process are encoded in a combinatorial data, the *exchange matrix*.

An *exchange matrix* is a matrix $B = (b_{ij}) \in M_{n,n+m}(\mathbb{Z})$ for some $m, n \geqslant 0$ such that the *principal part* of B, that is, the square submatrix $B^0 = (b_{ij})_{1 \leqslant i,j \leqslant n} \in M_n(\mathbb{Z})$ is *skew-symmetrisable*, that is, there exists a diagonal matrix $D \in M_n(\mathbb{Z})$

with positive diagonal entries such that DB^0 is skew-symmetric. Abusing terminology we say that B itself is *skew-symmetrisable*, or that it is *skew-symmetric* when B^0 is so.

Given a skew-symmetrisable exchange matrix $B \in M_{n,n+m}(\mathbb{Z})$, we denote by $\mathcal{A}_B$ the corresponding cluster algebra, see [FZ07] for details.

Definition 1.1 (Matrix mutation)**.** Let $B \in M_{n,n+m}(\mathbb{Z})$ be skew-symmetrisable. Then for any $1 \leqslant k \leqslant n$, the *mutation of* B *in the direction* k is the skew-symmetrisable matrix $\mu_k(B) = (b'_{ij}) \in M_{n,n+m}(\mathbb{Z})$ given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [b_{ik}]_-[b_{kj}]_- & \text{otherwise,} \end{cases}$$

where $[x]_+ = \max(x, 0)$ and $[x]_- = \min(x, 0)$ for any $x \in \mathbb{Z}$.

It is easy to see that $\mu_k(\mu_k(B)) = B$ for any $1 \leqslant k \leqslant n$ and that $\mu_k(B)$ is skew-symmetric if and only if B is skew-symmetric. In this latter case, we say that $\mathcal{A}_B$ is *simply-laced* and it is usually more convenient to use the formalism of *ice quivers* instead of exchange matrices.

1.2. **Ice quivers and their mutations.** An *ice quiver* is a pair (Q, F) where Q is a cluster quiver and $F \subset Q_0$ is a (possibly empty) subset of vertices called the *frozen vertices* such that there are no arrows between them. For simplicity, we always assume that $Q_0 = \{1, \ldots, n+m\}$ and that $F = \{n+1, \ldots, n+m\}$ for some integers $m, n \geqslant 0$. If F is empty, we simply write Q for $(Q, \varnothing)$.

We associate to (Q, F) its *adjacency matrix* $B(Q, F) = (b_{ij}) \in M_{n,n+m}(\mathbb{Z})$ such that

$$b_{ij} = |\{i \longrightarrow j \in Q_1\}| - |\{j \longrightarrow i \in Q_1\}|$$

for any $1 \leqslant i \leqslant n$ and any $1 \leqslant j \leqslant n+m$.

The map $(Q, F) \mapsto B(Q, F)$ induces a bijection from the set of ice quivers to the set of skew-symmetric exchange matrices. Therefore, to any ice quiver (Q, F) we can associate the cluster algebra $\mathcal{A}_{(Q,F)} = \mathcal{A}_{B(Q,F)}$.

Definition 1.2 (Quiver mutation)**.** Let (Q, F) be an ice quiver and $k \in Q_0$ be a non-frozen vertex. The *mutation of* Q *at* k is defined as the ice quiver $(\mu_k(Q), F)$ where $\mu_k(Q)$ is obtained from Q by applying the following modifications:

1. For any pair of arrows $i \xrightarrow{a} k \xrightarrow{b} j$ in Q, add an arrow $i \xrightarrow{[ab]} j$ in $\mu_k(Q)$;
2. Any arrow $i \xrightarrow{a} k$ in Q is replaced by an arrow $i \xleftarrow{a^*} k$ in $\mu_k(Q)$;
3. Any arrow $k \xrightarrow{b} j$ in Q is replaced by an arrow $k \xleftarrow{b^*} j$ in $\mu_k(Q)$;
4. A maximal collection of 2-cycles is removed.

Then it is easy to see that for any non-frozen vertex $k \in Q_0$, the ice quiver $\mu_k(Q, F)$ is the ice quiver corresponding to the skew-symmetric matrix $\mu_k(B(Q, F))$.

Example 1.3. Figure 1 shows an example of successive quiver mutations.

Two ice quivers are called *mutation-equivalent* if one can be obtained from the other by applying a finite number of successive mutations at non-frozen vertices. Since mutations are involutive, this defines an equivalence relation on the set of ice quivers. The equivalence class of an ice quiver (Q, F) is called its *mutation class* and is denoted by $\mathrm{Mut}(Q, F)$.

Two ice quivers (Q, F) and (Q', F) sharing the same set of frozen vertices are called *isomorphic as ice quivers* if there is an isomorphism of quivers $\phi : Q \longrightarrow Q'$ fixing F. In this case, we write $(Q, F) \simeq (Q', F)$ and we denote by $[(Q, F)]$ the isomorphism class of the ice quiver (Q, F).

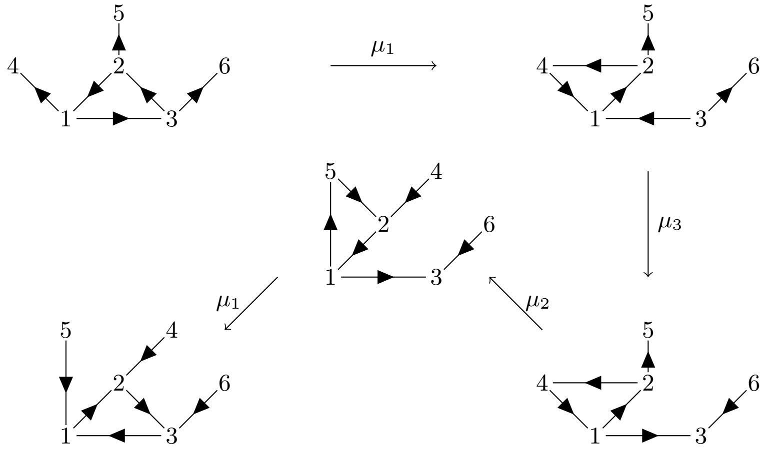


FIGURE 1. An example of quiver mutations.

1.3. **Green sequences.** From now on, Q will always denote a cluster quiver and we fix a copy $Q'_0 = \{i' \mid i \in Q_0\}$ of the set Q_0 of vertices in Q. We will identify Q_0 with the set of integers $\{1, \ldots, n\}$ and Q'_0 with $\{n+1, \ldots, 2n\}$ in such a way that for any $1 \leqslant i \leqslant n$, we have $i' = n + i$.

Definition 1.4 (Framed and coframed quivers)**.** The *framed quiver* associated to Q is the quiver $\widehat{Q}$ such that:

$$\widehat{Q}_0 = Q_0 \sqcup \left\{i' \mid i \in Q_0\right\},$$
$$\widehat{Q}_1 = Q_1 \sqcup \left\{i \longrightarrow i' \mid i \in Q_0\right\}.$$

The *coframed quiver* associated to Q is the quiver $\check{Q}$ such that:

$$\check{Q}_0 = Q_0 \sqcup \left\{i' \mid i \in Q_0\right\},$$
$$\check{Q}_1 = Q_1 \sqcup \left\{i' \longrightarrow i \mid i \in Q_0\right\}.$$

If Q is an arbitrary cluster quiver, both $\widehat{Q}$ and $\check{Q}$ are naturally ice quivers with frozen vertices Q'_0. Therefore, by $\mathrm{Mut}(\widehat{Q})$ we always mean the mutation class of the ice quiver $(\widehat{Q}, Q'_0)$.

Definition 1.5 (Green and red vertices)**.** Let $R \in \mathrm{Mut}(\widehat{Q})$. A non-frozen vertex $i \in R_0$ is called *green* if

$$\left\{j' \in Q'_0 \mid \exists\, j' \longrightarrow i \in R_1\right\} = \varnothing.$$

It is called *red* if

$$\left\{j' \in Q'_0 \mid \exists\, i \longrightarrow j' \in R_1\right\} = \varnothing.$$

If R is an ice quiver in $\mathrm{Mut}(\widehat{Q})$ with adjacency matrix $B = (b_{ij}) \in M_{n,2n}(\mathbb{Z})$, the submatrix $\mathbf{c}(R) = (b_{i,n+j})_{1 \leqslant i,j \leqslant n}$ is called the $\mathbf{c}$-*matrix* of R. For any non-frozen vertex $i \in Q_0$, its ith row $\mathbf{c}_i(R)$ is called the *ith* $\mathbf{c}$-*vector* of R and it encodes the number of arrows between i and the frozen vertices in R. For instance, we have $\mathbf{c}(\widehat{Q}) = I_n$ and $\mathbf{c}(\check{Q}) = -I_n$. For more details on $\mathbf{c}$-vectors, we refer the reader to [FZ07] where they were introduced and to [NZ12, NC12, ST12, Nag11, Kel12] where they were studied.

With this terminology, for a quiver $R \in \mathrm{Mut}(\widehat{Q})$, a vertex $i \in Q_0$ is green if and only if the ith $\mathbf{c}$-vector $\mathbf{c}_i(R)$ has only non-negative entries and it is red if and only if $\mathbf{c}_i(R)$ has only non-positive entries.

Given a quiver $R \in \mathrm{Mut}(\widehat{Q})$, we denote by $g(R)$ the number of green vertices in R. Note that this number only depends on $[R]$ so that we set $g([R]) = g(R)$.

Theorem 1.6. *Let Q be a cluster quiver and $R \in \mathrm{Mut}(\widehat{Q})$. Then any non-frozen vertex in R_0 is either green or red.*

Proof. Let $R \in \mathrm{Mut}(\widehat{Q})$. We need to prove that each row of the **c**-matrix of R is non-zero and its entries are either all non-negative or all non-positive. This result, known as the *sign-coherence for* **c***-vectors*, was established in the case of skew-symmetric exchange matrices in [DWZ10]. ☐

For skew-symmetrisable exchange matrices the sign-coherence for **c**-vectors is still conjectural, and so is the non-simply-laced analogue of Theorem 1.6.

Example 1.7. In $\widehat{Q}$, every non frozen vertex is green. In $\check{Q}$, any non-frozen vertex is red.

Definition 1.8 (Green sequences, [Kel11b])**.** A *green sequence for Q* is a sequence $\mathbf{i} = (i_1, \ldots, i_l) \subset Q_0$ such that i_1 is green in $\widehat{Q}$ and for any $2 \leqslant k \leqslant l$, the vertex i_k is green in $\mu_{i_{k-1}} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$. The integer l is called the *length* of the sequence **i** and is denoted by $\ell(\mathbf{i})$.

A green sequence $\mathbf{i} = (i_1, \ldots, i_l)$ is called *maximal* if every non-frozen vertex in $\mu_{\mathbf{i}}(\widehat{Q})$ is red, where $\mu_{\mathbf{i}}(\widehat{Q}) = \mu_{i_l} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$.

We denote by

$$\mathrm{green}(Q) = \{\mathbf{i} = (i_1, \ldots, i_l) \subset Q_0 \mid \mathbf{i} \text{ is a maximal green sequence for } Q\}$$

the set of all maximal green sequences for Q.

Example 1.9. Figure 2 shows that the sequence of mutations considered in Figure 1 is a maximal green sequence for the oriented triangle. Frozen vertices are coloured in blue, green vertices in green and red vertices in red.

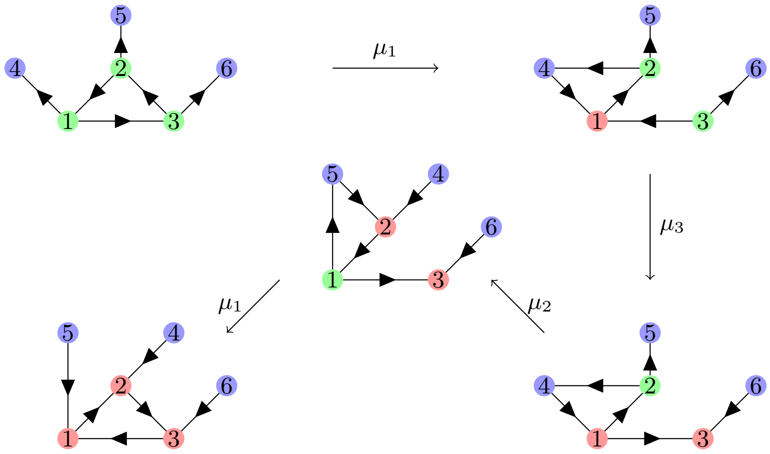


FIGURE 2. An example of a maximal green sequence.

We refer the reader willing to compute more examples to Bernhard Keller's java applet [Kel] or to the QUIVER MUTATION EXPLORER [DP12].

1.4. **The oriented exchange graph.** The following proposition will be proved in Section 7.

Proposition 1.10. *Let* Q *be a cluster quiver and let* $R \in \mathrm{Mut}(\widehat{Q})$*.*

1. *If all the non-frozen vertices in* R_0 *are green, then* $R \simeq \widehat{Q}$ *as ice quivers.*
2. *If all the non-frozen vertices in* R_0 *are red, then* $R \simeq \check{Q}$ *as ice quivers.*

Definition 1.11 (Oriented exchange graph)**.** The oriented exchange graph of Q is the oriented graph $\overrightarrow{\mathbf{EG}}(Q)$ whose vertices are the isomorphism classes $[R]$ of ice quivers $R \in \mathrm{Mut}(\widehat{Q})$ and where there is an arrow $[R] \longrightarrow [R']$ in $\overrightarrow{\mathbf{EG}}(Q)$ if and only if there exists a green vertex $k \in R_0$ such that $\mu_k(R) \simeq R'$.

In [FZ03], Fomin and Zelevinsky introduced the (unoriented) *exchange graph* of Q as the dual graph $\mathbf{EG}(Q)$ of the cluster complex $\Delta(\mathcal{A}_Q)$ of the cluster algebra $\mathcal{A}_Q$ associated with Q. Vertices in $\mathbf{EG}(Q)$ are labelled by the clusters in $\mathcal{A}_Q$ and two clusters in $\mathbf{EG}(Q)$ are joined by an edge if and only if they differ by a single cluster variable. Then $\overrightarrow{\mathbf{EG}}(Q)$ is an orientation of $\mathbf{EG}(Q)$ corresponding to the choice of an initial seed in $\mathcal{A}_Q$ with exchange matrix $B(Q)$. The orientation is defined as follows. Let $\mathbf{x}$ and $\mathbf{x}'$ be two adjacent clusters in $\mathbf{EG}(Q)$ corresponding respectively to $[R]$ and $[R']$ in $\overrightarrow{\mathbf{EG}}(Q)$. Assume that $\mathbf{x}$ and $\mathbf{x}'$ differ by a single cluster variable x_i, so that $R' \simeq \mu_i(R)$. Then the edge joining $\mathbf{x}$ and $\mathbf{x}'$ in $\mathbf{EG}(Q)$ is oriented towards $\mathbf{x}'$ if i is green in R and towards $\mathbf{x}$ otherwise.

As $\mathbf{EG}(Q)$ is an n-regular graph, if $[R]$ is a vertex in $\overrightarrow{\mathbf{EG}}(Q)$, then there are $g([R])$ arrows starting at $[R]$ in $\overrightarrow{\mathbf{EG}}(Q)$ and $n - g([R])$ arrows ending at $[R]$ in $\overrightarrow{\mathbf{EG}}(Q)$ (which, by Theorem 1.6, correspond to the red vertices in R).

Corollary 1.12. *Let* Q *be a cluster quiver. Then:*

1. $\overrightarrow{\mathbf{EG}}(Q)$ *has a unique source, which is* $[\widehat{Q}]$*.*
2. $\overrightarrow{\mathbf{EG}}(Q)$ *has a sink if and only if* $[\check{Q}]$ *is a vertex in* $\overrightarrow{\mathbf{EG}}(Q)$ *and in this case* $[\check{Q}]$ *is the unique sink.*

Proof. $[\widehat{Q}]$ belongs to $\overrightarrow{\mathbf{EG}}(Q)$ by construction and it is a source in $\overrightarrow{\mathbf{EG}}(Q)$ since all the vertices in $\widehat{Q}$ are green. If $[R]$ is another source, then all the vertices in R are green and then it follows from Proposition 1.10 that $R \simeq \widehat{Q}$, proving the first point. Now if $[R]$ is a sink in $\overrightarrow{\mathbf{EG}}(Q)$, then all its vertices are red and therefore, it follows from Proposition 1.10 that $R \simeq \check{Q}$, proving the second point. Conversely, if $[\check{Q}]$ is in $\overrightarrow{\mathbf{EG}}(Q)$, then it is a sink since all its non-frozen vertices are red. □

The following statement rephrases Corollary 1.12:

Proposition 1.13. *Let* Q *be a cluster quiver. Then* $\mathrm{green}(Q) \neq \varnothing$ *if and only if there is a sink in* $\overrightarrow{\mathbf{EG}}(Q)$*. In this case, there is a natural bijection between* $\mathrm{green}(Q)$ *and the set of oriented paths in* $\overrightarrow{\mathbf{EG}}(Q)$ *from its unique source to its unique sink.* □

As it is explained in Section 5, $\overrightarrow{\mathbf{EG}}(Q)$ is isomorphic to the Hasse graph of various partially ordered sets. In particular, it has the following essential property:

Proposition 1.14. *Let* Q *be a cluster quiver. Then* $\overrightarrow{\mathbf{EG}}(Q)$ *has no oriented cycles.* □

1.5. **Existence, finiteness and lengths.** Let Q be a cluster quiver. We recall that if $\mathbf{i} = (i_1, \dots, i_l)$ is a green sequence for Q, then the integer l is called the *length* of $\mathbf{i}$ and is denoted by $\ell(\mathbf{i})$. For any $l \geqslant 0$, we set

$$\mathrm{green}_l(Q) = \{\mathbf{i} \in \mathrm{green}(Q) \mid \ell(\mathbf{i}) = l\},$$

$$\mathrm{green}_{\leqslant l}(Q) = \{\mathbf{i} \in \mathrm{green}(Q) \mid \ell(\mathbf{i}) \leqslant l\}$$

and

$$\ell_{\min}(Q) = \min\{l \geqslant 0 \mid \mathrm{green}_l(Q) \neq \varnothing\} \in \mathbb{Z}_{\geqslant 0},$$
$$\ell_{\max}(Q) = \max\{l \geqslant 0 \mid \mathrm{green}_l(Q) \neq \varnothing\} \in \mathbb{Z}_{\geqslant 0} \sqcup \{\infty\},$$

with the conventions that $\ell_{\min}(Q) = \ell_{\max}(Q) = 0$ if green(Q) is empty.

It is clear that if Q and Q' are isomorphic quivers, then the isomorphism $\phi : Q \longrightarrow Q'$ induces an isomorphism $\overrightarrow{\mathbf{EG}}(Q) \longrightarrow \overrightarrow{\mathbf{EG}}(Q')$ so that $\mathrm{green}_l(Q) = \mathrm{green}_l(Q')$ for any $l \geqslant 1$. The following proposition shows a similar result for oppositions:

Proposition 1.15. *Let Q be a cluster quiver. Then for any $l \geqslant 1$, there exists a natural bijection*

$$\mathrm{green}_l(Q) \leftrightarrow \mathrm{green}_l(Q^{\mathrm{op}}).$$

Proof. Let $\mathbf{i} = (i_1, \ldots, i_l)$ be a maximal green sequence. Then there exists $\pi \in \mathfrak{S}_{Q_0}$ such that $\mu_{\mathbf{i}}(\widehat{Q}) = \pi \cdot \check{Q}$. Moreover, since π fixes the frozen vertices and since the only arrows between frozen and non-frozen vertices in $\check{Q}$ are the $i' \longrightarrow i$ for $i \in Q_0$, the permutation π is uniquely determined. Therefore we have $\mu_{\pi^{-1}(i_1)} \circ \cdots \circ \mu_{\pi^{-1}(i_l)}(\check{Q}) = \widehat{Q}$ where $\pi^{-1}(i_l)$ is red in $\check{Q}$ and for any $2 \leqslant k \leqslant l$, the vertex $\pi^{-1}(i_k)$ is red in $\mu_{\pi^{-1}(i_{k-1})} \circ \cdots \circ \mu_{\pi^{-1}(i_l)}(\check{Q})$. Since the mutations commute with taking opposite quivers, $\pi^{-1}(i_l)$ is green in $(\check{Q})^{\mathrm{op}}$, the vertex $\pi^{-1}(i_k)$ is green in $\mu_{\pi^{-1}(i_{k-1})} \circ \cdots \circ \mu_{\pi^{-1}(i_l)}((\check{Q})^{\mathrm{op}})$ for any $2 \leqslant k \leqslant l$ and $\mu_{\pi^{-1}(i_l)} \circ \cdots \circ \mu_{\pi^{-1}(i_l)}((\check{Q})^{\mathrm{op}})$ has only red vertices. Since $(\check{Q})^{\mathrm{op}} = \widehat{Q^{\mathrm{op}}}$, it follows that $(\pi^{-1}(i_l), \ldots, \pi^{-1}(i_l))$ is a maximal green sequence for Q^{op}. We therefore get a map $\mathrm{green}_l(Q) \longrightarrow \mathrm{green}_l(Q^{\mathrm{op}})$ and applying the same argument to Q^{op}, we get its inverse. Therefore, it is a bijection. □

Lemma 1.16. *Let Q be a cluster quiver and let $R, R' \in \mathrm{Mut}(\widehat{Q})$ such that $[R] \longrightarrow [R']$ in $\overrightarrow{\mathbf{EG}}(Q)$. Then $g([R']) \geqslant g([R]) - 1$.*

Proof. Without loss of generality, we can assume that $R' = \mu_k(R)$ for some green vertex k in R. In order to prove the statement, it is enough to prove that any green vertex in R which is different from k is also green in R'. We let $B = B(R)$ and $B' = B(R')$ be the corresponding adjacency matrices. Let i be a green vertex in R and let f be a frozen vertex in R. Since i is green in R, we have $b_{if} \geqslant 0$ and also, since k is green in R, we have $b_{kf} \geqslant 0$. Therefore,

$$\begin{aligned} b'_{if} &= b_{if} + [b_{ik}]_+[b_{kf}]_+ - [b_{ik}]_-[b_{kf}]_- \\ &= b_{if} + [b_{ik}]_+[b_{kf}]_+ \\ &\geqslant b_{if} \geqslant 0 \end{aligned}$$

so that i is green in R'. □

Remark 1.17. Note that under the hypothesis of Lemma 1.16, it may happen that $g([R']) > g([R]) - 1$ since a red vertex in R can turn green in R', see for instance the penultimate mutation in Figure 2.

Corollary 1.18. *Let Q be a cluster quiver. If* green$(Q) \neq \varnothing$, *then $\ell_{\min}(Q) \geqslant |Q_0|$.*

Proof. By definition, in a maximal green sequence $\mathbf{i} = (i_1, \ldots, i_l)$, we have $g(\mu_{\mathbf{i}}(\widehat{Q})) = 0$ whereas $g(\widehat{Q}) = |Q_0|$. Therefore, it follows from Lemma 1.16 that $l \geqslant |Q_0|$. □

Example 1.19.
(1) Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$. Then $\mathbf{i} = (123)$ is a maximal green sequence and therefore $\ell_{\min}(Q) = 3 = |Q_0|$.

(2) Let $Q' = \mu_2(Q)$ be the cyclic quiver with three vertices. Then it is easily verified that $\ell_{\min}(Q') = 4 > 3$ so that Corollary 1.18 only provides a lower bound for $\ell_{\min}$.

Note also that these examples show that $\ell_{\min}$ is not invariant under mutations. The same will appear to be true for $\ell_{\max}$.

We recall that for a quiver Q, an *admissible numbering of Q_0 by sources* (resp. *by sinks*) is an n-tuple $(i_1, \dots, i_n)$ such that $Q_0 = \{i_1, \dots, i_n\}$ and where i_1 is a source (resp. a sink) in Q and such that for any $2 \leqslant k \leqslant n$, the vertex i_k is a source (resp. a sink) in $\mu_{i_{k-1}} \circ \cdots \circ \mu_{i_1}(Q)$.

Lemma 1.20. *Let Q be an acyclic quiver. Then any admissible numbering of Q_0 by sources is a maximal green sequence. In particular,* $\text{green}(Q) \neq \emptyset$ *and* $\ell_{\min}(Q) = |Q_0|$.

Proof. Since Q is acyclic, it is well-known that there is at least one admissible numbering of Q_0 by sources. Let $\mathbf{i} = (i_1, \dots, i_n)$ be such a numbering. Without loss of generality, we assume that this admissible numbering is $(1, \dots, n)$. For any $1 \leqslant k \leqslant n$, we let $B^{(k)}$ be the adjacency matrix of $R^{(k)} = \mu_k \circ \cdots \circ \mu_1(\widehat{Q})$ and $Q^{(k)} = \mu_k \circ \cdots \circ \mu_1(Q)$. We prove by induction on k that the green vertices in $R^{(k)}$ are precisely $\{k+1, \dots, n\}$.

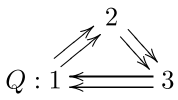
Let $i \neq k$ be non-frozen vertices and f be a frozen vertex. We have

$$b_{i,f}^{(k)} = b_{i,f}^{(k-1)} + [b_{i,k}^{(k-1)}]_+ [b_{k,f}^{(k-1)}]_+ - [b_{i,k}^{(k-1)}]_- [b_{k,f}^{(k-1)}]_-.$$

Since k is a source in $Q^{(k-1)}$, it follows that $b_{i,k}^{(k-1)} \leqslant 0$. Also, by induction hypothesis k is green in $R^{(k-1)}$ so that $b_{k,f}^{(k-1)} \geqslant 0$. Therefore, $b_{i,f}^{(k)} = b_{i,f}^{(k-1)}$ so that a non-frozen vertex $i \neq k$ is green (or red, respectively) in $R^{(k)}$ if and only if it is green (or red, respectively) in $R^{(k-1)}$. Moreover, $b_{k,k+n}^{(k)} = -b_{k,k+n}^{(k-1)} = -b_{k,k+n} = -1$ so that k is red in $R^{(k)}$ whereas it was green in $R^{(k-1)}$. Thus, the green vertices in $R^{(k)}$ are exactly $\{k+1, \dots, n\}$. In particular, $(1, \dots, n)$ is a maximal green sequence for Q. □

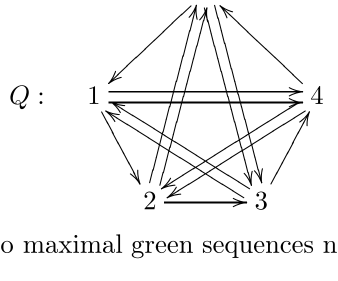
In general, it is not true that $\text{green}(Q) \neq \emptyset$ for an arbitrary quiver Q. For instance, we have the following proposition, which will be proved in Section 7:

Proposition 1.21. *The quiver*



has no maximal green sequences.

More generally, a representation-theoretic criterion for the non-existence of maximal green sequences is given in Proposition 7.1. This in particular enables us to show that the McKay quiver



considered in [TV10] has no maximal green sequences neither, see Example 7.2.

The quiver in Proposition 1.21 is the quiver associated with any triangulation of the once-punctured torus, see [FST08]. We will see in Section B.8 another example of a surface without boundary, namely the sphere with four punctures, for which there exist maximal green sequences.

1.6. **A conjecture on lengths.** Given a cluster quiver Q, the *empirical maximal length* is

$$\ell^0_{\max}(Q) = \max\{l \geqslant 1 \mid \mathrm{green}_k(Q) \neq \varnothing \text{ for any } k \text{ s.t. } \ell_{\min}(Q) \leqslant k \leqslant l\}$$

and we let

$$\mathrm{green}^0(Q) = \mathrm{green}_{\leqslant \ell^0_{\max}(Q)}(Q),$$

with the convention that $\ell^0_{\max}(Q) = 0$ if $\mathrm{green}(Q) = \varnothing$.

We always have $\ell^0_{\max}(Q) \leqslant \ell_{\max}(Q)$ and $\mathrm{green}^0(Q) \subset \mathrm{green}(Q)$ but based on the various examples we computed, we conjecture the following.

Conjecture 1.22. *Let Q be a cluster quiver. Then $\ell_{\max}(Q) = \ell^0_{\max}(Q)$ and* $\mathrm{green}(Q) = \mathrm{green}^0(Q)$.

In other words, the set

$$\{l \in \mathbb{Z}_{\geqslant 0} \mid \mathrm{green}_l(Q) \neq \varnothing\}$$

is an interval in $\mathbb{Z}$.

The motivation for introducing the empirical maximal length is that it is easy to determine in practice: let l be the smallest integer such that $\mathrm{green}_l(Q) \neq \varnothing$ and $\mathrm{green}_{l+1}(Q) = \varnothing$, then $l = \ell^0_{\max}(Q)$. Therefore, if Conjecture 1.22 holds, it is enough to find such an l to determine $\mathrm{green}(Q)$. This is the strategy we use in Appendix B.

Note that Conjecture 1.22 does not hold true in the non-simply-laced case, as it appears for instance in Appendices B.1 or B.3.

2. Maximal green sequences and BPS quivers

As we already mentioned, maximal green sequences appear independently in theoretical physics, implicitly in [GMN09] or more explicitly in [ACC$^+$11, CCV11]. In order to make the connection clear, we present in this short section a precise dictionary between the formal definition we gave in the previous section, and the definition given in [CCV11, §4.2].

We fix a cluster quiver Q. Vertices in Q are called *nodes* in [CCV11].

For simplicity, we identify the set Q_0 of vertices with $\{1, \dots, n\}$. We let $\{\gamma_i\}_{1\leqslant i\leqslant n}$ denote the canonical basis of $\mathbb{Z}^n$. In the terminology of [CCV11], for any $R \in \mathrm{Mut}(\widehat{Q})$ and for any $1 \leqslant i \leqslant n$, the ith $\mathbf{c}$-vector $\mathbf{c}_i(R) \in \mathbb{Z}^n$ is called the *charge at node i*. Therefore, the charges in $\widehat{Q}$ are $\gamma_1, \dots, \gamma_n$.

For any quiver $R \in \mathrm{Mut}(\widehat{Q})$ and for any $1 \leqslant k \leqslant n$, the charge at node k in R is $\mathbf{c}_k(R) = \sum_{i=1}^n c_{k;i}(R)\gamma_i$ where $c_{k;i}(R) \in \mathbb{Z}$ for any i. It follows from the sign-coherence for $\mathbf{c}$-vectors (see Theorem 1.6) that either $c_{k;i}(R) \leqslant 0$ for every i, in which case k is green in R, or $c_{k;i}(R) \geqslant 0$ for every i, in which case k is red in R. Moreover, if k is green in R, then the $\mathbf{c}$-vectors of $\mu_k(R)$ are precisely given in terms of those of R by the rule for charges given in [CCV11, (4.4)].

Now, the sequences of mutations considered in [CCV11] for capturing complete spectra of BPS particles are those for which:

(G1) the initial quiver appears with node charges γ_i;
(G2) the final quiver appears with node charges $-\gamma_i$;
(G3) At each step we may mutate on any node whose charge can be expressed as $\gamma = \sum_i c_i\gamma_i$ where $c_i \geqslant 0$ for any $1 \leqslant i \leqslant n$.

Therefore, with our terminology, (G1) implies that the initial quiver R has only green vertices, so that $[R] = [\widehat{Q}]$ according to Proposition 1.10, (G2) implies that the final quiver R' has only red vertices so that $[R] = [\check{Q}]$ according to Proposition 1.10. Finally, (G3) says that at each step in the mutation sequence $\widehat{Q} \simeq R \xrightarrow{\mu_{i_1}} R^{(1)} \xrightarrow{\mu_{i_l}} \cdots \xrightarrow{\mu_{i_1}} R^{(l)} \simeq \check{Q}$, we mutated at a green vertex. Therefore, the sequences considered in [CCV11] are precisely the maximal green sequences of Q.

3. The finite cluster type

It was proved in [FZ03] that a cluster algebra $\mathcal{A}_Q$ associated with a cluster quiver Q has finitely many cluster variables if and only if Q is mutation-equivalent to a Dynkin quiver $\overrightarrow{\Delta}$. In this case, Q is called of *finite cluster type* and it is known that the number of cluster variables in $\mathcal{A}_Q$ equals the number of *almost positive roots* of the Dynkin quiver $\overrightarrow{\Delta}$, where the set $\Phi_{\geqslant -1}(\overrightarrow{\Delta})$ of almost positive roots of $\overrightarrow{\Delta}$ is the disjoint union of the set $\Phi_+(\overrightarrow{\Delta})$ of positive roots with the set of negative simple roots.

Theorem 3.1. *Let Q be a quiver of finite cluster type. Then*

$$|Q_0| \leqslant |\mathrm{green}(Q)| < \infty.$$

Proof. Since Q is of finite cluster type, the exchange graph $\mathbf{EG}(Q)$ is finite. Moreover, we know from Proposition 1.14 that $\overrightarrow{\mathbf{EG}}(Q)$ is acyclic. Hence, it contains only finitely many oriented paths and thus it follows from Proposition 1.13 that $\mathrm{green}(Q)$ is finite.

Now since $\overrightarrow{\mathbf{EG}}(Q)$ is a finite acyclic oriented graph, it necessarily has at least one sink and one source and by Corollary 1.12, it has a unique sink, corresponding to $[\widehat{Q}]$, and a unique source, corresponding to $[\check{Q}]$. The underlying graph of $\overrightarrow{\mathbf{EG}}(Q)$ is $|Q_0|$-regular so that there are exactly $|Q_0|$ distinct arrows starting at $[\widehat{Q}]$. Since $\overrightarrow{\mathbf{EG}}(Q)$ is finite, each of these arrows gives rise to at least one oriented path from the unique sink to the unique source and therefore we obtain at least $|Q_0|$ distinct oriented paths from the unique source to the unique sink in $\overrightarrow{\mathbf{EG}}(Q)$, that is, $|Q_0| \leqslant |\mathrm{green}(Q)|$. ☐

Remark 3.2.
(1) If Q is a cluster quiver such that $|Q_0| = 1$, then clearly $|\mathrm{green}(Q)| = 1$.
(2) If Q is a (connected) cluster quiver such that $|Q_0| = 2$, then it is shown in Lemma 4.1 that $\mathrm{green}(Q)$ has two elements of respective lengths 2 and 3 in the finite cluster type and a unique element, necessarily of length two, in the other cases.
(3) If Q is a cluster quiver of finite cluster type such that $|Q_0| > 2$, then it will appear in the examples that $|\mathrm{green}(Q)| > |Q_0|$ in general and, as it is seen for instance in the Appendix B.2 for linearly oriented quivers Q_n of type A_n, the cardinality $|\mathrm{green}(Q_n)|$ grows exponentially as a function of n.

Remark 3.3. If Q is of finite cluster type, then a rough analysis provides an upper bound for $\ell_{\max}(Q)$. Namely, if we set

$$\chi(Q) = \left|\left\{[R] \mid R \in \mathrm{Mut}(\widehat{Q})\right\}\right|,$$

then we have

$$\ell_{\max}(Q) \leqslant |Q_0| \cdot (|Q_0| - 1)^{\chi(Q)-2}.$$

Indeed, an oriented path on $\overrightarrow{\mathbf{EG}}(Q)$ starts at $[\widehat{Q}]$ where we have $|Q_0|$ choices of directions and then, it passes at most once through any vertex in $\overrightarrow{\mathbf{EG}}(Q)$ distinct from $[\widehat{Q}]$ and $[\check{Q}]$. There are $\chi(Q) - 2$ such vertices and at each such vertex $[R]$, there are at most $|Q_0|-1$ possible directions (since in order to leave $[R]$, we cannot use backwards the arrow we just used in order to arrive at $[R]$).

In general, these upper and lower bounds are not optimal but in the acyclic case, we can sharpen the result with the following theorem whose proof is given in Section 8:

Theorem 3.4. *Let Q be a Dynkin quiver. Then:*

(1) $\ell_{\min}(Q) = |Q_0|$,
(2) $\ell_{\max}(Q) = |\Phi_+(Q)|$,

where $\Phi_+(Q)$ is the set of positive roots *of Q.*

Example 3.5. We show below the oriented exchange graphs for the quivers in the mutation class of type A_3 (up to isomorphisms and opposition). The labels on the faces correspond to denominators of the cluster variables in the corresponding clusters expressed in the seed with exchange matrix $B(Q)$. The unique source is circled in green and the unique sink is circled in red.

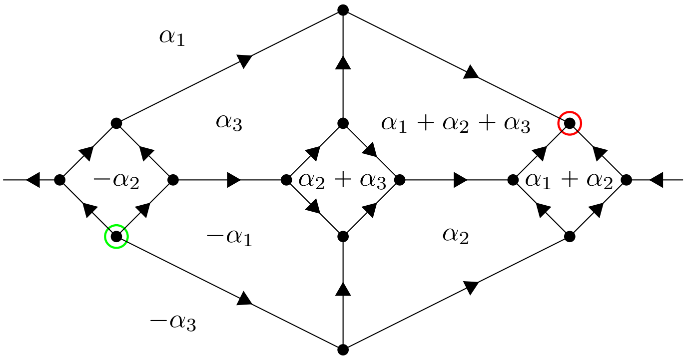


FIGURE 3. The oriented exchange graph of $1 \longrightarrow 2 \longrightarrow 3$.

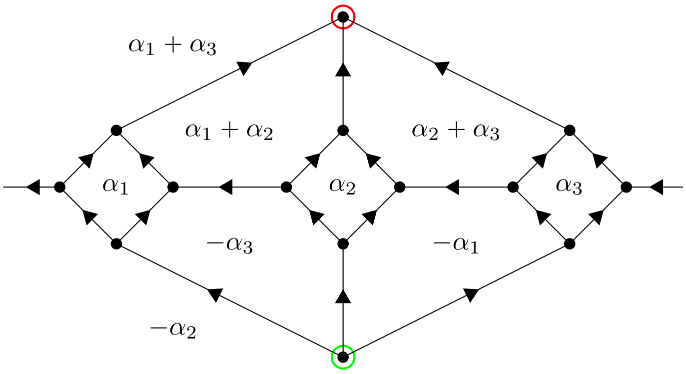


FIGURE 4. The oriented exchange graph of the cyclic quiver with 3 vertices.

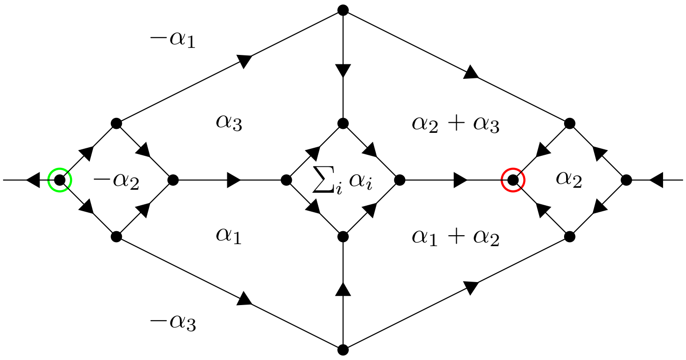


Figure 5. The oriented exchange graph of $1 \longleftarrow 2 \longrightarrow 3$.

We refer to Appendices B.2 and B.3 for additional examples.

4. The infinite cluster type

If Q is not of finite cluster type, then $\overrightarrow{\mathbf{EG}}(Q)$ is an infinite oriented graph and it is not known whether green(Q) is a finite set or not. Moreover, we have already seen in Proposition 1.21 that green(Q) can be empty in the general case. When Q is acyclic, we know from Corollary 1.18 that green(Q) is non-empty so that we will now focus on this case.

It is proved in [FZ03] that an acyclic quiver is of finite cluster type if and only if it is an orientation of a Dynkin diagram or, in representation-theoretic terms, if it is of finite representation type. Representation-infinite quivers are partitioned into two sets: *affine quivers*, which are acyclic orientations of extended Dynkin diagrams of types $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$, and *wild quivers*, which are the acyclic quivers which are neither Dynkin nor affine.

The following lemma will be proved in Section 9.

Lemma 4.1. *Let Q be a (connected) cluster quiver with two vertices. Then:*

(1) *either Q is of type A_2 and* green$(Q) = 2$, *$\ell_{\min}(Q) = 2$ and $\ell_{\max}(Q) = 3$,*
(2) *or Q is representation-infinite and* green$(Q) = 1$ *and $\ell_{\min}(Q) = \ell_{\max}(Q) = 2$.*

4.1. The affine case.

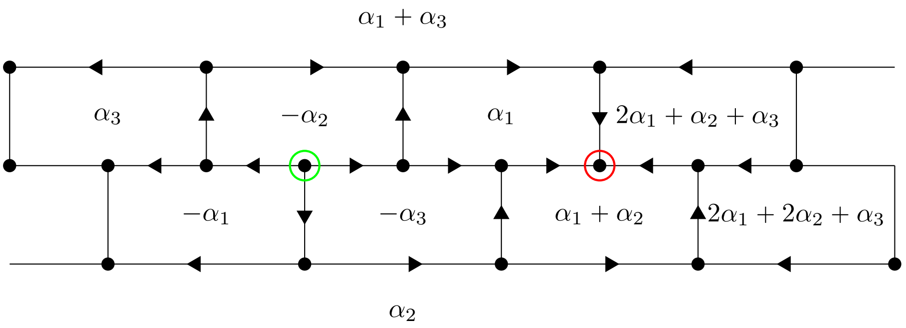
In the affine case, our main theorem is:

Theorem 4.2. *Let Q be an affine quiver. Then* green(Q) *is finite and non-empty.*

Example 4.3. Consider the quiver

$$\begin{tikzcd}
 & 2 \arrow[dr] & \\
Q: \quad 1 \arrow[ur] \arrow[rr] & & 3
\end{tikzcd}$$

of affine type $\widetilde{A}_{2,1}$. Then locally around $[\widehat{Q}]$, the oriented exchange graph $\overrightarrow{\mathbf{EG}}(Q)$ can be depicted as follows where the unique source is circled in green and the unique sink is circled in red. Here the faces are labelled by the denominator vectors of the cluster variables in the corresponding clusters, when expressed in the initial seed corresponding to $[\widehat{Q}]$.

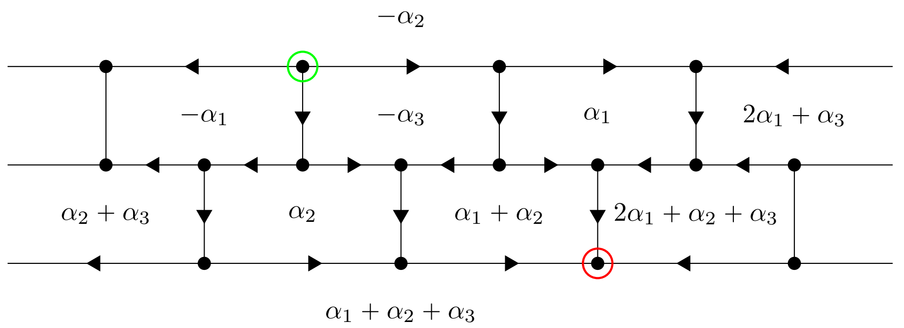


We see that, in this case, there are exactly five maximal green sequences.

Now, if we consider its mutation $Q' = \mu_2(Q)$ given by

$$\begin{tikzcd} Q' : & & 2 \arrow[dl] & \\ & 1 \arrow[rr, shift left] \arrow[rr, shift right] & & 3, \arrow[ul] \end{tikzcd}$$

then locally around $[\widehat{Q'}]$, the oriented exchange graph $\overrightarrow{\mathbf{EG}}(Q')$ looks as follows.



Note that in this case, there are also five maximal green sequences.

For additional examples, we refer the reader to Appendix B.5.

4.2. **Wild quivers with three vertices.** For the wild case, the situation appears to be more complicated. It is in fact known that for any (connected) wild quiver Q with at least three vertices, there exist regular tilting $\mathbf{k}Q$-modules [Rin88]. Therefore, the proof of Theorem 4.2 cannot be reproduced for wild quivers. However, we will prove in Proposition 9.3 that for quivers with three vertices, regular tilting $\mathbf{k}Q$-modules do not appear along maximal green sequences so that we are still able to deduce the finiteness of $\text{green}(Q)$ in this case.

Theorem 4.4. *Let Q be an acyclic quiver with three vertices. Then* $\text{green}(Q)$ *is finite and non-empty.*

The proof is given in Section 9.

If Q is a wild quiver with at least four vertices, we do not know whether $\text{green}(Q)$ is finite or not. In this case, we can only provide a few examples which yield some evidence for the finiteness of this number.

If Conjecture 1.22 holds, in order to prove that $\text{green}(Q)$ is a finite set for a given quiver Q, it would be enough to find the smallest $l \geqslant 1$ such that $\text{green}_l(Q) \neq \varnothing$ and such that $\text{green}_{l+1}(Q) = \varnothing$; in this case $\text{green}(Q) = \text{green}_{\leqslant l}(Q)$. We now provide examples of wild quivers for which we found such integers. These were computed with the computer program Quiver Mutation Explorer [DP12] whose algorithm will be outlined in Appendix A.

Example 4.5. Consider the quiver

$$Q : 1 \to 2 \to 3 \rightrightarrows 4.$$

Then

l	$\lvert\mathrm{green}_l(Q)\rvert$
4	1
5	7
6	6
7	7
8	0

Therefore, $\ell_{\min}(Q) = 4$, $\ell^0_{\max}(Q) = 7$ and $\lvert\mathrm{green}^0(Q)\rvert = 21$.

Example 4.6. Consider the quiver

$$Q : 1 \rightrightarrows 2 \to 3 \rightrightarrows 4.$$

Then

l	$\lvert\mathrm{green}_l(Q)\rvert$
4	1
5	4
6	0

Therefore, $\ell_{\min}(Q) = 4$, $\ell^0_{\max}(Q) = 5$ and $\lvert\mathrm{green}^0(Q)\rvert = 5$.

5. Silting, tilting, cluster-tilting and t-structures

As it was already mentioned, given a cluster quiver Q, the oriented exchange graph $\overrightarrow{\mathbf{EG}}(Q)$ we are studying in this article is an orientation of the cluster exchange graph $\mathbf{EG}(Q)$ of the cluster algebra $\mathcal{A}_Q$, which is the dual graph of the cluster complex $\Delta(\mathcal{A}_Q)$ introduced in [FZ03]. The same exchange graph also arises naturally in representation theory. This was first observed in [BMR$^+$06, CCS06] where it was proved that if Q is an acyclic quiver, then the clusters in $\mathcal{A}_Q$ correspond bijectively to the cluster-tilting objects in the so-called *cluster category* $\mathcal{C}_Q$ of Q in such a way that cluster mutations correspond to mutations of cluster-tilting objects in $\mathcal{C}_Q$. This generalises to arbitrary skew-symmetric cluster algebras by considering the cluster-tilting theory of certain generalised cluster categories, see [Ami09, Pla11a]. The aim of this section is to recall how $\mathbf{EG}(Q)$ and $\overrightarrow{\mathbf{EG}}(Q)$ arise in the context of additive categorifications and related topics in representation theory.

In the particular case where Q is acyclic, identifying $\operatorname{mod} \mathbf{k}Q$ with a subcategory of the cluster category $\mathcal{C}_Q$, the tilting $\mathbf{k}Q$-modules become cluster-tilting objects in $\mathcal{C}_Q$ and therefore, the cluster complex $\Delta(\mathcal{A}_Q)$ contains a certain subcomplex whose maximal simplices correspond to the tilting $\mathbf{k}Q$-modules. Already in 1987 Ringel observed that the set $\mathcal{T}_A$ of tilting modules over a finite dimensional algebra A carries the structure of a simplicial complex. The study of this complex and of a poset structure on $\mathcal{T}_A$ was initiated in [RS91] and further carried out by Happel and Unger [Ung96a, Ung96b, HU05]. We refer to the contributions of Ringel and of Unger in the Handbook of tilting theory for further details [Rin07, Ung07].

In the first part of this section we recall the related notions for tilting modules, and describe then some generalisation to the setup of derived categories. We finally explain the link to cluster categories.

Throughout, we fix an algebraically closed field $\mathbf{k}$ and all the algebras we consider are $\mathbf{k}$-algebras. If there is no risk of confusion, for a finite-dimensional algebra A, we denote by $\mathcal{D} = \mathcal{D}^b(\operatorname{mod} A)$ its bounded derived category with shift functor [1].

5.1. **Tilting modules and their mutations.** Let A be a basic connected finite-dimensional $\mathbf{k}$-algebra with n non-isomorphic simple modules.

Definition 5.1 (Tilting modules)**.** A finitely generated A-module T is called *tilting* if

1. $\operatorname{pdim} T \leqslant 1$,
2. $\operatorname{Ext}^i_A(T, T) = 0$ for all $i > 0$,
3. A admits a coresolution in $\operatorname{mod} A$ by A-modules in $\operatorname{add} T$.

A poset structure on the set $\mathcal{T}_A$ of isomorphism classes of basic tilting modules is defined in [RS91] by setting

$$T \leqslant T' \Leftrightarrow T^{\perp} \subseteq T'^{\perp}$$

where

$$T^{\perp} = \{X \in \operatorname{mod} A \mid \operatorname{Ext}^i(T, X) = 0 \text{ for all } i > 0\}.$$

We denote by $\overrightarrow{\mathcal{K}}_{\operatorname{mod} A}$ the Hasse graph of this poset of tilting A-modules. It is shown in [HU05] that the unoriented graph underlying this Hasse graph is the dual graph of the complex of tilting A-modules: there is an arrow $T \to T'$ in $\overrightarrow{\mathcal{K}}_{\operatorname{mod} A}$ precisely when $T = \bigoplus_j T_j$ and $T' = \mu_k^+(T) = (T/T_k) \oplus T'_k$ where T'_k is the *forward mutation* of T at some i defined as the cokernel of a minimal left $\operatorname{add}(T/T_k)$-approximation $T_k \to M$ (we usually slightly abuse notations and write T/T_k for $\bigoplus_{j \neq k} T_j$).

The poset $\mathcal{T}_A$ has A as unique maximal element, and in case the algebra A is Gorenstein, it has DA as unique minimal element.

5.2. **Silting objects and their mutations.** The Hasse graph of the poset of tilting A-modules is not n-regular since not all tilting modules admit mutations. There is a number of ways to fix this problem: Iyama and Reiten propose to study support *τ-tilting modules* (since every sincere almost tilting module has a completion) [IR12], and there are various ways to extend the notion of tilting module to a larger class of objects where mutations are always possible. We refer to [KY12] for a more complete picture on those various concepts, and we just recall the concept of silting objects here:

Let $\mathcal{D}$ denote the bounded derived category of $\operatorname{mod} A$ with shift functor [1].

Definition 5.2 (Silting objects, [KV88])**.** An object T in $\mathcal{D}$ is called *silting* if:

1. $\operatorname{Hom}_{\mathcal{D}}(T, T[i]) = 0$ for any $i > 0$,
2. $\operatorname{thick}(T) = \mathcal{D}$

where $\operatorname{thick}(T)$ denotes the thick subcategory generated by T in $\mathcal{D}$.

It is shown in [AI10] that the set $\mathcal{T}_{\mathcal{D}}$ of isomorphism classes of basic silting objects is turned into a poset by setting

$$T \leqslant T' \Leftrightarrow T^{\perp} \subseteq T'^{\perp},$$

where as for modules

$$T^{\perp} = \{X \in \mathcal{D} \mid \operatorname{Hom}_{\mathcal{D}}(T, X[i]) = 0 \text{ for all } i > 0\}.$$

Aihara and Iyama also show in [AI10] that the unoriented graph underlying the Hasse graph $\overrightarrow{\mathcal{K}}_{\mathcal{D}}$ of $\mathcal{T}_{\mathcal{D}}$ is the dual graph of the complex of silting objects in $\mathcal{D}$: there is an arrow $T \to T'$ in $\overrightarrow{\mathcal{K}}_{\mathcal{D}}$ precisely when $T = \bigoplus_j T_j$ and $T' = \mu_k^+(T) = (T/T_k) \oplus T'_k$ where T'_k is the *forward mutation* of T at some k defined as

$$T'_k = \operatorname{Cone}(T_k \to \bigoplus_{j \neq k} \operatorname{Irr}(T_k, T_j)^* \otimes T_j).$$

A *tilting object* in $\mathcal{D}$ is a silting object T such that $\operatorname{Hom}_{\mathcal{D}}(T, T[i]) = 0$ for any $i \neq 0$. In particular, any tilting A-module T viewed as a stalk complex in $\mathcal{D}$ is a tilting object in $\mathcal{D}$, and therefore a silting object in $\mathcal{D}$. It follows immediately from the definition that if T and T' are tilting A-modules such that $T' = \mu_k^+(T)$ as A-modules, then $T' = \mu_k^+(T)$ as silting objects in $\mathcal{D}$.

5.3. ***t*-structures and their mutations.** Let T be a silting object in $\mathcal{D}$ and consider the full subcategories in $\mathcal{D}$:

$$\mathcal{D}_T^{\leqslant 0} = \{N \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(T, N[i]) = 0 \text{ for all } i > 0\}$$

$$\mathcal{D}_T^{\geqslant 0} = \{N \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(T, N[i]) = 0 \text{ for all } i < 0\}.$$

Then $(\mathcal{D}_T^{\leqslant 0}, \mathcal{D}_T^{\geqslant 0})$ is a bounded t-structure on $\mathcal{D}$ with length heart $\mathcal{H}_T$, see for instance [KY12]. The simple *forward mutation* (also called *forward tilt*) of a heart of a bounded t-structure in $\mathcal{D}$ defined in [HRS96] corresponds to the forward mutation of the respective silting object in $\mathcal{D}$, see [AI10, KY12].

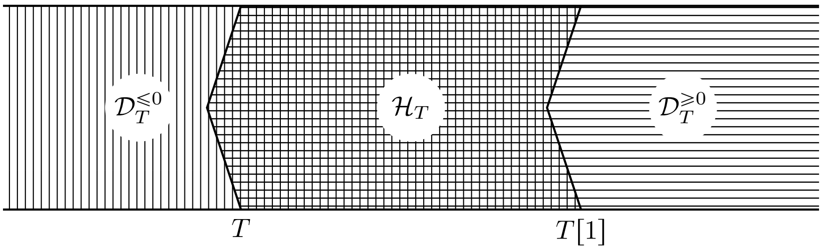


FIGURE 6. The bounded t-structure on $\mathcal{D}$ associated with a silting object.

Also from these papers, we summarise the situation as follows: isomorphism classes of basic silting objects in $\mathcal{D}$ correspond bijectively to bounded t-structures with length heart in $\mathcal{D}$. The t-structures are ordered by inclusion of their left aisles, and the forward mutation describes the arrows in the Hasse graph of these posets, see Figure 7.

Since there is always an infinite number of silting objects in the derived category, we restrict our study to an interval with maximal element A and minimal element $A[1]$, thus slightly larger than the poset of tilting A-modules. We denote by $\overrightarrow{\mathbf{EG}}_{\mathcal{D}}(A, A[1])$ the Hasse graph of the interval formed by the silting objects which are between A and $A[1]$ for this partial order. This interval, which was already considered in [KQ12] appears to be relevant for the purpose of maximal green sequences, dilogarithm identities or for BPS quivers theory [Kel11b, CCV11, BD12].

5.4. **Cluster-tilting objects and their mutations.**

Definition 5.3 (Cluster-tilting objects)**.** A *cluster-tilting object* T in a triangulated category $\mathcal{C}$ is an object T such that for any X in $\mathcal{C}$, we have

$$\mathrm{Ext}^1_{\mathcal{C}}(T, X) = 0 \Leftrightarrow X \in \mathrm{add}\, T.$$

Cluster-tilting objects were first considered in [BMR$^+$06] where it was proved that the combinatorics of cluster-tilting objects in cluster categories were governed by mutations in (simply-laced) acyclic cluster algebras.

Given an acyclic quiver Q, its path algebra $\mathbf{k}Q$ is a finite-dimensional hereditary algebra. We denote by Γ the *Ginzburg dg-algebra* associated with the quiver with potential $(Q, 0)$. It is a 3-Calabi-Yau dg-algebra concentrated in negative degrees, see [Kel11a]. We denote by $\mathcal{D}\Gamma$ the derived category of dg-Γ-modules, by $\mathrm{per}\,\Gamma$ its perfect subcategory and by $\mathcal{D}_{\mathrm{fd}}\Gamma$ the full subcategory of $\mathcal{D}\Gamma$ formed by those dg-modules with finite-dimensional total homology. The *cluster category* of Q is defined in [Ami09] as the triangulated quotient $\mathcal{C}_Q = \mathrm{per}\,\Gamma/\mathcal{D}_{\mathrm{fd}}\Gamma$. It is a Hom-finite

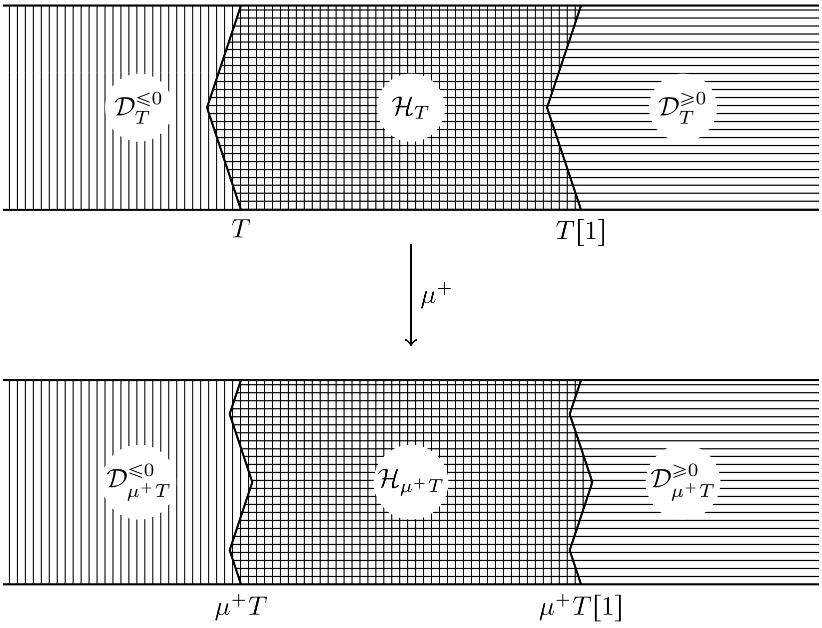


FIGURE 7. Forward mutation of a silting object in $\mathcal{D}$ and the inclusion of the corresponding left aisles.

triangulated 2-CY category which is naturally triangle-equivalent to the cluster category defined as an orbit category in [BMR$^+$06].

Then $\overrightarrow{\mathbf{EG}}(Q) = \overrightarrow{\mathbf{EG}}_{\mathcal{D}\mathbf{k}Q}(\mathbf{k}Q, \mathbf{k}Q[1])$ is an orientation of the graph of mutations of the cluster-tilting objects in $\mathcal{C}_Q$ and the unique source corresponds to the image of Γ under the canonical morphism $\operatorname{per}\Gamma \longrightarrow \mathcal{C}_Q$, see [KN10] and also [KQ12, KY12, Qiu12].

For a general cluster quiver Q and a non-degenerate potential W on Q, it is still possible to form the triangulated quotient $\mathcal{C}_{Q,W} = \operatorname{per}\Gamma_{Q,W}/\mathcal{D}_{\mathrm{fd}}\Gamma_{Q,W}$ where $\Gamma_{Q,W}$ is the Ginzburg dg-algebra associated with the quiver with potential (Q,W). Then $\overrightarrow{\mathbf{EG}}(Q)$ is an orientation of the connected component of the graph of mutations of cluster-tilting objects in $\mathcal{C}_{Q,W}$ which contains the image of $\Gamma_{Q,W}$ under the canonical morphism $\operatorname{per}\Gamma_{Q,W} \longrightarrow \mathcal{C}_{Q,W}$.

If Σ denotes the suspension functor in $\mathcal{D}_{\mathrm{fd}}\Gamma_{Q,W}$, a maximal green sequence for Q corresponds in this context to a sequence of forward mutations from the canonical heart $\mathcal{H}$ of $\mathcal{D}_{\mathrm{fd}}\Gamma_{Q,W}$ to its shift $\Sigma\mathcal{H}$, see [Kel11b].

5.5. **Patterns.** Let Q be a cluster quiver with n vertices and let $\mathbb{T}_n$ denote the n-regular tree so that the edges adjacent to any vertex in $\mathbb{T}_n$ are labelled by $\{1,\dots,n\}$. Let t_0 be a vertex in that graph. To any vertex t in $\mathbb{T}_n$ we can associate an ice quiver $Q(t)$ such that $Q(t_0) = \widehat{Q}$ and such that t and t' are joined by an edge labelled by k in $\mathbb{T}_n$ if and only if $Q(t') = \mu_k(Q(t))$. This endows $\mathbb{T}_n$ with a structure of an oriented graph $\overrightarrow{\mathbb{T}}_n$ by orienting the edge $t \overset{k}{\text{———}} t'$ towards t' if and only if k is green in $Q(t)$.

Let W be a non-degenerate potential on Q and Γ be the corresponding Ginzburg dg-algebra. The category $\mathcal{D}_{\mathrm{fd}}\Gamma$ (with suspension functor Σ) is endowed with a natural t-structure with length heart $\mathcal{H}$. As explained in [Kel12], we can associate to any vertex t in $\mathbb{T}_n$ a heart $\mathcal{H}(t)$ in $\mathcal{D}_{\mathrm{fd}}\Gamma$ such that $\mathcal{H}(t_0) = \mathcal{H}$ and such that there is an arrow $t \xrightarrow{k} t'$ in $\overrightarrow{\mathbb{T}}_n$ if and only if $\mathcal{H}(t')$ is obtained from $\mathcal{H}(t)$ by a forward mutation at the simple $S_k(t)$ in $\mathcal{H}(t)$, see also [KQ12, KY12]. This is called the *pattern of hearts in* $\mathcal{D}_{\mathrm{fd}}\Gamma$. Then a maximal green sequence $(i_1, \ldots, i_n)$ corresponds to a path $t_0 \xrightarrow{i_1} t_1 \xrightarrow{i_2} \cdots \xrightarrow{i_n} t_n$ in $\overrightarrow{\mathbb{T}}_n$ such that $\mathcal{H}(t_n) \simeq \Sigma\mathcal{H}(t)$ and $\mathcal{H}(t_k) \not\simeq \Sigma\mathcal{H}(t)$ for any $k < n$.

If Q is acyclic, $H = \mathbf{k}Q$ and $\mathcal{D} = \mathcal{D}^b(\operatorname{mod} H)$ has shift functor [1], we can also associate to any vertex t in $\mathbb{T}_n$ a silting object $T(t)$ in $\mathcal{D}$ such that $T(t_0) = \mathbf{k}Q$ and such that there is an arrow $t \xrightarrow{k} t'$ in $\overrightarrow{\mathbb{T}}_n$ if and only if $T(t')$ is obtained from $T(t)$ by a forward mutation at $T_k^{(t)}$. This is called the *silting pattern on* $\mathcal{D}$. Then it follows from [KQ12] that a maximal green sequence $(i_1, \ldots, i_n)$ corresponds to a path $t_0 \xrightarrow{i_1} t_1 \xrightarrow{i_2} \cdots \xrightarrow{i_n} t_n$ in $\overrightarrow{\mathbb{T}}_n$ such that $T(t_n) \simeq H[1]$ and $T(t_k) \not\simeq H[1]$ for any $k < n$.

6. More on Happel-Unger's poset

We assume in this section that Q is a cluster quiver which admits a non-degenerate potential W which is Jacobi-finite, that is, the Jacobian algebra $A = \mathcal{J}(Q, W)$ is finite-dimensional. These conditions are clearly satisfied when Q is acyclic (with zero potential so that $\mathcal{J}(Q, 0) = \mathbf{k}Q$) or when (Q, W) is given by an unpunctured surface, see [ABCP10, Lab09].

The Jacobian algebra A is Gorenstein [KR07], thus we know from Section 5 that the oriented exchange graph $\overrightarrow{\mathcal{K}}_{\operatorname{mod} A}$ of tilting modules has A as unique maximal element and DA as unique minimal element. We note that $\overrightarrow{\mathcal{K}}_{\operatorname{mod} A}$ is in general not connected, even in cases where the exchange graph of silting objects is connected: Happel and Unger have shown that for an affine acyclic quiver Q, the graph $\overrightarrow{\mathcal{K}}_{\operatorname{mod} A}$ is connected precisely when Q is not of type $\widetilde{A}_{1,s}$ with $s \geqslant 1$, or $\widetilde{A}_{2,2}$ with alternating orientation [HU05].

Theorem 6.1. *Let Q be an acyclic quiver and $H = \mathbf{k}Q$. Then $\overrightarrow{\mathcal{K}}_{\operatorname{mod} H}$ is a full convex oriented subgraph of $\overrightarrow{\mathbf{EG}}(Q)$.*

Proof. Let T and T' be two tilting H-modules and consider a path

$$T \longrightarrow T^{(1)} \longrightarrow \cdots \longrightarrow T^{(l)} \longrightarrow T'$$

in $\overrightarrow{\mathbf{EG}}(Q)$. Then we have the following chain of inclusions of the left aisles of the corresponding t-structures in $\mathcal{D} = \mathcal{D}(\operatorname{mod} H)$:

$$\mathcal{D}_T^{\leqslant 0} \subset \mathcal{D}_{T^{(1)}}^{\leqslant 0} \subset \cdots \subset \mathcal{D}_{T^{(l)}}^{\leqslant 0} \subset \mathcal{D}_{T'}^{\leqslant 0}.$$

If there is some $1 \leqslant k \leqslant l$ such that $T^{(k)}$ is not a tilting H-module, then it is a silting object in $\overrightarrow{\mathbf{EG}}_{\mathcal{D}}(H, H[1])$ and therefore, it has a summand of the form $P_i[1]$. Thus we have $P_i[1] \in \mathcal{D}_{T^{(k)}}^{\leqslant 0}$ and we get $P_i[1] \in \mathcal{D}_{T'}^{\leqslant 0}$ which implies

$$0 = \operatorname{Hom}_{\mathcal{D}}(P_i[1], T'[1]) = \operatorname{Hom}_{\mathcal{D}}(P_i, T') = \operatorname{Hom}_H(P_i, T')$$

so that T' is not sincere, which is a contradiction since every tilting H-module is sincere. Therefore, for any $1 \leqslant k \leqslant l$, the silting object $T^{(k)}$ is a tilting H-module. And as it was already mentioned, the forward mutation of a tilting module T in $\operatorname{mod} H$ coincides with the forward mutation of T viewed as a silting object in $\mathcal{D}$. Therefore, $\overrightarrow{\mathcal{K}}_{\operatorname{mod} H}$ is a full convex oriented subgraph of $\overrightarrow{\mathbf{EG}}(Q)$. □

Example 6.2. Figures 8 and 9 show how the Happel-Unger's poset embeds in the oriented exchange graph of type A_2. In both cases, the unique source is circled in green and the unique sink is circled in red.

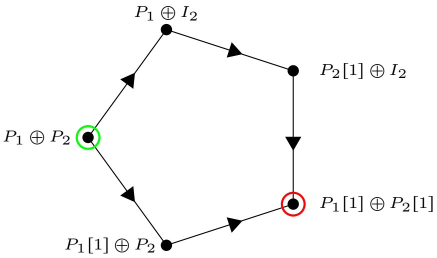


FIGURE 8. The oriented exchange graph of $1 \longrightarrow 2$.

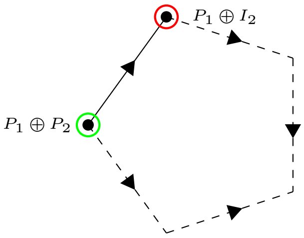


FIGURE 9. Happel-Unger's poset, sitting in the oriented exchange graph of $1 \longrightarrow 2$.

Example 6.3. Figures 10, 11, 12, 13 14 and 15 show how the Happel-Unger's posets embed in the oriented exchange graphs of type A_3. In any case, the sources are circled in green and the sinks are circled in red.

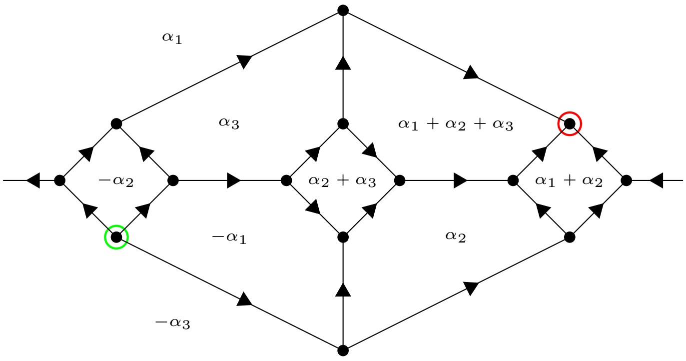


FIGURE 10. The oriented exchange graph of the quiver $1 \longrightarrow 2 \longrightarrow 3$, labelled with denominator vectors.

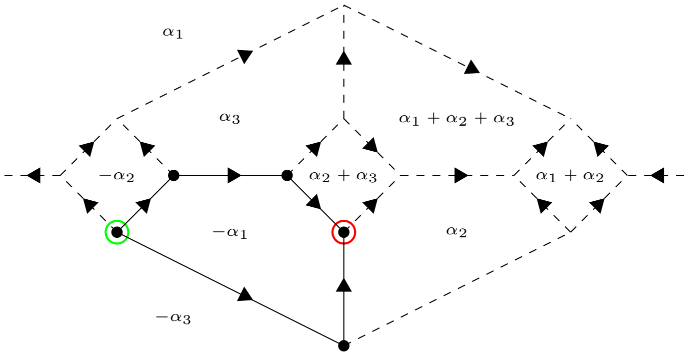


FIGURE 11. Happel-Unger's poset for the quiver $1 \longrightarrow 2 \longrightarrow 3$, sitting in the poset of maximal green sequences.

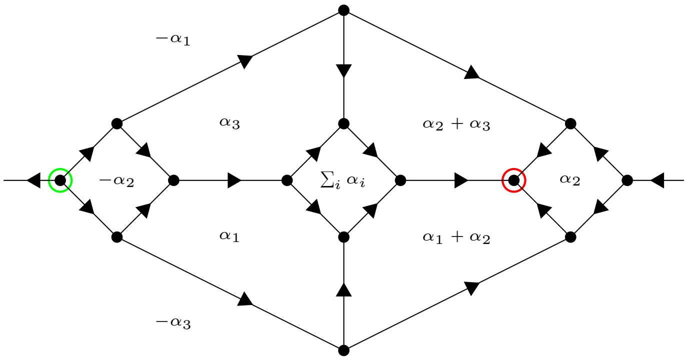


FIGURE 12. The oriented exchange graph of $1 \longleftarrow 2 \longrightarrow 3$, labelled with denominator vectors.

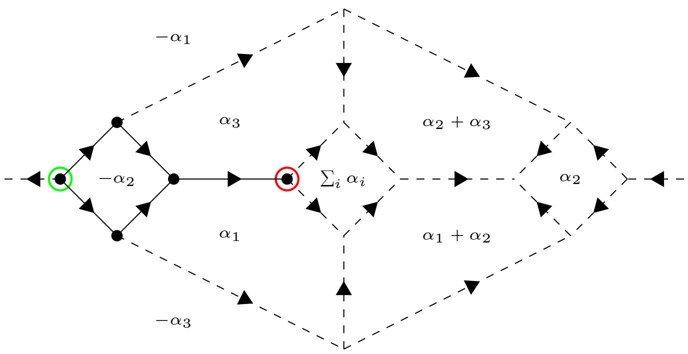


FIGURE 13. Happel-Unger's poset, sitting in the oriented exchange graph of $1 \longleftarrow 2 \longrightarrow 3$.

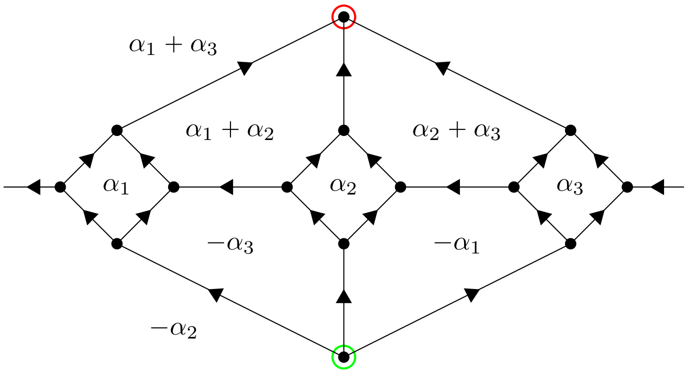


FIGURE 14. The oriented exchange graph of the cyclic quiver with 3 vertices, labelled with denominator vectors.

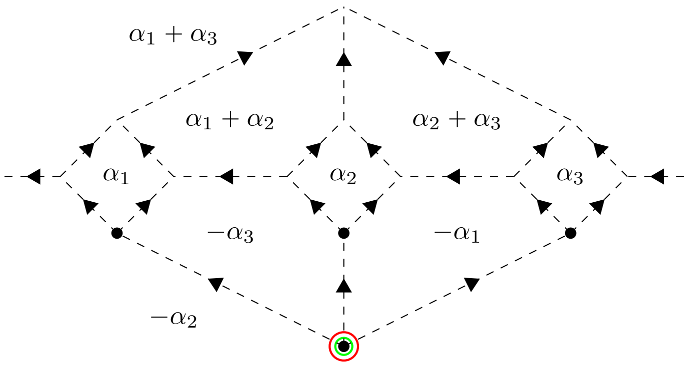


FIGURE 15. Happel-Unger's poset for the non-hereditary cluster-tilted algebra of type A_3, sitting in the corresponding oriented exchange graph.

Remark 6.4. Example 6.3 shows a phenomenon which was already observed in [HU05], namely that the Happel-Unger's poset is not stable under derived equivalences. It also shows that the oriented exchange graph is *not* stable under derived equivalences neither. Indeed, in the linear case (Figure 11) there are 9 maximal green sequences whereas there are 10 in the alternating case (Figure 12).

Lemma 6.5. *Let Q be an acyclic quiver and $H = \mathbf{k}Q$. Let $(i_1, \ldots, i_n)$ be an admissible numbering of Q_0 by sinks. Assume that there is a path from H to DH in $\overrightarrow{\mathcal{K}}_{\operatorname{mod} H}$ and let $(v_1, \ldots, v_l)$ be the corresponding green sequence for Q. Then $(v_1, \ldots, v_l, i_1, \ldots, i_n) \in \operatorname{green}_{l+n}(Q)$.*

Proof. Since DH is a tilting H-module, it is in particular a tilting object in $\mathcal{D}(\operatorname{mod} H)$ and therefore a silting object in $\mathcal{D} = \mathcal{D}^b(\operatorname{mod} H)$. Let $(i_1, \ldots, i_n)$ be an admissible numbering of Q_0 by sinks. The endomorphism algebra of DH has Gabriel quiver Q^{op} and $(i_1, \ldots, i_n)$ is an admissible sequence of sources for Q^{op}. Therefore, considering successive APR-tilts at sources (see [APR79]), we obtain a sequence of tilting objects in $\mathcal{D}$:

$$DH \xrightarrow{\mu_{i_1}} T^{(1)} \xrightarrow{\mu_{i_2}} \cdots \xrightarrow{\mu_{i_{n-1}}} T^{(n-1)} \xrightarrow{\mu_{i_n}} H[1]$$

where

$$T^{(k)} = \left(\bigoplus_{j<k} I_{i_j}\right) \oplus \left(\bigoplus_{l\leqslant k} \tau^{-1} I_{i_j}\right) = \left(\bigoplus_{j<k} I_{i_l}\right) \oplus \left(\bigoplus_{l\geqslant k} P_{i_l}[1]\right)$$

for any $1 \leqslant k \leqslant n$. In particular, we have proper inclusions of left aisles

$$\mathcal{D}^{\leqslant 0}_{DH} \subset \mathcal{D}^{\leqslant 0}_{T^{(1)}} \subset \mathcal{D}^{\leqslant 0}_{T^{(2)}} \subset \cdots \subset \mathcal{D}^{\leqslant 0}_{T^{(n-1)}} \subset \mathcal{D}^{\leqslant 0}_{H[1]}$$

so that we obtain a path of length n from DH to $H[1]$ in $\mathbf{EG}(Q)$. By Theorem 6.1, a path of length l from H to DH in $\overrightarrow{\mathcal{K}}_{\operatorname{mod} H}$ gives rise to a path of length l from H to DH in $\overrightarrow{\mathbf{EG}}(Q)$, composing this path with the above path from DH to $H[1]$ gives a path from H to $H[1]$ of length $n+l$ in $\overrightarrow{\mathbf{EG}}(Q)$, and therefore an element in $\operatorname{green}_{l+n}(Q)$. Figure 16 illustrates the proof.

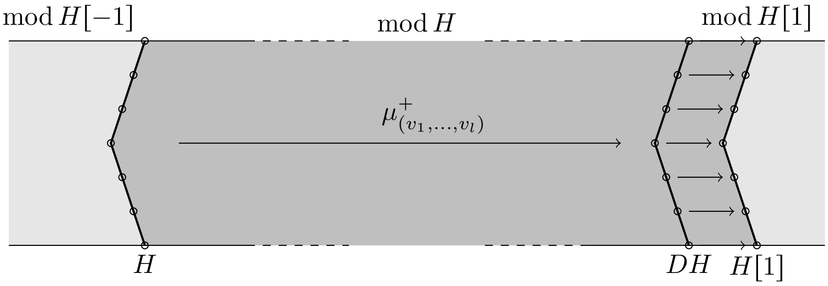


Figure 16. Extending a path from H to DH to a maximal green sequence.

□

The statement of Lemma 6.5 fails if Q is not acyclic: from Figure 15 in Example 6.3, we see that in case Q is the 3-cycle with non-degenerate potential that 3-cycle, the Jacobian algebra A is self-injective and so the poset $\overrightarrow{\mathcal{K}}_{\operatorname{mod} A}$ consists only of one point. The minimal length of a maximal green sequence is 4, thus the statement of lemma 6.5 does not hold.

Remark 6.6. Lemma 6.5 provides a criterion for the non-existence of paths from H to DH in $\overrightarrow{\mathcal{K}}_{\operatorname{mod} H}$. For instance, consider the quivers

$$Q_1 : 1 \to 2 \to 3 \mathrel{\substack{\to\\ \to\\ \to}} 4 \quad\text{and}\quad Q_2 : 1 \mathrel{\substack{\to\\ \to\\ \to}} 2 \to 3 \mathrel{\substack{\to\\ \to\\ \to}} 4.$$

Since $\mathbf{k}Q_1$ and $\mathbf{k}Q_2$ are wild hereditary algebras, they have no projective-injective modules. Therefore, for any $i = 1, 2$, a path from $\mathbf{k}Q_i$ to $D\mathbf{k}Q_i$ in $\overrightarrow{\mathcal{K}}_{\operatorname{mod} \mathbf{k}Q_i}$ must have a length at least 4. However, as we saw in Examples 4.5 and 4.6, we have $\ell^0_{\max}(Q_1) = 7$ and $\ell^0_{\max}(Q_2) = 5$. Therefore, if Conjecture 1.22 holds, then $\operatorname{green}_l(Q_1)$ and $\operatorname{green}_l(Q_2)$ would be empty for $l \geqslant 8$ and there would be no paths from $\mathbf{k}Q_i$ to $D\mathbf{k}Q_i$ in $\overrightarrow{\mathcal{K}}_{\operatorname{mod} \mathbf{k}Q_i}$.

7. Proofs of Section 1

7.1. Proof of Proposition 1.10.

Proof. Let W be a generic potential on Q and let Γ be the Ginzburg dg-algebra associated to the quiver with potential (Q, W). The category $\mathcal{D}_{\mathrm{fd}}\Gamma$ is endowed with a natural t-structure and we denote by $\mathcal{H}$ the corresponding heart with simples S_i, $i \in Q_0$. Let $\mathbb{T}_{|Q_0|}$ denote the $|Q_0|$-regular tree and consider the pattern of tilts $t \mapsto \mathcal{H}(t)$ with $\mathcal{H}(T_0) = \mathcal{H}$, see Section 5.5 or [Kel12, §7.7].

Since $R \in \mathrm{Mut}(\widehat{Q})$, there is some vertex t in $\mathbb{T}_{|Q_0|}$ such that $R = Q(t)$ and since all the non-frozen vertices in R are green, it means that all the simples in $\mathcal{H}(t)$ are in $\mathcal{H}$. Therefore, there exists a permutation $\pi \in \mathfrak{S}_{Q_0}$ such that $S_i(t) \simeq S_{\pi(i)}$ for any $i \in Q_0$ and it follows from [Kel12, Corollary 7.11] that π induces the wanted isomorphism of ice quivers.

Similarly, if all the non-frozen vertices in $R = Q(t)$ are red, it means that all the simples in $\mathcal{H}(t)$ are in $\Sigma\mathcal{H}$. Therefore, there exists a permutation $\pi \in \mathfrak{S}_{Q_0}$ such that $S_i(t) \simeq \Sigma S_{\pi(i)}$ for any $i \in Q_0$ and it follows from [Kel12, Corollary 7.11] that π induces the wanted isomorphism of ice quivers. ☐

7.2. Proof of Proposition 1.21.

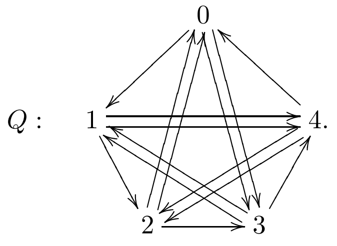
Proof. We know from Plamondon's thesis [Pla11b, Example 4.3] that there exists a (Jacobi-finite) non-degenerate potential W on Q such that there is no sequence of mutations in the cluster category $\mathcal{C}_{Q,W}$ joining the cluster-tilting object $\Gamma_{Q,W}$ to the cluster-tilting object $\Sigma\Gamma_{Q,W}$. Therefore, $\Gamma_{Q,W}$ and $\Sigma\Gamma_{Q,W}$ are in two different connected components of the mutation graph of cluster-tilting objects in $\mathcal{C}_{Q,W}$. In particular, if $\mathcal{H}$ denotes the canonical heart in $\mathcal{D}_{\mathrm{fd}}\Gamma_{Q,W}$, there is no sequence of forward mutations from $\mathcal{H}$ to $\Sigma\mathcal{H}$ and therefore, there is no path from $[\widehat{Q}]$ to $[\check{Q}]$ in $\overrightarrow{\mathbf{EG}}(Q)$. Hence, $\mathrm{green}(Q) = \varnothing$. ☐

7.3. On Jacobi-infinite quivers with potential. In this short section we give a criterion for the non-existence of maximal green sequences. We recall that a quiver with potential is called Jacobi-infinite if the corresponding (completed) Jacobian algebra is infinite dimensional over $\mathbf{k}$.

Proposition 7.1. *Let (Q, W) be a Jacobi-infinite quiver with potential. Assume that it is non-degenerate. Then* $\mathrm{green}(Q) = \varnothing$.

Proof. Let Γ be the Ginzburg dg-algebra corresponding to (Q, W). Let $\mathcal{H}$ denote the canonical heart in $\mathcal{D}_{\mathrm{fd}}\Gamma$. We claim that $\Sigma\mathcal{H}$ is not reachable from $\mathcal{H}$ by iterated forward mutations of hearts in $\mathcal{D}_{\mathrm{fd}}\Gamma$. Indeed, if so, we would obtain a sequence of mutations from Γ to $\Sigma\Gamma$ in the generalised cluster category $\mathcal{C}$ associated to (Q, W), see [Kel12, Pla11a]. Therefore, Γ and $\Sigma\Gamma$ are in the same connected component of the cluster-tilting graph of $\mathcal{C}$. Fix thus a sequence $\mathbf{i} = (i_1, \ldots, i_l)$ such that $\Gamma = \mu_{\mathbf{i}}(\Sigma\Gamma)$. Then it follows from [Pla11a], that this gives a sequence of mutations of decorated representation in the sense of [DWZ10] from the decorated representation of (Q, W) corresponding to $\mathrm{Hom}_{\mathcal{C}}(\Gamma, \Sigma\Gamma) = 0$ to the decorated representation of $\mu_{\mathbf{i}}(Q, W)$ of $\mathrm{Hom}_{\mathcal{C}}(\Gamma, \Gamma) \simeq \mathrm{End}_{\mathcal{C}}(\Gamma) \simeq \mathcal{J}(Q, W)$ which is infinite dimensional by hypothesis. However, mutations of finite-dimensional decorated representations of quivers with potential are finite-dimensional, a contradiction. Thus, there is no sequence of forward mutations from $\mathcal{H}$ to $\Sigma\mathcal{H}$ and therefore, there is no path from $[\widehat{Q}]$ to $[\check{Q}]$ in $\overrightarrow{\mathbf{EG}}(Q)$. Hence, $\mathrm{green}(Q) = \varnothing$. ☐

Example 7.2. Consider the McKay quiver



Then the main theorem of [TV10] asserts that Q admits a non-degenerate potential such that the corresponding Jacobian algebra is infinite dimensional. Therefore, it follows from Proposition 7.1 that Q has no maximal green sequences.

8. Proofs of Section 3

8.1. **Proof of Theorem 3.4.**

Proof. A Dynkin quiver is acyclic so that the first point follows from Lemma 1.20.

For the second point, consider the Weyl group W associated to Q with simple reflections s_i, with $i \in Q_0$. Let w_0 be the longest element in W and fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_r}$ so that $r = |\Phi_+(Q)|$. Then it is well-known that $\mathbf{i} = (i_1, \ldots, i_r)$ is an admissible sequence of sinks in Q. For any $1 \leqslant k \leqslant r$, we set

$$T^{(k)} = \mu^+_{i_k} \circ \cdots \circ \mu^+_{i_1}(H)$$

with the convention that $T^{(0)} = H$.

Since $\mathbf{i}$ is an admissible sequence of sinks, for any $1 \leqslant k \leqslant r$, the vertex i_k is a sink in the quiver of the endomorphism ring of $T^{(k-1)}$ so that $T^{(k)}$ is obtained from $T^{(k-1)}$ by a simple APR-tilt (see [APR79]). Therefore, the left aisles $\mathcal{D}^{\leqslant 0}_{T^{(k-1)}}$ and $\mathcal{D}^{\leqslant 0}_{T^{(k)}}$ differ by a single indecomposable object, namely $T^{(k-1)}_{i_k}$. Moreover, it is well-known that $T^{(r)} \simeq H[1]$. Therefore, we obtained a sequence of forward mutations

$$H \xrightarrow{\mu^+_{i_1}} T^{(1)} \xrightarrow{\mu^+_{i_2}} \cdots \xrightarrow{\mu^+_{i_r}} T^{(r)} \simeq H[1]$$

which is the longest possible. Thus $\mathbf{i}$ is a maximal green sequence of the longest possible length and we have $\ell_{\max}(Q) = r = |\Phi_+(Q)|$. □

9. Proofs of Section 4

9.1. **Proof of Lemma 4.1.**

Proof. The first point follows from Theorem 3.4. We now prove the second point. Let (i_1, i_2) be an admissible numbering of Q_0 by sources. Then it was proved in Lemma 1.20 that (i_1, i_2) is a maximal green sequence for Q. If there exists another maximal green sequence, then it is necessarily obtained by iterated mutations at sinks of the form $i_2 i_1 i_2 i_1 \cdots$. Let $H = \mathbf{k}Q$, $T^{(0)} = H$ and for any $k \geqslant 1$, let

$$T^{(k)} = \begin{cases} \mu^+_{i_2}(T^{(k-1)}) & \text{if } k \text{ is odd,} \\ \mu^+_{i_1}(T^{(k-1)}) & \text{if } k \text{ is even.} \end{cases}$$

Then, as in the proof of Theorem 3.4, the left aisles $\mathcal{D}^{\leqslant 0}_{T^{(k-1)}}$ and $\mathcal{D}^{\leqslant 0}_{T^{(k)}}$ differ by a single indecomposable object, namely $T^{(k-1)}_{i_k}$. However, the left aisle $\mathcal{D}^{\leqslant 0}_{H[1]}$ contains infinitely many more objects than the left aisle $\mathcal{D}^{\leqslant 0}_{H}$ so that $T^{(k)} \not\simeq H[1]$ for any $k \geqslant 1$. Therefore, there is no maximal green sequence beginning with i_2 and thus $\mathrm{green}(Q) = \{(i_1, i_2)\}$. □

9.2. **Proof of Theorem 4.2.** Let Q be an affine quiver and let $H = \mathbf{k}Q$. The aim of this subsection is to prove that $\mathrm{green}(Q)$ is a finite set. Before we can prove this we will need some technical results.

We let $S_1, \ldots, S_n$ denote the simple H-modules and for any $1 \leqslant i \leqslant n$, we denote by P_i the projective cover of S_i and by I_i its injective hull.

As usual, we let $\mathcal{D}$ be the bounded derived category of $\operatorname{mod} H$ with shift functor [1]. We denote by $\Gamma(\mathcal{D})$ the Auslander-Reiten quiver of $\mathcal{D}$, by $\mathcal{P}$ the preprojective component of $\Gamma(\mathcal{D})$, that is, the connected component containing the projective H-modules and by $\mathcal{I}$ the preinjective component, that is, the connected component of $\Gamma(\mathcal{D})$ containing the injective H-modules.

We start with a general lemma:

Lemma 9.1. *Let H be a representation-infinite connected hereditary algebra. Then there exists $N \geqslant 0$ such that for any $k \geqslant N$, for any projective H-module P and for any injective H-module I, the H-modules $\tau^{-k}P$ and $\tau^{k}I$ are sincere.*

Proof. For any $1 \leqslant i \leqslant n$, it is known that the sets $\{\tau^{-k}P_i\}_{k\geqslant 0}$ and $\{\tau^{k}I_i\}_{k\geqslant 0}$ contain only finitely many non-sincere modules, see [ASS05, Ch. IX, Proposition 5.6]. Therefore, there exists $N_i \geqslant 0$ such that $\tau^{-k}P_i$ and $\tau^{k}I_i$ are sincere for any $k \geqslant N_i$. Then $N = \max\{N_i \mid 1 \leqslant i \leqslant n\}$ is as wanted. □

Proposition 9.2. *Assume that H is tame. Let T be a tilting object with its indecomposable summands in $\mathcal{P}$ and let and T' be a tilting object with its indecomposable summands in $\mathcal{I}$. Then the number of oriented paths from T to T' in $\overrightarrow{\mathcal{K}}_{\mathcal{D}}$ is finite.*

Proof. We can write

$$T = \bigoplus_{j=1}^{n} \tau^{k_j} P_j \text{ and } T' = \bigoplus_{i=1}^{n} \tau^{l_i} I_i$$

with $k_j, l_i \in \mathbb{Z}$ for any $1 \leqslant i, j \leqslant n$.

Fix an oriented path $T = T^{(0)} \longrightarrow T^{(1)} \longrightarrow \cdots \longrightarrow T^{(p-1)} \longrightarrow T^{(p)} = T'$ in $\overrightarrow{\mathcal{K}}_{\mathcal{D}}$ and let $T^{(m)}$ be the first silting object in that path with a direct summand in the component $\mathcal{I}$. We denote by $\tau^{l_i+l}I_i$ with $l \geqslant 0$ this indecomposable direct summand. Since H is tame, any tilting object in $\mathcal{D}$ has at most $n-2$ indecomposable regular modules as direct summands, see [Rin84]. The same holds for any silting object S such that $T \leqslant S \leqslant T'$. Thus, $T^{(m)}$ has at least one direct summand in $\mathcal{P}$, which we denote by $\tau^{k_j-k}P_j$ with $k \geqslant 0$.

We have

$$\begin{aligned} \operatorname{Ext}^1_{\mathcal{D}}(\tau^{l_i+l}I_i, \tau^{k_j-k}P_j) &\simeq D\operatorname{Hom}_{\mathcal{D}}(\tau^{k_j-k}P_j, \tau^{l_i+l+1}I_i) \\ &\simeq D\operatorname{Hom}_{\mathcal{D}}(P_j, \tau^{l_i-k_j+l+k+1}I_i), \end{aligned}$$

and this is non-zero for $k+l \geqslant N + k_j - l_i - 1$ according to Lemma 9.1. Therefore, there exists $K > 0$ and $L > 0$ such that any tilting object on an oriented path from T to T' in $\mathcal{K}_{\mathcal{D}}$ has its indecomposable summands in $\mathcal{P}$ of the form $\tau^{k_j-k}P_j$ for some $1 \leqslant j \leqslant n$ and $0 \leqslant k \leqslant K$ and its indecomposable summands in $\mathcal{I}$ of the form $\tau^{l_i+l}I_i$ for some $1 \leqslant i \leqslant n$ and $0 \leqslant l \leqslant L$.

Then, if $\tau^{k_{j'}-k'}P_{j'}$ is another indecomposable summand of a silting object on an oriented path from T to T' in $\overrightarrow{\mathcal{K}}_{\mathcal{D}}$, we have $k' \geqslant 0$ and

$$\operatorname{Ext}^1_{\mathcal{D}}(\tau^{k_{j'}-k'}P_{j'}, \tau^{k_j-k}P_j) \simeq D\operatorname{Hom}_{\mathcal{D}}(P_j, \tau^{k_{j'}-k_j+k-k'}P_j),$$

which is non-zero for $k' \geqslant N + k + k_{j'} - k_j$ according to Lemma 9.1. Therefore, the number of isomorphism classes of indecomposable summands in $\mathcal{P}$ of silting objects arising on an oriented path from T to T' in $\overrightarrow{\mathcal{K}}_{\mathcal{D}}$ is finite.

Dually, the number of isomorphism classes of indecomposable summands in $\mathcal{I}$ of tilting objects arising on an oriented path from T to T' in $\mathcal{K}_{\mathcal{D}}$ is finite.

Since there are only finitely many rigid regular H-modules, the number of isomorphism classes of regular indecomposable summands of tilting object arising on an oriented path from T to T' in $\mathcal{K}_{\mathcal{D}}$ is also finite. Therefore, the number of isomorphism classes of silting objects arising on an oriented path from T to T' in $\mathcal{K}_{\mathcal{D}}$ is finite, proving the theorem. Figure 17 sums up the situation.

□

We can now prove Theorem 4.2.

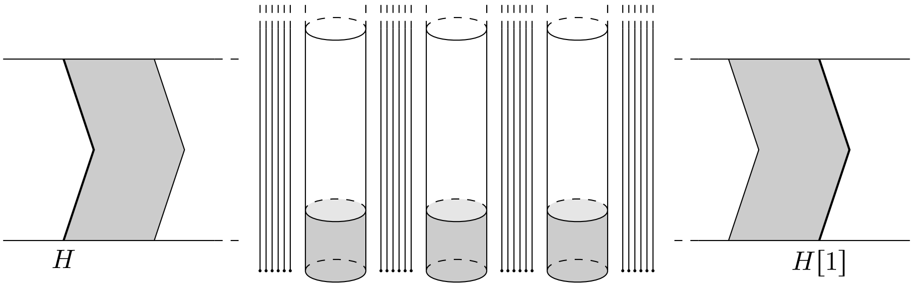


FIGURE 17. The Auslander-Reiten quiver of $\mathcal{D}^b(\operatorname{mod} H)$ for a tame hereditary algebra. Shaded areas correspond to where silting objects located on a path from H to $H[1]$ can have their indecomposable summands.

Proof of Theorem 4.2: Let Q be an affine quiver and $H = \mathbf{k}Q$. Then a maximal green sequence for Q is an oriented path from H to $H[1]$ in $\overrightarrow{\mathcal{K}}_{\mathcal{D}}$. Since H has its indecomposable summands in $\mathcal{P}$ and $H[1]$ has its indecomposable summands in $\mathcal{I}$, the result is a direct consequence of Proposition 9.2. □

9.3. **Proof of Theorem 4.4.** In this section, we want to prove that green(Q) is a finite set for an acyclic quiver with 3 vertices.

We first show the following proposition.

Proposition 9.3. *Let Q be a connected wild quiver with three vertices and $H = \mathbf{k}Q$. Assume that T is a silting object arising on a path from H to $H[1]$ in $\overrightarrow{\mathbf{EG}}(Q)$. Then T is not a regular tilting H-module.*

Proof. Consider a path from H to $H[1]$ in $\overrightarrow{\mathbf{EG}}(Q)$ containing a regular tilting H-module and let R' be the first regular tilting H-module arising on this path. Therefore, this path contains an arrow $R \xrightarrow{\mu_v} R'$ where $R' = R/R_v \oplus R_v^*$ with R_v is preprojective, say $R_v \simeq \tau^{-s}P_j$, and R_v^*, R/R_v are regular.

Since R is tilting, we get

$$\begin{aligned} 0 &= \operatorname{Ext}^1_{\mathcal{D}}(R/R_v, R_v) \\ &\simeq D\operatorname{Hom}_{\mathcal{D}}(R_v, \tau(R/R_v)) \\ &\simeq D\operatorname{Hom}_{\mathcal{D}}(\tau^{-s}P_j, \tau(R/R_v)) \\ &\simeq D\operatorname{Hom}_{\mathcal{D}}(P_j, \tau^{s+1}(R/R_v)) \\ &\simeq D\operatorname{Hom}_{H}(P_j, \tau^{s+1}(R/R_v)). \end{aligned}$$

Therefore, $\tau^{s+1}(R/R_v)$ is an $A/(e_j)$-module which is rigid (since $\tau^{s+1}(R/R_v)$ is rigid as an A-module) and, since it has $|Q_0| - 1$ indecomposable summands, it is tilting as a $A/(e_j)$-module and thus, $\tau^{s+1}R' = \tau^{s+1}(R/R_v) \oplus \tau^{s+1}R_v^*$ is a regular tilting H-module satisfying the hypothesis of [Ung96a, Theorem 4.3]. Hence, any tilting module in the same connected component of $\overrightarrow{\mathcal{K}}_{\operatorname{mod} H}$ as $\tau^{s+1}R'$ contains at least two τ-sincere indecomposable summands. These indecomposables are in particular regular H-modules. Therefore, the connected component of $\overrightarrow{\mathcal{K}}_{\operatorname{mod} H}$ containing $\tau^{s+1}R'$ does not contain any preprojective, nor preinjective module.

Now if there is a path $R' \longrightarrow \cdots \longrightarrow H[1]$ in $\overrightarrow{\mathbf{EG}}(Q)$, since none of the modules in that path have a projective direct summand, we get a path $\tau R' \longrightarrow \cdots \longrightarrow \tau H[1] \simeq DH$ in $\overrightarrow{\mathbf{EG}}(Q)$ and, by convexity of $\overrightarrow{\mathcal{K}}_{\operatorname{mod} H}$ inside $\overrightarrow{\mathbf{EG}}(Q)$ (Theorem 6.1), we obtain a path $\tau R' \longrightarrow \cdots \longrightarrow DH$ in $\overrightarrow{\mathcal{K}}_{\operatorname{mod} H}$, a contradiction. □

We can now prove Theorem 4.4.

Proof of Theorem 4.4: Without loss of generality we can restrict to the case where Q is connected. If it is Dynkin or affine, the result is known, see Theorems 3.1 and 4.2. We can thus restrict to the case where Q is wild. We let $H = \mathbf{k}Q$. The number of maximal green sequences for Q equals the number of paths from H to $H[1]$ in $\overrightarrow{\mathbf{EG}}(Q)$.

Consider such a path and let T be the last silting object along this path which is without summand of the form $\tau^l I_i$ for $l \geqslant -1$ and $i \in Q_0$. The next silting object in the path is thus $\mu_k(T) = T/T_k \oplus T_k^*$ with $T_k^* \simeq \tau^l I_i$ for some $l \geqslant -1$ and some $i \in Q_0$.

Assume first that T/T_k is a regular H-module. If T/T_k is sincere, since $\mu_k(T)$ is tilting, we get

$$\begin{aligned} 0 &= \mathrm{Ext}^1_{\mathcal{D}}(T_k^*, T/T_k) \\ &\simeq D\mathrm{Hom}_{\mathcal{D}}(T/T_k, \tau T_k^*) \\ &\simeq D\mathrm{Hom}_{\mathcal{D}}(T/T_k, \tau^{l+1} I_i) \\ &\simeq D\mathrm{Hom}_{\mathcal{D}}(\tau^{-(l+1)}(T/T_k), I_i) \\ &\simeq D\mathrm{Hom}_{H}(\tau^{-(l+1)}(T/T_k), I_i) \end{aligned}$$

so that $\tau^{-(l+1)}$ is almost complete, non-sincere and regular. Therefore, it follows from [HU05, Proposition 7.3] that its unique complement, which is $\tau^{-(l+1)}T_k$, is regular. Therefore, T_k is also regular and so is $T = T/T_k \oplus T_k$, which contradicts Proposition 9.3.

If T/T_k is non-sincere, then it is almost complete, non-sincere and regular and it again follows from [HU05, Proposition 7.3] that $\tau^t(T/T_k)$ is sincere for any $t \neq 0$. According to Proposition 9.3, T is not regular so that necessarily $T_k \simeq \tau^{-l} P_j$ for some $j \in Q_0$ and some $l \geqslant 0$. Since T is silting, we get

$$\begin{aligned} 0 &= \mathrm{Ext}^1_{\mathcal{D}}(T/T_k, T_k) \\ &\simeq \mathrm{Ext}^1_{\mathcal{D}}(T/T_k, \tau^{-l} P_j) \\ &\simeq D\mathrm{Hom}_{\mathcal{D}}(\tau^{-l} P_j, \tau(T/T_k)) \\ &\simeq D\mathrm{Hom}_{\mathcal{D}}(P_j, \tau^{l+1}(T/T_k)). \end{aligned}$$

But we know that $\tau^t(T/T_k)$ is sincere for any $t \neq 0$. Therefore, $l = -1$ and thus $T_k \simeq \tau P_j \simeq I_i[-1]$ is in $\mathrm{mod}\, H[-1]$, which is a contradiction. Hence, T/T_k cannot be regular.

Thus, T contains at most one regular direct summand and dually, the first silting object without preprojective direct summand contains at most one regular direct summand. Then, as in the proof of Proposition 9.2, it follows from Lemma 9.1 that the non-regular summands of silting objects between H and $H[1]$ run over a finite set of isomorphism classes. And since we cannot mutate twice consecutively at the same vertex along a maximal green sequence, we cannot mutate twice a regular summand consecutively. Therefore, there is necessarily a finite number of maximal green sequences for Q. □

Appendix A. Outline of the algorithm

A.1. **Motivations.** Given a cluster quiver Q, we would like to answer the two following questions:

(1) Does there exist a maximal green sequence for Q, *i.e* is $\mathrm{green}(Q) \neq \varnothing$?
(2) If yes, how many maximal green sequences of each length are there in $\mathrm{green}(Q)$, *i.e* what is $|\mathrm{green}_l(Q)|$ for any $\ell_{\min}(Q) \leqslant l \leqslant \ell_{\max}(Q)$.

It appears that a computational approach is well suited to answer these two questions. The second one requires an enumeration of a considerable number of possibilities, either explicit as the first following subsection shows, or implicit for greater speeds as explained in Section A.3.

A.2. **Maximal green sequences enumeration principles.** In order to answer these two questions, a direct approach consists in trying every possible green sequences. Starting from the initial cluster quiver, we will mutate at every green vertex of the corresponding framed quiver, get a new set of ice quivers, pick one, mutate at every green vertex. . . and so on, up to finding a quiver without green vertices. The detailed algorithm is given in Algorithm 1. This algorithm is implemented and available, see [DP12].

In this algorithm:

- the `last` method applied to a list pops the last element of the chained list;
- `append`, applied to a list, adds an element to the end;
- `getNextGreenVertex`, applied to a quiver, pops one green vertex from the set of unexplored green vertices, returns NULL when all vertices are explored;
- `mutate(i)`, applied to a quiver Q, returns the quiver obtained from mutating Q on the vertex `i`;
- `mutationLength()`, applied to a quiver, returns the length of the list of mutations applied to the quiver.

This algorithm is a typical depth-first search: using the green vertices list of the quivers as the branching element, it will consider the initial quiver as the root of a search tree and explore branches constructed by sequences of green mutations.

Algorithm 1 Depth-first search algorithm.

```
Require: $\widehat{Q}$ the framed quiver of a cluster quiver $Q$, L an empty chained list green, an array of integers, with all values equal to 0
Ensure: $\forall i, green[i] = |\mathrm{green}_i(\widehat{Q})|$
  L.append($\widehat{Q}$);
  while $L \neq \varnothing$ do
    $w \leftarrow L.last()$
    if $g(w) \neq 0$ then
      while $i \leftarrow w.getNextGreenVertex()$ do
        $x \leftarrow w.mutate(i)$
        if $g(x) \neq 0$ then
          L.append(x);
        else
          $green[x.mutationLength()] \leftarrow green[x.mutationLength()] + 1$
        end if
      end while
    else
      $green[w.mutationLength()] \leftarrow green[w.mutationLength()] + 1$
    end if
  end while
```

Typical problems with such an algorithm arise with green sequences of infinite lengths (these may only appear if Q is not of finite cluster type). These will render the algorithm inefficient as it has no way of detecting them. Practical, yet imperfect, solutions include limiting the exploration up to a certain depth and/or the absolute values of the entries of the **c**-matrix.

Even if this algorithm is efficient in terms of memory footprint, and has provided some preliminary results, its CPU usage remains overwhelming and limits its usage to small instances. However, due to its explicit and exhaustive approach it allows to list all the maximal green sequences encountered, an additional information which might be of interest.

Remark A.1. Translating the definition of a maximal green sequence in the language of **c**-vectors and exchange matrices instead of that of ice quivers, it is possible to define maximal green sequences for a skew-symmetrisable exchange matrix $B \in M_n(\mathbb{Z})$. There is however a slight difference due to the fact that the sign-coherence for **c**-vectors is still conjectural in the skew-symmetrisable case, and therefore, so is the analogue of Theorem 1.6. Thus instead of mutating at green vertices, we will mutate at vertices which are not red. With this modification, since the input of Algorithm 1 only deals with the adjacency matrix $B(Q)$ of the cluster quiver Q and since the implemented mutation rule is the matrix mutation rule given in Definition 1.1, Algorithm 1 also applies to skew-symmetrisable exchange matrices. Some results obtained by this mean are provided in Appendices B.3 and B.7.

A.3. **Isomorphism discrimination.** The CPU intensive nature of Algorithm 1 could be greatly reduced using the fact that along the exploration tree many nodes may be isomorphic. Hence, branches can be cut and the computation can be reduced. This however complexifies the algorithm:

(1) Explored quivers must be stored in memory, in order to be able to test isomorphims;
(2) To limit the cost of searches in memory, complex data structures must be set up;
(3) Specific algorithms must be unrolled when an isomorphism is found with a quiver along a path leading to a maximal green sequence.

While the first two points are pure computer science, the last one requires explanations: let $\mathbf{i} = (i_1, \ldots, i_l)$ be a maximal green sequence. For $1 \leqslant k \leqslant l$, we let $\widehat{Q}^{(k)} = \mu_{i_k} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$. Let $\mathbf{j} = (j_1, \ldots, j_p)$ be a green sequence such that $\mu_{\mathbf{j}}(\widehat{Q})$ is isomorphic to $\widehat{Q}^{(k)}$ and $\mathbf{j} \neq (i_1, \ldots, i_k)$, then it can be asserted that $\mathbf{j}$ is the beginning of another maximal green sequence for Q with length $p+l-k$. Additionnal care must be taken when branches are cut because of isomorphisms to quivers which do not lead to maximal green sequences.

Implementing this algorithm allows a quick walk of the exploration tree. However, if the initial two questions are answered, the possibility to list all the maximal green sequences explicitly is lost: the enumeration becomes implicit.

Remark A.2. The implementation of this algorithm provided in [DP12] relies on NAUTY for the detection of isomorphims [McK81]. Therefore, it only works for adjacency matrices of (cluster) quivers, that is for skew-symmetric matrices and not for valued quivers. Therefore, the (optional) feature of isomorphism detection cannot be used for skew-symmetrisable exchange matrices and one has to use the implementation of Algorithm 1 in this case.

APPENDIX B. EXAMPLES

B.1. **Rank two oriented exchange graphs.** Any connected valued quiver with two vertices is either of infinite type or of type A_2, B_2, C_2 or G_2. We list below the corresponding oriented exchange graphs.

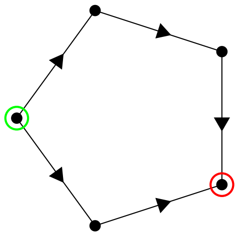


Figure 18. The oriented exchange graph of type A_2.

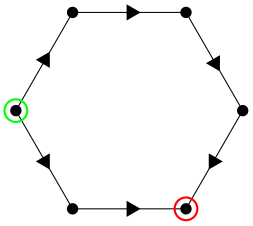


Figure 19. The oriented exchange graph of type B_2 or C_2.

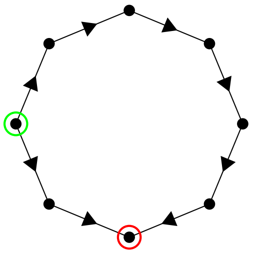


Figure 20. The oriented exchange graph of type G_2.

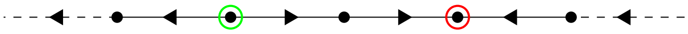


Figure 21. Oriented exchange graph of of rank two in infinite type.

B.2. **Examples of simply-laced cluster finite quivers.**

B.2.1. *Dynkin type A.* Figures 22, 23 and 24 show lengths of maximal green sequences for certain quivers of finite cluster type A.

B.2.2. *Dynkin type D.* Figure 25 shows the lengths of the maximal green sequences for the cluster finite quivers of type D_4.

Q	$1 \longrightarrow 2 \longrightarrow 3$	$1 \longrightarrow 2 \longleftarrow 3$	$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$
$\lvert\mathrm{green}(Q)\rvert$	9	10	9

Length			
3	1	2	
4	4	2	6
5	2	2	3
6	2	4	

Figure 22. Maximal green sequences for quivers in the mutation class of type A_3.

Q	$1 \to 2 \to 3 \to 4$	$1 \leftarrow 2 \to 3 \to 4$	$1 \to 2 \leftarrow 3 \to 4$	$\begin{tikzcd} & 2 \arrow[dl] & & \\ 1 \arrow[rr] & & 3 \arrow[ul] \arrow[r] & 4 \end{tikzcd}$
$\lvert\mathrm{green}(Q)\rvert$	98	141	179	101

Length				
4	1	3	5	
5	10	11	9	12
6	22	13	9	21
7	22	18	16	33
8	18	25	28	25
9	13	30	42	10
10	12	41	70	

Figure 23. Maximal green sequences for quivers in the mutation class of type A_4.

B.2.3. *Dynkin type E.* Consider the following quiver of type E_6:

$$Q: \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5. \\ & & 6 & & \end{tikzcd}$$

Then we have $\ell_{\min}(Q) = 6$, $\ell_{\max}(Q) = 36$ and $\lvert\mathrm{green}(Q)\rvert = 253\ 085\ 705\ 387$.

Consider the following quiver of type E_7:

$$Q: \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] \arrow[d] & 5 \arrow[r] & 6. \\ & & & 7 & & \end{tikzcd}$$

Then we have $\ell_{\min}(Q) = 7$, $\ell_{\max}(Q) = 63$ and

$$\lvert\mathrm{green}(Q)\rvert = 372\ 133\ 972\ 845\ 031\ 649\ 851\ 164.$$

Consider the following quiver of type E_8:

$$Q: \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] \arrow[d] & 6 \arrow[r] & 7. \\ & & & & 8 & & \end{tikzcd}$$

Then we have $\ell_{\min}(Q) = 8$, $\ell_{\max}(Q) = 120$ and $\lvert\mathrm{green}(Q)\rvert \sim 5.641 \cdot 10^{51}$.

B.3. **Examples of non-simply-laced cluster finite quivers.**

Q	A_2	A_3	A_4	A_5	A_6	A_7
$\|\text{green}(Q)\|$	2	9	98	2 981	340 549	216 569 887
Length						
2	1					
3	1	1				
4		4	1			
5		2	10	1		
6		2	22	20	1	
7			22	112	35	1
8			18	232	392	56
9			13	382	1 744	1 092
10			12	348	4 474	9 220
11				456	8 435	40 414
12				390	12 732	123 704
13				420	17 337	276 324
14				334	21 158	550 932
15				286	27 853	917 884
16					33 940	1 510 834
17					41 230	2 166 460
18					45 048	3 370 312
19					50 752	4 810 150
20					41 826	7 264 302
21					33 592	10 435 954
22						15 227 802
23						20 089 002
24						27 502 220
25						32 145 952
26						36 474 460
27						30 474 332
28						23 178 480
29						

FIGURE 24. Maximal green sequences for linearly oriented quivers of Dynkin type A_n, with $n \leqslant 7$.

B.3.1. *Dynkin type B.* Figure 26 shows the lengths of the maximal green sequences for the cluster finite valued quivers of type B_3.

B.4. **Dynkin type F_4.** Figure 27 shows the maximal green sequences for valued quivers of type F_4.

B.5. **Examples of simply-laced affine types.** For a quiver Q of affine type, we have (non-maximal) green sequences of infinite lengths. However, if Conjecture 1.22 holds, in order to list all the maximal green sequences, it is enough to find some $l \geqslant 1$ for which $0 < |\text{green}_{\leqslant l}(Q)| = |\text{green}_{\leqslant l+1}(Q)|$ and then $\text{green}(Q) = \text{green}_{\leqslant l}(Q)$. In the tables below, the empty cells should be read as zeros.

B.5.1. *Examples in types $\widetilde{A}$.* Figure 28 shows the maximal green sequences for certain affine quivers of type $\widetilde{A}$. We note that for all the values of n for which we could perform the computations, the empirical maximal length of a quiver of type $\widetilde{A}_{n,1}$ is $\frac{n(n+3)}{2}$.

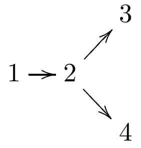
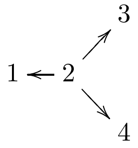
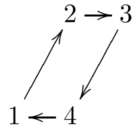
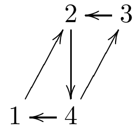
Q				
$\lvert\text{green}(Q)\rvert$	250	468	112	150
Length				
4	2	6		
5	10	6		6
6	10	6	32	24
7	26	18	44	40
8	16	24	20	22
9	18	24	16	18
10	24	24		16
11	72	144		24
12	72	216		
13				
14				
15				

FIGURE 25. Maximal green sequences for quivers in the mutation class of type D_4.

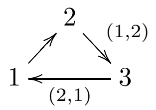
Q	$1 \to 2 \xrightarrow{(1,2)} 3$	$1 \leftarrow 2 \xleftarrow{(2,1)} 3$	$1 \to 2 \xleftarrow{(2,1)} 3$	$1 \leftarrow 2 \xrightarrow{(1,2)} 3$	
$\lvert\text{green}(Q)\rvert$	14	7	18	18	12
Length					
2					
3	1	3	2	2	
4	2	3	1	1	2
5	2		1	1	4
6	2	1	2	2	4
7					
8	3		4	4	2
9	4		8	8	
10					

FIGURE 26. Maximal green sequences for valued quivers in the mutation class of type B_3.

Q	$1 \to 2 \xrightarrow{(1,2)} 3 \to 4$	$1 \to 2 \xleftarrow{(2,1)} 3 \to 4$
$\lvert \mathrm{green}(Q) \rvert$	40 366	163 282
Length		
3		
4	1	5
5	6	8
6	14	2
7	10	8
8	8	4
9	18	10
10	44	48
11	32	48
12	20	39
13	48	136
14	35	65
15	78	100
16	181	330
17	136	260
18	665	1 104
19	1 668	4 072
20	2 002	5 843
21	4 592	12 672
22	11 643	42 391
23	13 420	62 676
24	5 741	33 461

FIGURE 27. Maximal green sequences for valued quivers of type F_4.

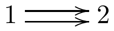
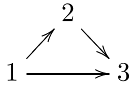
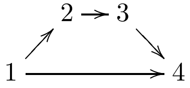
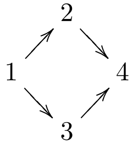
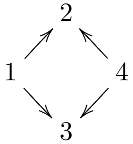
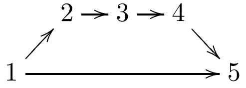
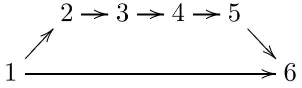
Q							
$\lvert\mathrm{green}^0(Q)\rvert$	1	5	75	100	100	4 882	1 645 136
Length							
2	1						
3		1					
4		2	1	2	4		
5		2	8	4	4	1	
6			9	12	8	18	1
7			11	24	20	73	33
8			22	18	16	116	314
9			24	16	16	162	1 036
10				24	32	290	2 375
11						520	4 176
12						1 076	7 734
13						1 380	15 830
14						1 246	34 178
15							72 986
16							143 626
17							252 023
18							371 780
19							397 012
20							342 032
21							

FIGURE 28. Maximal green sequences for type $\widetilde{A}$ quivers.

B.6. **Examples in types $\widetilde{D}$.** Figure 29 shows the lengths of maximal green sequences for the affine quivers of type $\widetilde{D}_4$. It is interesting to note in that example that even if the number of maximal green sequences depend on the orientations, the minimal lengths and the empirical maximal lengths do not.

	$\begin{tikzcd} & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[r] \arrow[dr] & 4 \\ & & 5 \end{tikzcd}$	$\begin{tikzcd} 1 \arrow[dr] & & 3 \\ & 2 \arrow[ur] \arrow[dr] & \\ 4 \arrow[ur] & & 5 \end{tikzcd}$	$\begin{tikzcd} 1 \arrow[dr] & & 3 \arrow[dl] \\ & 2 & \\ 4 \arrow[ur] & & 5 \arrow[ul] \end{tikzcd}$
$\lvert\mathrm{green}^0(Q)\rvert$	286 068	210 284	641 496
Length			
5	6	4	24
6	36	24	24
7	36	40	24
8	108	168	72
9	150	144	120
10	252	272	240
11	348	400	384
12	1 266	1 144	960
13	2 394	1 720	2 400
14	2 208	1 792	4 224
15	3 192	2 912	5 760
16	5 976	4 928	9 792
17	10 512	8 192	19 584
18	13 056	9 984	24 192
19	16 704	12 672	27 648
20	38 016	31 104	69 120
21	98 496	72 576	228 096
22	93 312	62 208	248 832

FIGURE 29. Maximal green sequences for affine quivers of type $\widetilde{D}_4$.

For any $n \geqslant 4$, we denote by Q_n the quiver of affine type $\widetilde{D}_n$ given by

$$\begin{tikzcd} & 1 \arrow[dr] & & & & n \\ Q_n : & & 3 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[ur] \arrow[dr] & \\ & 2 \arrow[ur] & & & & n+1. \end{tikzcd}$$

Then Figure 30 shows the values of $\ell_{\min}(Q_n)$, $\ell^0_{\max}(Q_n)$ and $\lvert\mathrm{green}^0(Q_n)\rvert$ for certain small values of n. We note that for all the values of n for which we performed the computations, we obtained $\ell^0_{\max}(Q_n) = 2n^2 + 6n + 2$.

B.6.1. *Examples in types $\widetilde{E}$.* For the following quiver of type $\widetilde{E}_6$

$$\begin{tikzcd} & & & 7 \arrow[d] & & \\ & & & 6 \arrow[d] & & \\ Q : & 1 & 3 \arrow[l] & 6 \arrow[l] \arrow[r] & 4 \arrow[r] & 2 \end{tikzcd}$$

n	$\ell_{\min}(Q_n)$	$\ell^0_{\max(Q_n)}$	$\lvert\mathrm{green}^0(Q_n)\rvert$
4	5	22	210 284
5	6	38	371 667 875 684
6	7	58	528 229 038 497 072 158 920
7	8	82	1 334 686 668 231 927 938 739 442 459 338 512

Figure 30. Maximal Green Sequences for affine quivers of type $\widetilde{D}_n$.

we obtain

$$\ell_{\min}(Q) = 7, \quad \ell^0_{\max}(Q) = 78$$

and

$$\mathrm{green}^0(Q) = 212\,876\,586\,503\,402\,188\,760\,490\,821\,544.$$

For the following quiver of type $\widetilde{E}_7$

$$\begin{tikzcd} & & & & 9 & & & \\ Q: & 1 & 3 \arrow[l] & 5 \arrow[l] & 8 \arrow[l] \arrow[r] \arrow[u] & 6 \arrow[r] & 4 \arrow[r] & 2 \end{tikzcd}$$

we obtain

$$\ell_{\min}(Q) = 8, \quad \ell^0_{\max}(Q) = 159$$

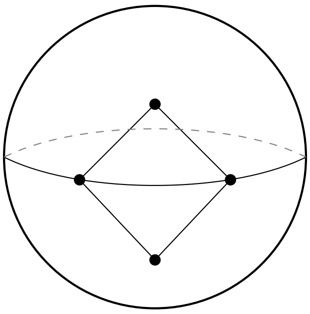
and $\mathrm{green}^0(Q) \sim 1.976 \cdot 10^{69}$.

B.7. **Examples of non-simply-laced affine types.** For affine types, we have (non-maximal) green sequences of arbitrary lengths. Therefore, we could only compute $\mathrm{green}_{\leqslant l}(Q)$ for various values of l. In the table below we chose $l = 25$. The empty or non-appearing cells for $l \leqslant 25$ should be read as zeros.

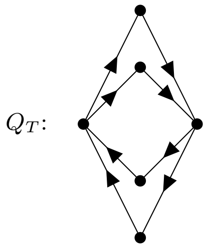
	$1 \xrightarrow{(2,1)} 2 \xrightarrow{(1,2)} 3$	$1 \xrightarrow{(2,1)} 2 \xleftarrow{(2,1)} 3$	$1 \xleftarrow{(1,2)} 2 \xrightarrow{(1,2)} 3$	$\begin{tikzcd} & 2 \arrow[dr, "(1,2)"] & \\ 1 \arrow[ur, "(2,1)"] & & 3 \arrow[ll] \end{tikzcd}$
$\lvert\mathrm{green}_{\leqslant 25}(Q)\rvert$	7	6	6	7
Length				
2				
3	1	2	2	
4				
5	4	2	2	4
6				
7	2	2	2	3
8				

Figure 31. Maximal green sequences for valued quivers in the mutation class of type $\widetilde{B}_2$.

B.8. **An example from a surface without boundary.** Consider the following triangulation T of the sphere with four punctures.



As defined in [FST08], the quiver Q_T corresponding to this triangulation is the following.



Then a direct computation shows that

$$\ell_{\min}(Q_T) = 12,\ \ell^0_{\max}(Q_T) = 46$$

and

$$|\mathrm{green}^0(Q_T)| = 1\ 044\ 863\ 666\ 576.$$

B.9. **An exceptional mutation-finite type.** Consider the following quiver

$$\begin{tikzcd}
 & & 2 \arrow[r, shift left] \arrow[r, shift right] & 3 \arrow[dl] \\
Q: & 6 & 1 \arrow[l] \arrow[u] \arrow[d] & \\
 & & 4 \arrow[r, shift left] \arrow[r, shift right] & 5 \arrow[ul]
\end{tikzcd}$$

which first appeared in [DO08] as an example of mutation-finite quiver which is not arising from a surface. Then we obtained

$$\ell_{\min}(Q) = 10,\ \ell^0_{\max}(Q) = 30 \text{ and } |\mathrm{green}^0(Q)| = 119\ 819\ 022.$$

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