The Higher Relation Bimodule

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Dedicated to Idun Reiten for her 70th birthday

Abstract

Given a finite dimensional algebra A of finite global dimension, we consider the trivial extension of A by the $A-A$-bimodule $\oplus_{i\geq 2}\operatorname{Ext}^i_A(DA,A)$, which we call the higher relation bimodule. We first give a recipe allowing to construct the quiver of this trivial extension in case A is a string algebra and then apply it to prove that, if A is gentle, then the tensor algebra of the higher relation bimodule is gentle.

1 Introduction

The objective of this paper is to describe a new class of algebras, which we call higher relation extensions. Our motivation comes from the study of cluster-tilted algebras, introduced by Buan, Marsh and Reiten in [BMR], and in [CCS] for type $\mathbb{A}$. Indeed, it was shown in [ABS] that an algebra A is cluster-tilted if and only if there exists a tilted algebra C such that A is isomorphic to the trivial extension of C by the $C-C$-bimodule $\operatorname{Ext}^2_C(DC,C)$. Moreover, a recipe for constructing the quiver of this trivial extension was given in [ABS, Theorem 2.6]. The proof of the latter result rests on the fact that tilted algebras have global dimension 2.

Here, we consider the more general case of an algebra A having an arbitrary finite global dimension and consider its trivial extension by the bimodule $\bigoplus_{i\geq 2}\operatorname{Ext}^i_A(DA,A)$, which we call the higher relation bimodule. We believe that this class of algebras, which we call higher relation extensions, will be useful in the study of m-cluster-tilted algebras (see [FPT] [B]). Our first objective is to describe the ordinary quiver of the higher relation extension of A in the case where A is a string algebra in the sense of Butler and Ringel [BR]. We also assume that the quiver of A is a tree. This is no restriction, because the universal cover of a string algebra is a string tree [G]. Our theorem reads as follows.

Theorem 1.1 *Let $A = kQ/I$ be a string tree algebra. Then there exist two sequences $(c_\ell), (z_\ell)$ of points of Q such that the arrows in the quiver of the higher relation extension are exactly those of Q plus one additional arrow from each z_ℓ to c_ℓ.*

Our proof is constructive, in the sense that we give an algorithm allowing to construct explicitly the sequences (c_ℓ) and (z_ℓ) and thus the quiver of the higher relation extension.

We then consider the particular case where A is a gentle algebra. Gentle algebras form an important subclass of the class of string algebras. Part of their importance comes from the fact that this subclass is stable under derived equivalences [SZ]. While, as we show, the higher relation extension algebra of a gentle algebra is monomial but not necessarily gentle, we prove using our Theorem 1.1, that the tensor algebra of the higher relation bimodule is gentle.

Theorem 1.2 *Let $A = kQ/I$ be a gentle algebra, then the tensor algebra of the higher relation bimodule $\bigoplus_{i\geq 2} \mathrm{Ext}^i_A(DA, A)$ is gentle.*

The paper is organised as follows. In section 2, we fix the notation and recall some facts and results about string and gentle algebras. Section 3 is devoted to the computation of projective resolutions and injective coresolutions of uniserial modules over a string algebra. We study the top of the higher extension bimodule in section 4 and we prove Theorem 1.1 in section 5. Sections 6 and 7 are devoted to the case of gentle algebras.

2 Preliminaries

2.1 Notation

Throughout this paper, algebras are basic and connected finite dimensional algebras over an algebraically closed field k. Given an algebra A, there always exists a (unique) quiver $Q = (Q_0, Q_1)$ and (at least) an isomorphism $A \cong kQ/I$, where kQ is the path algebra of Q, and I is an admissible ideal of kQ, see, for instance, [ASS]. Such an isomorphism is called a **presentation** of the algebra. Given an algebra A, we denote by $\mathrm{mod}\, A$ the category of finitely generated right A-modules, and by $D = \mathrm{Hom}_k(-, k)$ the standard duality between $\mathrm{mod}\, A$ and $\mathrm{mod}\, A^{op}$. For a point x in the quiver Q of A, we denote by $P(x), I(x), S(x), e_x$ respectively, the corresponding indecomposable projective module, injective module, simple module and primitive idempotent. We recall that a module M can be equivalently considered as a bound quiver representation $M = (M_i, M_\alpha)_{i\in Q_0, \alpha\in Q_1}$. The projective, or injective, dimension of a module M is denoted by $\mathrm{pd}\, M$, or $\mathrm{id}\, M$, respectively. The global dimension of A is denoted by $\mathrm{gldim}\, A$. For facts about the category $\mathrm{mod}\, A$, we refer the reader to [ARS] or [ASS].

2.2 Trivial extensions

Let A be an algebra and M an $A - A$-bimodule . The trivial extension of A by M is the algebra $A \ltimes M$ with underlying k-vector space

$$A \oplus M = \{(a, m)|a \in A,\, m \in M\}$$

and the multiplication defined by

$$(a, m) \cdot (a', m') = (aa', am' + ma')$$

for $a, a' \in A$ and $m, m' \in M$.

For instance, an algebra A is cluster-tilted if and only if there exists a tilted algebra C such that A is the trivial extension of C by the so-called relation bimodule $\mathrm{Ext}^2_C(DC, C)$, see [ABS].

The ordinary quiver of a trivial extension is computed as follows (see, for instance, [ABS]): let M be an $A-A$ bimodule, then the quiver $Q_{A\ltimes M}$ of $A \ltimes M$ is given by

1) $\left(Q_{A\ltimes M}\right)_0 = (Q_A)_0$

2) For $z, c \in (Q_A)_0$, the set of arrows in $Q_{A\ltimes M}$ from z to c equals the set of arrows in Q_A from z to c plus

$$\dim_k \frac{e_z M e_c}{e_z M(\operatorname{rad} A)e_c + e_z(\operatorname{rad} A)Me_c}$$

additional arrows from z to c.

The latter arrows are called **new** arrows, while the former are the **old** arrows.

2.3 String algebras

Recall from [BR] (see also [WW]) that an algebra A is called a **string algebra** if there exists a presentation $A = kQ/I$ (called **a string presentation**) such that:

S1) I is generated by a set of paths (thus A is monomial).

S2) Each point in Q is the source of at most two arrows and the target of at most two arrows.

S3) For an arrow α, there is at most one arrow β and at most one arrow γ such that $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

Whenever we deal with a string algebra A, we always assume that it is given by a string presentation $A = kQ/I$. We assume moreover that the relations (that is, the generators of I) are of minimal length.

A reduced walk ω in Q is called a **string** if it contains no zero relations.To each string ω in Q, we can associate a so-called **string module** [BR] in the following way. If ω is the stationary path at j, then $M(\omega) = S(j)$. Let $\omega = \omega_1\omega_2\cdots\omega_t$ be a string, with each ω_i an arrow or the inverse of an arrow. For each i such that $0 \le i \le t$, let $V_i = k$; and for $1 \le i \le t$, let V_{ω_i} be the identity map sending $x \in V_i$ to $x \in V_{i+1}$ if ω_i is an arrow and otherwise the identity map sending $x \in V_{i+1}$ to $x \in V_i$. The string module $M(\omega)$ is then defined as follows: for each $j \in Q_0$, $M(\omega)_j$ is the direct sum of the vector spaces V_i such that the source of ω_i is j if j appears in ω, and otherwise $M(\omega)_j = 0$; for each $\alpha \in Q_1$, $M(\omega)_\alpha$ is the direct sum of the maps V_{ω_i} such that $\omega_i = \alpha$ or $\omega_i^{-1} = \alpha$ if α appears in ω, and otherwise $M(\omega)_\alpha = 0$.

A non-zero path ω in Q for a to b will sometimes be denoted by $[a, b]$, whenever there is no ambiguity. Then, the corresponding string module is denoted by $M(\omega) = M[a, b]$.

We also recall that the endomorphism ring of a projective module over a string tree algebra A (a full subcategory of A) is also a string tree algebra.

2.4 Gentle algebras

Recall from [AS] that a string algebra $A = kQ/I$ is called **gentle** if in addition to $(S1)$, $(S2)$, $(S3)$, the bound quiver (Q, I) satisfies:

G1) For an arrow α, there is at most one arrow β and at most one arrow γ such that $\alpha\beta \in I$ and $\gamma\alpha \in I$.

G2) I is quadratic (that is, I is generated by paths of length 2).

For instance, cluster-tilted algebras of types $\mathbb{A}$ and $\tilde{\mathbb{A}}$ are gentle [ABCP].

3 Resolutions of uniserial modules

In this section, we compute minimal projective resolutions of an injective module, and dually minimal injective coresolutions of a projective module over a string algebra. Throughout, we let $A = kQ/I$ be a string presentation.

Definition 3.1 *Let $[x_0, y_0]$ be a non-zero path from x_0 to y_0 in Q. We define inductively the right maximal sequence of $[x_0, y_0]$ as follows. This is a finite sequence of non-zero paths $[x_{i_1 i_2 \cdots i_t}, y_{i_1 i_2 \cdots i_t}]$ with $i_1 = 0$ and $i_j \in \{0, 1\}$ such that*

1) Let $[x_0, y_{00}]$,$[x_0, y_{01}]$ be the maximal non-zero paths starting at x_0 (where we agree that $[x_0, y_0]$ is contained in $[x_0, y_{00}]$).

Then we set

$$[x_{00}, y_{00}] = [x_0, y_{00}] \backslash [x_0, y_0]$$

and

$$[x_{01}, y_{01}] = [x_0, y_{01}] \backslash [x_0, y_0] = [x_0, y_{01}] \backslash \{x_0\}.$$

Observe that the path $[x_0, y_{01}]$ may be empty (that is, the point y_{01} does not exist).

2) Inductively, assume that $[x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}}]$ has been defined. Let $[x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}0}]$ and $[x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}1}]$ be the maximal non-zero paths starting at $x_{0i_2 \cdots i_{t-1}}$, where we agree that $[x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}}]$ is contained in $[x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}0}]$.

Then we set

$$[x_{0i_2 \cdots i_{t-1}0}, y_{0i_2 \cdots i_{t-1}0}] = [x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}0}] \backslash [x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}}]$$

and

$$[x_{0i_2 \cdots i_{t-1}1}, y_{0i_2 \cdots i_{t-1}1}] = [x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}1}] \backslash \{x_{0i_2 \cdots i_{t-1}}\}.$$

Observe that one or both of the paths $[x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}0}]$ and $[x_{0i_2 \cdots i_{t-1}}, y_{0i_2 \cdots i_{t-1}1}]$ may be empty and in this case the corresponding point may not exist.

The left maximal sequence of a non-zero path is defined dually. However, we do it explicity for the convenience of the reader.

Definition 3.2 *Let $[r_0, s_0]$ be a non-zero path from r_0 to s_0 in Q. We define inductively the left maximal sequence of $[r_0, s_0]$ as follows. This is a finite sequence of non-zero paths $[r_{i_1 i_2 \cdots i_t}, s_{i_1 i_2 \cdots i_t}]$ with $i_1 = 0$ and $i_j \in \{0, 1\}$ such that*

1) *Let $[r_{00}, s_0]$,$[r_{01}, s_0]$ be the maximal non-zero paths ending at s_0, where we agree that $[r_0, s_0]$ is contained in $[r_{00}, s_0]$. Then we set*

$$[r_{00}, s_{00}] = [r_{00}, s_0] \backslash [r_0, s_0]$$

and

$$[r_{01}, s_{01}] = [r_{01}, s_0] \backslash [r_0, s_0] = [r_{01}, s_0] \backslash \{s_0\}.$$

Observe that the path $[r_{01}, s_0]$ may be empty (that is, the point r_{01} does not exist).

2) *Inductively, assume that $[r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1}}]$ has been defined. Let $[r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1} 0}]$ and $[r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1} 1}]$ be the maximal non-zero paths ending at $s_{0i_2 \cdots i_{t-1}}$, where we agree that $[r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1}}]$ is contained in $[r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1} 0}]$. Then we set*

$$[r_{0i_2 \cdots i_{t-1} 0}, s_{0i_2 \cdots i_{t-1} 0}] = [r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1} 0}] \backslash [r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1}}]$$

and

$$[r_{0i_2 \cdots i_{t-1} 1}, s_{0i_2 \cdots i_{t-1} 1}] = [r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1} 1}] \backslash \{s_{0i_2 \cdots i_{t-1}}\}.$$

Observe that one or both of the paths $[r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1} 0}]$ and $[r_{0i_2 \cdots i_{t-1}}, s_{0i_2 \cdots i_{t-1} 1}]$ may be empty and in this case the corresponding point may not exist.

Our first result follows directly from the above definitions.

Theorem 3.3 *Let $A = kQ/I$ be a string algebra.*

a) *If $[x_0, y_0]$ is a non-zero path in Q and*

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M[x_0, y_0] \to 0$$

is a minimal projective resolution then, for $l \geq 1$,

$$P_{l-1} = \bigoplus P(x_{i_1 i_2 \cdots i_l})$$

where the direct sum is taken over all l-tuples $(0, i_2, \cdots, i_l)$ such that $i_k \in \{0, 1\}$ for all k with $2 \leq k \leq l$ and the point $x_{0i_2 \cdots i_l}$ in definition 3.1 exists.

b) *If $[r_0, s_0]$ is a non-zero path in Q and*

$$0 \to M[r_0, s_0] \to I_0 \to I_1 \to I_2 \to I_3 \to \cdots$$

is a minimal injective coresolution then, for $l \geq 1$,

$$I_{l-1} = \bigoplus I(s_{i_1 i_2 \cdots i_l})$$

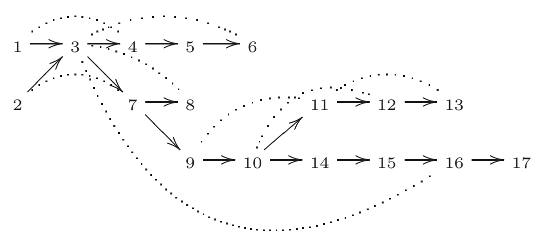
where the direct sum is taken over all l-tuples $(0, i_2, \cdots, i_l)$ such that $i_k \in \{0, 1\}$ for all k with $2 \leq k \leq l$ and the point $s_{0i_2 \cdots i_l}$ in definition 3.2 exists.

Proof. We only prove a), since the proof of b) is dual.
Clearly, the projective cover of the uniserial module $M[x_0, y_0]$ is $P(x_0)$, whose support consists of the (at most two) maximal non-zero paths $[x_0, y_{01}]$ and $[x_0, y_{01}]$ starting at x_0. Then,

$$\Omega^1 M[x_0, y_0] = M[x_{00}, y_{00}] \oplus M[x_{01}, y_{01}]$$

where x_{00} and x_{01} are defined as above. The rest follows from an easy induction. $\square$

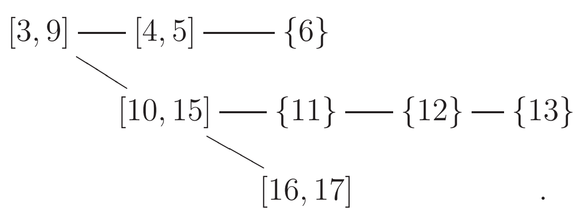
Example 3.4 *Suppose that the string algebra is given by the following bound quiver.*



Here, and in the sequel, dotted lines indicate relations.
Considering the path $[x_0, y_0] = [3, 9]$, *the right maximal sequence is*

$$[3, 9], [10, 15], [4, 5], [16, 17], [11, 11] = \{11\}, [6, 6] = \{6\}, [12, 12] = \{12\}, [13, 13] = \{13\}.$$

This sequence may be conveniently shown in the following diagram



The minimal projective resolution of $M[3, 9]$ *is the following (compare with the above diagram)*

$$0 \to P(13) \to P(12) \to$$

$$\to P(16) \oplus P(11) \oplus P(6) \to P(10) \oplus P(4) \to P(3) \to M[3, 9] \to 0,$$

where the morphisms are induced by the corresponding paths.
Similarly, taking $[r_0, s_0] = [3, 9]$, *the left maximal sequence is*

$$\{1\} — [3, 9]$$

from which we deduce the minimal injective coresolution

$$0 \to M[3, 9] \to I(9) \to I(1) \to 0.$$

We are interested in computing resolutions of injective and projective indecomposable modules. These modules are usually not uniserial, neither are in general their first syzygy or cosyzygy, respectively. In order to apply Theorem 3.3, the next lemma shows that we must start from the second.

Lemma 3.5 *a) The second syzygy of an indecomposable injective module is the direct sum of at most six uniserial modules.*

b) The second cosyzygy of an indecomposable projective module is the direct sum of at most six uniserial modules.

Proof. Let $I(c)$ be an indecomposable injective A-module. If $I(c)$ is uniserial, then there is nothing to prove because of Theorem 3.3. Otherwise, let $\operatorname{top} I(c) = S(a_0) \oplus S(a_1)$. Then the projective cover of $I(c)$ is $P(a_0) \oplus P(a_1)$. Let $[a_i, b_i]$ and $[a_i, b_i']$ be the two maximal non-zero paths starting at a_i (with $i = 0, 1$), where we agree that $[a_0, c]$ is contained in $[a_0, b_0']$ and $[a_1, c]$ is contained in $[a_1, b_1']$. Let d_i be the direct successor of a_i on the path $[a_i, b_i]$ then

$$\Omega^1 I(c) = M[d_0, b_0] \oplus M[d_1, b_1] \oplus M$$

where M is an indecomposable module, usually non-uniserial, such that $\operatorname{top} M = S(c)$ and $\operatorname{soc} M = S(b_0') \oplus S(b_1')$. Hence, the projective cover of $\Omega^1 I(c)$ is $P(d_0) \oplus P(d_1) \oplus P(c)$, and $\Omega^2 I(c)$ is the direct sum of at most six uniserial modules obtained as follows.
Let $[d_i, b_{i0}], [d_i, b_{i1}]$ be the maximal non-zero paths starting in d_i (with $i = 0, 1$), where we agree that $[d_i, b_i]$ is contained in $[d_i, b_{i0}]$. Then let

$$[d_{i0}, b_{i0}] = [d_i, b_{i0}] \backslash [d_i, b_i]$$

$$[d_{i1}, b_{i1}] = [d_i, b_{i1}] \backslash [d_i, b_i] = [d_i, b_{i1}] \backslash \{d_i\}.$$

Let also $[c, c_0]$ and $[c, c_1]$ be the maximal non-zero paths starting at c, where we agree that $[c, b_0']$ is contained in $[c, c_0]$ and $[c, b_1']$ is contained in $[c, c_1]$.
We let

$$[c_0', c_0] = [c, c_0] \backslash [c, b_0']$$

and

$$[c_1', c_1] = [c, c_1] \backslash [c, b_1'].$$

It is then clear that

$$\begin{aligned} \Omega^2 I(c) &= M[d_{00}, b_{00}] \oplus M[d_{01}, b_{01}] \\ &\oplus M[d_{10}, b_{10}] \oplus M[d_{11}, b_{11}] \\ &\oplus M[c_0', c_0] \oplus M[c_1', c_1] \end{aligned}$$

which establishes a). Statement b) is dual. □

Corollary 3.6 *a) Let $I(c)$ be an indecomposable injective module such that* $\operatorname{top}(I(c)) = S(a_0) \oplus S(a_1)$. *Then $I(c)$ has the following minimal projective resolution*

$$\cdots \to \bigoplus_{j;(0,i_2,i_3)} P(x^j_{0i_2i_3}) \to \bigoplus_{j;(0,i_2)} P(x^j_{0i_2}) \to \bigoplus_j P(x^j_0) \to$$

$$\to P(d_0) \oplus P(c) \oplus P(d_1) \to P(a_0) \oplus P(a_1) \to I(c) \to 0$$

with the morphisms induced by the paths, where $\{x_0^j \mid 1 \leq j \leq 6\} = \{d_{00}, d_{01}, d_{10}, d_{11}, c'_0, c'_1\}$ *and* $i_j \in \{0, 1\}$.

b) *Let* $P(z)$ *be an indecomposable projective module such that* $\mathrm{soc}(P(z)) = S(w_0) \oplus S(w_1)$. *Then* $P(z)$ *has the following minimal injective coresolution*

$$0 \to P(z) \to I(w_0) \oplus I(w_1) \to I(v_1) \oplus I(z) \oplus I(v_2) \to \bigoplus_j I(s_0^j) \to$$

$$\to \bigoplus_{j;(0,i_2)} I(s_{0i_2}^j) \to \bigoplus_{j;(0,i_2,i_3)} I(s_{0i_2i_3}^j) \to \dots$$

with the morphisms induced by the paths, where $\{s_0^j \mid 1 \leq j \leq 6\}$ *are as above and* $i_j \in \{0, 1\}$.

Proof. This follows from Lemma 3.5 and Theorem 3.3. □

Corollary 3.7 *With the above notations*

a) *All the points* $x_0^j, x_{0i_2}^j, \cdots, x_{0i_2\dots i_l}^j$ *are targets of relations.*

b) *All the points* $s_0^j, s_{0i_2}^j, \cdots, s_{0i_2\dots i_l}^j$ *are sources of relations.*

Proof. This follows from the construction of these points. □

4 The top of the higher relation bimodule

Definition 4.1 *Let* A *be a finite dimensional algebra of finite global dimension. The* $A-A$*-bimodule* $\left(\bigoplus_{i\geq 2} \mathrm{Ext}_A^i(DA, A)\right)$ *with the natural action is called the* **higher relation bimodule**. *The trivial extension*

$$A \ltimes \Big(\bigoplus_{i\geq 2} \mathrm{Ext}_A^i(DA, A)\Big)$$

of A *by its higher relation bimodule is called the* **higher relation extension** *of* A.

If $\mathrm{gldim}\, A \leq 2$, then the higher relation extension of A coincides with its relation extension, as defined in [ABS].

Our objective in this section is to construct the ordinary quiver of the higher relation extension of a string algera A of finite global dimension.

As mentioned in the introduction, we also assume that the ordinary quiver Q_A of A is a tree.

Let thus $A = kQ/I$ be a string algebra, with Q a tree and M an $A-A$-bimodule. We have

$$\mathrm{rad}\, M = M(\mathrm{rad}\, A) + (\mathrm{rad}\, A)M$$

and then

$$\operatorname{top} M = M/[M(\operatorname{rad} A) + (\operatorname{rad} A)M].$$

If $M = \bigoplus_{i\geq 2} \operatorname{Ext}^i_A(DA, A)$, then, clearly, $\operatorname{top} M = \bigoplus_{i\geq 2} \operatorname{top} \operatorname{Ext}^i_A(DA, A)$. In order to describe this top, we start by describing the modules $\operatorname{top}_A \operatorname{Ext}^i_A(I(c), A)$ and $\operatorname{top} \operatorname{Ext}^i_A(DA, P(z))_A$ for all points $c, z \in (Q_A)_0$.

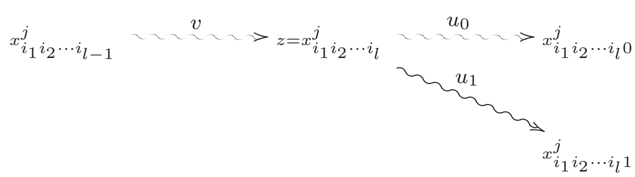
In the following, we use the notation of section 3.

Proposition 4.2 *Let $A = kQ/I$ be a string tree algebra and $l \geq 0$. Then $\operatorname{Ext}^{l+2}_A(I(c), P(z)) \neq 0$ if and only if one of the following two conditions hold:*

a) there exists a non-zero path $\omega : z \rightsquigarrow x^j_{i_1 i_2 \cdots i_{l+1}}$ not passing through $x^j_{i_1 i_2 \cdots i_l}$ and whose compositions with $x^j_{i_1 i_2 \cdots i_{l+1}} \rightsquigarrow x^j_{i_1 i_2 \cdots i_{l+2}}$ are both zero.

b) $z = x^j_{i_1 i_2 \cdots i_l}$ and $x^j_{i_1 i_2 \cdots i_l 0}$, $x^j_{i_1 i_2 \cdots i_l 1}$ both exist. In this case, a non-zero element is induced from the difference of the two paths $x^j_{i_1 i_2 \cdots i_l} \rightsquigarrow x^j_{i_1 i_2 \cdots i_l 0}$ and $x^j_{i_1 i_2 \cdots i_l} \rightsquigarrow x^j_{i_1 i_2 \cdots i_l 1}$.

Remark 4.3 *Observe that in case (b), we have the following situation*

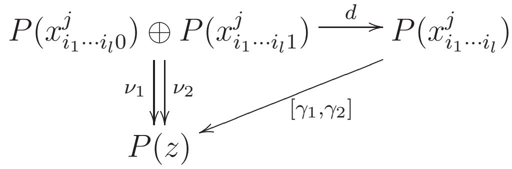


where vu_0, vu_1 are zero paths.

Proof. Let

$$\cdots \to \oplus P(x^j_{i_1 \cdots i_{l+2}}) \overset{d_{l+3}}{\to} \oplus P(x^j_{i_1 \cdots i_{l+1}}) \overset{d_{l+2}}{\to} \oplus P(x^j_{i_1 \cdots i_l}) \to \cdots \to P_c \to I(c) \to 0$$

be a minimal projective resolution of $I(c)$. Recall that the morphisms d_k are induced from the paths in Q.
If condition (a) holds then it follows from the definition of $\operatorname{Ext}^{l+2}_A(I(c), P(z))$ that ω induces a non-zero element in $\operatorname{Ext}^{l+2}_A(I(c), P(z))$.
If condition (b) holds, then $P_{l+2} = \oplus P(x^j_{i_1 \cdots i_{l+1}})$ has two indecomposable summands $P(x^j_{i_1 \cdots i_l 0})$, $P(x^j_{i_1 \cdots i_l 1})$, whose images $d(P(x^j_{i_1 \cdots i_l 0}))$ and $d(P(x^j_{i_1 \cdots i_l 1}))$ lie in the same indecomposable summand $P(x^j_{i_1 \cdots i_l})$ of P_{l+1}, together with two non-zero morphisms $\nu_i : P(x^j_{i_1 \cdots i_l i_{l+1}}) \to P(z)$ $(i_{l+1} = 0, 1)$ such that there exist two morphisms $\gamma_i : P(x^j_{i_1 \cdots i_l}) \to P(z)$ with $\nu_i = \gamma_i d$.

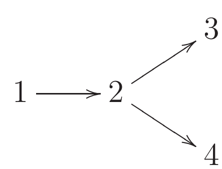


In this case $[\nu_1 \ \ -\nu_2]^t : P(x^j_{i_1\cdots i_l 0}) \oplus P(x^j_{i_1\cdots i_l 1}) \to P(z)$ does not factor through $P(x^j_{i_1\cdots i_l})$ because $\dim_{\mathrm{k}} \mathrm{Hom}(P(x^j_{i_1\cdots i_l 0}) \oplus P(x^j_{i_1\cdots i_l 1}), P(z)) = 2$ while $\dim_{\mathrm{k}} \mathrm{Hom}(P(x^j_{i_1\cdots i_l}), P(z)) = 1$, since the algebra is a tree algebra. This shows that $\mathrm{Ext}^{l+2}_A(I(c), P(z)) \neq 0$.

Conversely, suppose that $\mathrm{Ext}^{l+2}_A(I(c), P(z))$ contains a non-zero element $[f]$. Then $[f]$ is in the class of a morphism $f : \oplus P(x^j_{i_1\cdots i_{l+1}}) \to P(z)$ such that $f d_{l+3} = 0$. Since A is a tree string algebra, there are at most two indecomposable summands on which f is non-zero, because otherwise there are non-zero paths from z to three points $x^j_{i_1\cdots i_{l+1}}$ and these induce a full subcategory of type $\mathbb{D}_4$ which contradicts the fact that A is string. Thus we get a morphism $f : P(x^j_{i_1\cdots i_{l+1}}) \oplus P(x^{j'}_{i'_1\cdots i'_{l+1}}) \to P(z)$ which does not factor through d_{l+2}. If $z = x^j_{i_1\cdots i_l}$ then we must have $j = j'$, $i = i', \cdots, i_l = i'_l$ and $i_{l+1} \neq i'_{l+1}$. Suppose $z \neq x^j_{i_1\cdots i_l}$. If both non-zero paths $z \rightsquigarrow x^j_{i_1\cdots i_{l+1}}$, $z \rightsquigarrow x^{j'}_{i'_1\cdots i'_{l+1}}$ which induce f pass through $x^j_{i_1\cdots i_l}$ then we have a contradiction to A being string. If one non-zero path $z \rightsquigarrow x^j_{i_1\cdots i_{l+1}}$ passes through $x^j_{i_1\cdots i_l}$ then the other satisfies condition (a). Indeed, the composition with $x^{j'}_{i'_1 i'_2\cdots i'_{l+1}} \rightsquigarrow x^{j'}_{i'_1 i'_2\cdots i'_{l+2}}$ vanishes because our original path corresponds to an element of $\mathrm{Ext}^{l+2}_A(I(c), P(z))$. Similarly, if $z \rightsquigarrow x^j_{i_1\cdots i_{l+1}}$ does not pass through neither $x^j_{i_1\cdots i_l}$ nor $x^{j'}_{i'_1\cdots i'_{l+1}}$ then both paths satisfy condition (a). $\square$

The following example ilustrates condition (b).

Example 4.4 *Let A be given by the quiver*



bound by $\mathrm{rad}^2 A = 0$. *Then the minimal projective resolution of $I(1)$ is*

$$0 \to P(3) \oplus P(4) \to P(2) \to P(1) \to I(1) \to 0.$$

Let $j_1 : P(3) \to P(2)$ and $j_2 : P(4) \to P(2)$ be the canonical inclusions, then it is easily seen that the morphism

$$[j_1 \quad -j_2]^t : P(3) \oplus P(4) \to P(2)$$

induces a non-zero element of $\mathrm{Ext}^2_A(I(1), P(2))$.

Corollary 4.5 *Assume A is a gentle tree algebra, then $\mathrm{Ext}^{l+2}_A(I(c), P(z)) \neq 0$ if and only if there exists a non-zero path $\omega : z \rightsquigarrow x^j_{i_1 i_2\cdots i_{l+1}}$ not passing through $x^j_{i_1 i_2\cdots i_l}$ and whose compositions with $x^j_{i_1 i_2\cdots i_{l+1}} \rightsquigarrow x^j_{i_1 i_2\cdots i_{l+2}}$ are both zero.*

Proof. Indeed, if A is gentle, then condition (b) cannot occur as shown in the remark preceding the proof. $\square$

Corollary 4.6 *a) Let $\omega : z \rightsquigarrow x^j_{i_1 i_2\cdots i_{l+1}}$ be a non-zero path as in Proposition 4.2 a).*

a1) Assume that a point $x^j_{i_1 i_2 \cdots i_{l+2}}$ exists, then w induces an element of the top of ${}_A\operatorname{Ext}^{l+2}_A(I(c),A)$ if and only if z is the starting point of a relation of the form $\omega\omega'$ when $\omega' : x^j_{i_1 i_2 \cdots i_{l+1}} \rightsquigarrow x^j_{i_1 i_2 \cdots i_{l+2}}$

a2) Assume that no point $x^j_{i_1 i_2 \cdots i_{l+2}}$ exists, then w induces an element of the top of ${}_A\operatorname{Ext}^{l+2}_A(I(c),A)$ if and only if $z = x^j_{i_1 i_2 \cdots i_{l+1}}$ and ω is the stationary path in z.

b) In the situation of Proposition 4.2 b), the class of the difference of the paths $x^j_{i_1 \ldots i_l} \rightsquigarrow x^j_{i_1 \ldots i_l 0}$ and $x^j_{i_1 \ldots i_l} \rightsquigarrow x^j_{i_1 \ldots i_l 1}$ in $\operatorname{Ext}^{l+2}_A(I(c),P(z))$ lies in the top of ${}_A\operatorname{Ext}^{l+2}_A(I(c),A)$ if and only if there are two minimal relations $z \rightsquigarrow x^j_{i_1 \ldots i_l 0} \rightsquigarrow x^j_{i_1 \ldots i_l 0 i_{l+2}}$ and $z \rightsquigarrow x^j_{i_1 \ldots i_l 1} \rightsquigarrow x^j_{i_1 \ldots i_l 1 i_{l+2}}$.

Proof.

a1) The morphism $f : P_{l+2} = \oplus P(x^j_{i_1 \ldots i_l i_{l+1}}) \to P(z)$ induced by ω factors through $P(s)$ where s is the source of a relation ending at $x^j_{i_1 i_2 \cdots i_{l+2}}$ and such that s lies on the path ω. So, f induces an element on the top of ${}_A\operatorname{Ext}^{l+2}_A(I(c),A)$ if and only if $s = z$.

a2) This follows from the fact that the morphism $f : P_{l+2} = \oplus P(x^j_{i_1 \ldots i_l i_{l+1}}) \to P(z)$ factors through the identity of $P(x^j_{i_1 i_2 \cdots i_{l+1}})$.

b) Let f be a representative of the class of the difference of paths $x^j_{i_1 \ldots i_l 0} \rightsquigarrow x^j_{i_1 \ldots i_l}$ and $x^j_{i_1 \ldots i_l 1} \rightsquigarrow x^j_{i_1 \ldots i_l}$ in $\operatorname{Ext}^{l+2}_A(I(c),P(x^j_{i_1 \ldots i_l}))$. Then

$$f = [f_0 \;\; f_1 \;\; 0] : P(x^j_{i_1 \ldots i_l 0}) \oplus P(x^j_{i_1 \ldots i_l 1}) \oplus \overline{P}_{\ell+2} \longrightarrow P(x^j_{i_1 \ldots i_l}).$$

Suppose first that there is no relation $z \rightsquigarrow x^j_{i_1 \ldots i_l 0} \rightsquigarrow x^j_{i_1 \ldots i_l 0 i_{l+2}}$. Then any relation ending at $x^j_{i_1 \ldots i_l 0 i_{l+2}}$ must start at a successor y of z. Therefore there exists $g : P(x^j_{i_1 \ldots i_l 0}) \to P(y)$ such that f_0 factors through g, whence $f = [hg \;\; f_1 \;\; 0]$, for some morphism h. So $[f]$ is not in the top of ${}_A\operatorname{Ext}^{l+2}_A(I(c),A)$.

Conversely, if we have two minimal relations as in the statement, and $[f]$ is not in the top of ${}_A\operatorname{Ext}^{l+2}_A(I(c),A)$, then $[f] = [h][g]$ for some $[g] \in \operatorname{Ext}^{l+2}_A(I(c),P(y))$, which is represented by a morphism

$$g : P(x^j_{i_1 \ldots i_l 0}) \oplus P(x^j_{i_1 \ldots i_l 1}) \longrightarrow P(y).$$

Then y lies on the non-zero path $z \rightsquigarrow x^j_{i_1 \ldots i_l 0}$ or $z \rightsquigarrow x^j_{i_1 \ldots i_l 1}$ (or both) and $y \neq z$. But then $g\, d_{l+3,0} : P(x^j_{i_1 \ldots i_l 0 i_{l+2}}) \to P(y)$ is non-zero, because it is given by the non-zero path $y \rightsquigarrow x^j_{i_1 \ldots i_l 0} \rightsquigarrow x^j_{i_1 \ldots i_l 0 i_{l+2}}$, and this contradicts the fact that $[g]$ belongs to $\operatorname{Ext}^{l+2}_A(I(c),P(y))$. □

We summarise the results in the theorem below.

For each point c in a string algebra $A = kQ/I$, we compute the minimal projective resolution of $I(c)$ given in Corollary 3.6. Then for all $l \geq 0$, the $l+2$-nd term in the minimal projective resolution of $I(c)$ is given by $P_{l+2} = \bigoplus_{j,(i_1,i_2,\cdots,i_{l+1})} P(x^j_{i_1 i_2 \cdots i_{l+1}})$.

Whenever the point $x^j_{i_1 i_2 \cdots i_{l+2}}$ exists, let $z^j_{i_1 i_2 \cdots i_{l+2}}$ be the source of the relation ending in $x^j_{i_1 i_2 \cdots i_{l+2}}$ and passing through $x^j_{i_1 i_2 \cdots i_{l+1}}$.

For each $j \in \{1, \cdots, 6\}$ and for each $l \geq 0$, define

$$\mathbf{Z}^j_{i_1 i_2 \cdots i_{l+1}} = \begin{cases} z^j_{i_1 i_2 \cdots i_{l+1} 0} & \text{if } x^j_{i_1 i_2 \cdots i_{l+1} 0} \text{ exists;} \\ x^j_{i_1 i_2 \cdots i_{l+1}} & \text{otherwise,} \end{cases}$$

and let

$$\zeta^j_{i_1 \ldots i_{l+1}} = \begin{cases} [x^j_{i_1 \cdots i_l}, \mathbf{Z}^j_{i_1 i_2 \cdots i_{l+1}}] & \text{if } x^j_{i_1 i_2 \cdots i_l 0},\, x^j_{i_1 i_2 \cdots i_l 1} \text{ both exist;} \\ \left[x^j_{i_1 \cdots i_l}, \mathbf{Z}^j_{i_1 i_2 \cdots i_{l+1}}\right] \setminus \{x^j_{i_1 \cdots i_l}\} & \text{otherwise.} \end{cases}$$

Dually, whenever the point $s^j_{i_1 i_2 \cdots i_{l+2}}$ exists, let $c^j_{i_1 i_2 \cdots i_{l+2}}$ be the target of the relation starting in $s^j_{i_1 i_2 \cdots i_{l+2}}$ and passing through $s^j_{i_1 i_2 \cdots i_{l+1}}$. For each $j \in \{1, \cdots, 6\}$ and for each $l \geq 0$, define

$$\mathbf{C}^j_{i_1 i_2 \cdots i_{l+1}} = \begin{cases} c^j_{i_1 i_2 \cdots i_{l+1} 0} & \text{if } s^j_{i_1 i_2 \cdots i_{l+1} 0} \text{ exists;} \\ s^j_{i_1 i_2 \cdots i_{l+1}} & \text{otherwise,} \end{cases}$$

and let

$$\Theta^j_{i_1 \ldots i_{l+1}} = \begin{cases} [\mathbf{C}^j_{i_1 i_2 \cdots i_{l+1}}, s^j_{i_1 \cdots i_l}] & \text{if } s^j_{i_1 i_2 \cdots i_l 0},\, s^j_{i_1 i_2 \cdots i_l 1} \text{ both exist;} \\ \left[\mathbf{C}^j_{i_1 i_2 \cdots i_{l+1}}, s^j_{i_1 \cdots i_l}\right] \setminus \{s^j_{i_1 \cdots i_l}\} & \text{otherwise.} \end{cases}$$

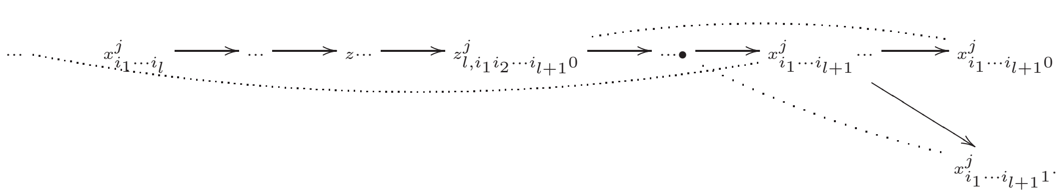
Theorem 4.7 *Let $A = kQ/I$ be a string tree algebra and $l \geq 0$. The following are equivalent*

(a) $\operatorname{Ext}^{l+2}_A(I(c), P(z)) \neq 0$;

(b) *there exists j such that $z \in \zeta^j_{i_1 i_2 \cdots i_{l+1}}$;*

(c) *there exists j such that $c \in \Theta^j_{i_1 i_2 \cdots i_{l+1}}$.*

Proof. The equivalence of (a) and (b) follows from Proposition 3.6 and from the definition of $\zeta^j_{i_1 i_2 \cdots i_{l+1}}$, using the fact that if both points $x^j_{i_1 i_2 \cdots i_{l+2}}$ exist then we have the following situation in the quiver



The equivalence of (a) and (c) follows from the dual argument. □

Remark 4.8 *One can easily compute the top of ${}_A \operatorname{Ext}^{l+2}_A(I(c), A)$ using Corollary 4.6.*

5 The quiver of the higher relation extension

Knowing how to compute $\operatorname{top}{}_A\operatorname{Ext}^i_A(I(c),A)$ and $\operatorname{top}\operatorname{Ext}^i_A(DA,P(z))_A$ allows us to find the new arrows of the higher relation extension of a string tree algebra A since they are in bijection with a basis of

$$\begin{aligned}\operatorname{top}\operatorname{Ext}^i_A(DA,A) &= \operatorname{Ext}^i_A(DA,A)/\operatorname{rad}(\operatorname{Ext}^i_A(DA,A))\\ &= \frac{\operatorname{Ext}^i_A(DA,A)}{(\operatorname{rad}A)\operatorname{Ext}^i_A(DA,A)+\operatorname{Ext}^i_A(DA,A)(\operatorname{rad}A)}.\end{aligned}$$

Note that $\operatorname{Ext}^i_A(DA,A)\cdot e_c = \operatorname{Ext}^i_A(I(c),A)$ is a left A-module and that $e_z\cdot\operatorname{Ext}^i_A(DA,A) = \operatorname{Ext}^i_A(DA,P(z))$ is a right A-module.

Given a right (left) A-module M, we denote by $P_i(M)$ the i-th term in a minimal projective resolution of M and by $I_i(M)$ the i-th term in a minimal injective coresolution of M.

If we represent the elements of $\operatorname{Ext}^i_A(DA,A)$ as classes $[f_{cz}]$ of morphisms $f_{cz} : P_i(I(c)) \to P(z)$ such that the composition of f_{cz} with the map $P_{i+1}(I(c)) \to P_i(I(c))$ of the projective resolution is zero, then we are considering the left A-module structure of $\operatorname{Ext}^i_A(DA,A)$. Therefore, $[f_{cz}]$ lies in $(\operatorname{rad}A)\operatorname{Ext}^i_A(DA,A)$ if and only if $[f_{cz}] \in \operatorname{Ext}^i_A(I(c),A)$ lies in the radical of the left A-module $\operatorname{Ext}^i_A(I(c),A)$.

In terms of morphisms, $[f_{cz}]$ is in $\operatorname{rad}\operatorname{Ext}^i_A(I(c),A)$ if and only if f_{cz} factors non-trivially through another morphism $f_{cy} : P_i(I(c)) \to P(y)$ such that the following diagram is commutative

$$\begin{array}{ccc} P_i(I(c)) & \xrightarrow{\quad f_{cz}\quad} & P(z)\\ & {\scriptstyle f_{cy}}\searrow \qquad \nearrow{\scriptstyle h} & \\ & P(y) & \end{array}$$

where the map h is given by the left-multiplication by a path in Q from z to y, and the composition of f_{cy} with $P_{i+1}(I(c)) \to P_i(I(c))$ is zero.

Dually, we can represent the elements of $\operatorname{Ext}^i_A(DA,A)$ as classes $[g_{cz}]$ of morphisms $g_{cz} : I(c) \to I_i(P(z))$ such that the composition of g_{cz} with the map $I_i(I(c)) \to I_{i+1}(I(c))$ of the injective coresolution is zero. This corresponds to the right A-module structure of $\operatorname{Ext}^i_A(DA,A)$. Therefore, $[g_{cz}]$ lies in $\operatorname{Ext}^i_A(DA,A)(\operatorname{rad}A)$ if and only if $[g_{cz}] \in \operatorname{Ext}^i_A(DA,P(z))$ lies in the radical of the left A-module $\operatorname{Ext}^i_A(DA,P(z))$.

In terms of morphisms, $[g_{cz}]$ is in $\operatorname{rad}\operatorname{Ext}^i_A(DA,P(z))$ if and only if g_{cz} factors non-trivially through another morphism $g_{bz} : I(b) \to I_i(P(z))$ such that the following diagram is commutative

$$\begin{array}{ccc} I(c) & \xrightarrow{\quad g_{cz}\quad} & I_i(P(z))\\ & {\scriptstyle h'}\searrow \qquad \nearrow{\scriptstyle g_{bz}} & \\ & I(b) & \end{array}$$

where the map h' is given by the right-multiplication by a path in Q from b to c, and the composition of g_{bz} with $I_i(P(c)) \to P_{i+1}(P(c))$ is zero.

Moreover there is an isomorphism of vector spaces

$$LR : \bigoplus_{c \in Q_0} \operatorname{Ext}^i_A(I(c), A) \longrightarrow \bigoplus_{z \in Q_0} \operatorname{Ext}^i_A(DA, P(z))$$

such that $LR([f_{cz}])$ and f_{cz} induce the same class in $\operatorname{Ext}^i_A(DA, A)$. Thus, $[f_{cz}]$ is in $\operatorname{rad} \operatorname{Ext}^i_A(DA, A)$ if and only if $[f_{cz}] \in \operatorname{rad} \operatorname{Ext}^i_A(I(c), A)$ or $LR([f_{cz}]) \in \operatorname{rad} \operatorname{Ext}^i_A(DA, P(z))$.

Algorithm 5.1

- *Compute* $\operatorname{top}_A \operatorname{Ext}^i(I(c), A)$ *for all* $c \in Q_0$ *using Theorem 4.7 and Corollary 4.6. (For efficiency we can restrict to the points that are the source or the target of a relation because of Corollary 4.6 and Corollary 3.7).*
- *For each* $c, z \in Q_0$, *let* $\{\rho_{cz1}, \rho_{cz2}, \cdots\}$ *be a basis for* $e_z \cdot \operatorname{top}_A \operatorname{Ext}^i(I(c), A)$.
- *Let* $B_0^i = \{\rho_{czj} : \ c, z \in Q_0, c$ *the source or target of relations*$\}$ *be the set that spans the vector space* $\operatorname{top} \operatorname{Ext}^i(DA, A)$.
- *Compute* $\operatorname{top} \operatorname{Ext}^i(DA, P(z))_A$ *for each* z *such that* $\rho_{czj} \in B_0^i$ *using Theorem 4.7 and the dual statements of Corollary 4.6.*
- *A basis of* $\operatorname{top} \operatorname{Ext}^i(DA, A)$ *is*

$$B^i = B_0^i \setminus \{\rho_{czj} \in \operatorname{rad} \operatorname{Ext}^i(DA, P(z))_A; \ c, z, j\}.$$

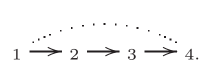
Each element of B^i has a triple subcript czj, and each such element gives rise to exactly one new arrow $z \to c$ in the quiver of the higher relation extension.

Theorem 5.2 *Let* $A = kQ/I$ *be a string tree algebra. Then the algorithm 5.1 computes two sequences* $(c_l), (z_l)$ *of vertices of* Q_A *such that the arrows in the quiver of the higher relation extension are exactly those of* Q_A *plus one additional arrow from each* z_l *to* c_l.

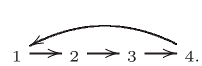
Proof. This follows from the discussion preceding the algorithm. □

Remark 5.3 *The vertices* $(c_l), (z_l)$ *are not necessarily distinct, there may be repetitions.*

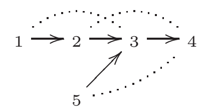
Example 5.4 *Let* $A = kQ/I$ *be the string algebra given by the following bound quiver:*



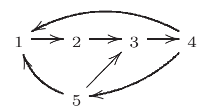
Then there exists an element $\rho_{2,4} \in e_4 \cdot \operatorname{top}_A \operatorname{Ext}^2_A(I(2), A)$ *which is not in* $\operatorname{top} \operatorname{Ext}^2_A(DA, P(4))_A \cdot e_2$ *and therefore not in* $\operatorname{top}_A \operatorname{Ext}^2_A(DA, A)_A$. *Thus the quiver of the higher relation extension is*



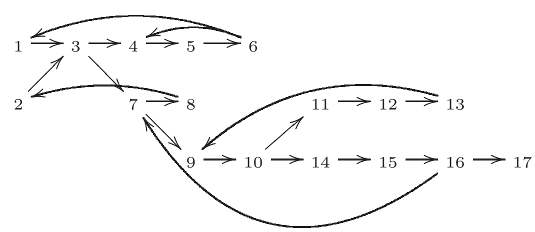
Example 5.5 *In this example, the higher relation extension contains an* Ext^2*-arrow* $5 \to 1$ *although there is no relation between the points 5 and 1. Let* $A = kQ/I$ *be the string algebra given by the bound quiver:*



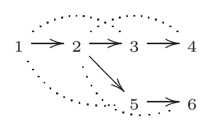
Then the quiver of the higher relation extension of A is the following:



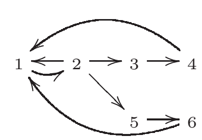
Example 5.6 *Let A be the string algebra of Example 3.4. Then the quiver of the higher relation extension of A is the following:*



Example 5.7 *This example illustrates the situation in Corollary 4.6 (b). Let* $A = kQ/I$ *be the string algebra given by the bound quiver:*



Then the quiver of the higher relation extension of A is the following:



Note the existence of a 2-cycle.

6 The higher relation bimodule for gentle algebras

Recall that a set of monomial relations $\{\kappa_i\}_{i=1,..,t}$ is called an **overlapping** if the paths κ_i and κ_{i+1} have a common subpath ϑ such that $\kappa_i = \vartheta_i\vartheta$ and $\kappa_{i+1} = \vartheta\vartheta_{i+1}$, for all $i = 1, .., t-1$. A **maximal** t**-overlapping** is an overlapping $\{\kappa_i\}_{i=1,..,t}$ such that there exists no monomial relation κ such that the sets $\{\kappa, \kappa_i,\ i = 1, \cdots, t\}$ and $\{\kappa_i,\ i = 1, \cdots, t, \kappa\}$ are an overlapping, see [GHZ, Gu].

Lemma 6.1 *Let $\kappa = (\kappa_1, \cdots, \kappa_t)$ be the following maximal t-overlapping over a gentle algebra $A = kQ/I$:*

Then, for the injective $I(1)$ associated to the vertex 1, the sequence of $x_{i_1 i_2 \cdots i_t}$ is:

$$x_0 = 3,\ x_{00} = 4,\ x_{000} = 5,\ \cdots, x_{i_1 i_2 \cdots i_t} = x_{00\cdots 0} = t + 2.$$

Proof. This follows from the construction of the points $x_{i_1 i_2 \cdots i_t}$ given in section 3. $\square$

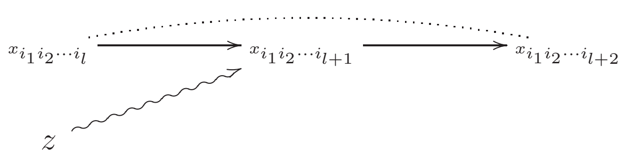
Remark 6.2 *Observe that there may be other points $x_{i_1 i_2 \cdots i_t}$ where some $i_j \neq 0$. In the Lemma we only consider one branch of the quiver which contains all the points $x_{00\cdots 0}$.*

Proposition 6.3 *For every maximal t-overlapping $\kappa = (\kappa_1, \cdots, \kappa_t)$ from c to z there is exactly one new arrow $\alpha(\kappa) : z \to c$ in the higher relation extension which is induced by an element of $\mathrm{Ext}_A^{l+1}(I(c), P(z))$ and these are the only new arrows in the higher relation extension. Moreover, we have the following relations:*

(a) $\alpha(\kappa)\alpha_1 = 0$ and $\alpha_{t+1}\alpha(\kappa) = 0$, where α_1 and α_{t+1} denote the first and the last arrow of κ;

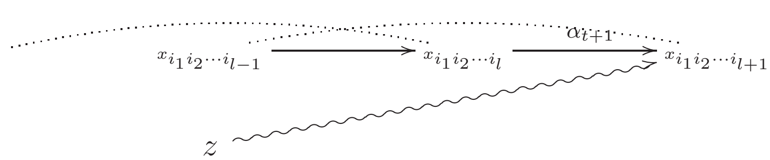
(b) $\zeta\rho\zeta'$ where ζ, ζ' are new arrows and ρ is a path consisting of old arrows.

Proof. By Corollary 4.5, $\mathrm{Ext}_A^{l+2}(I(c), P(z)) \neq 0$ if and only if there is a non-zero path $\omega : z \rightsquigarrow x_{i_1 i_2 \cdots i_{l+1}}$ not passing through $x_{i_1 i_2 \cdots i_l}$ and such that the compositions with the non-zero paths $\omega'_{i_{l+2}} : x_{i_1 i_2 \cdots i_{l+1}} \rightsquigarrow x_{i_1 i_2 \cdots i_{l+2}}$ are both zero if i_{l+2} exists, see figure.



.

But the previous Lemma implies that $x_{i_1 i_2 \cdots i_l} \to x_{i_1 i_2 \cdots i_{l+1}} \to x_{i_1 i_2 \cdots i_{l+2}}$ is a relation of length 2, contradicting that A is gentle. Therefore i_{l+2} does not exist, that is, $\mathrm{pd}\, I(c) = l + 2$. Then we have the situation



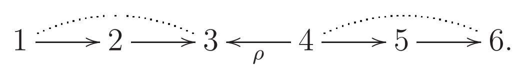
and $x_{i_1 i_2 \cdots i_{l+1}}$ is the target of an overlapping ω.
Thus by Corollary 4.5, $\mathrm{Ext}_A^{l+2}(I(c), P(z)) \neq 0$ if and only if there is a non-zero path ω from z to $x_{i_1 i_2 \cdots i_{l+1}}$ not passing through $x_{i_1 i_2 \cdots i_l}$. Then, by Corollary 4.6 a2), ω induces an element of the top of ${}_A \mathrm{Ext}_A^i(I(c), A)$ if and only if $z = x_{i_1 i_2 \cdots i_{l+1}}$.

To check whether $\omega : z \to x_{i_1 i_2 \cdots i_{l+1}}$ induces an element of top $\operatorname{Ext}_A^{l+2}(DA, A)$, we can apply the algorithm 5.1. Hence, ω induces an element of the top $\operatorname{Ext}_A^{l+2}(DA, A)$ if and only if z is the starting point of the overlapping κ.
The result about the new arrows now follows from the algorithm.

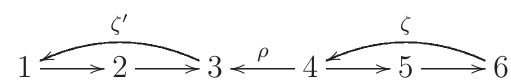
Using the fact that $\operatorname{Ext}_A^{l+2}(I(c), P(z)) \neq 0$ if and only if there is a non-zero path ω from z to $x_{i_1 i_2 \cdots i_{l+1}}$ not passing through $x_{i_1 i_2 \cdots i_l}$ with $z = x_{i_1 i_2 \cdots i_l}$ shows that $\alpha_{t+1}\alpha(\kappa) = 0$. Dually, one proves that $\alpha(\kappa)\alpha_1 = 0$, and the relations of the form $\zeta\rho\zeta'$ occur since we are dealing with a trivial extension. □

The following example shows that the higher relation extension of a gentle algebra is not necessarily gentle.

Example 6.4 *Let A be given by the bound quiver*



Then the higher relation extension coincides with the relation extension and has the quiver



bound by relations of length 2 and the relation $\zeta\rho\zeta'$, which is of length 3.

Corollary 6.5 *The tensor algebra of the higher relation bimodule has the same quiver as the higher relation extension and has the relations in Proposition 6.3 (a). In particular its relation ideal is quadratic.* □

7 The tensor algebra of a gentle algebra

Theorem 7.1 *Let A be a gentle algebra.*

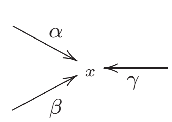
(a) The tensor algebra $T_A(\bigoplus_{i\geq 2} \operatorname{Ext}_A^i(DA, A))$ is gentle.

(b) The higher relation extension $A \ltimes (\bigoplus_{i\geq 2} \operatorname{Ext}_A^i(DA, A))$ is monomial.

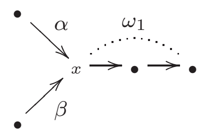
Proof. Since the universal cover of a gentle algebra is a gentle tree, we may assume that A is a tree. We prove the conditions S1), S2), S3), G1) and G2) of section 2.

S2) At every point there are at most two incoming arrows (dually, outcoming arrows).

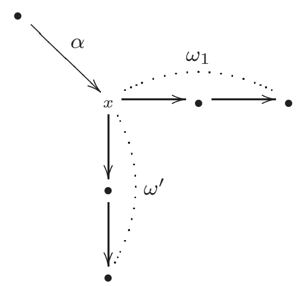
Suppose there are three arrows α, β, γ with target x (see figure)



Then at least one, say γ, is a new arrow. Hence, γ corresponds to an overlapping $\omega = (\omega_1, \omega_2, \cdots)$ with source x and there is no relation involving α or β and overlapping with ω_1, that is,



Because A is gentle, at least one of the arrows α and β is new. Assume α is old and β is new. Then we have two such overlappings ω, ω' and no relation involving α and overlapping with ω or ω', that is we have the following situation in the bound quiver of A.



which yields a contradiction.

Finally, if all three arrows α, β, γ are new, we get three overlappings starting at x. Because A is gentle, condition G1) implies that we have three arrows having x as a source, a contradiction.

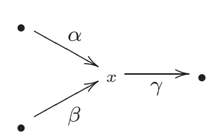
S1,G2) Suppose we have a minimal relation involving at least two paths in the sense of [MP]. Then, in the higher relation extension we have at least two paths c_1, c_2 starting and ending at the same point with at least one new arrow in each of these paths. Let $c_i = c_{i1}\alpha_i c_{i2}$ where α_i is a new arrow, $i = 1, 2$.

Assume first that there is exactly one new arrow on each path c_i. Then each α_i corresponds to an overlapping ω_i in A starting at the target of α_i and ending at its source, and this contradicts the assumption that A is a tree.

If c_i contains several new arrows, the same argument as before applies.

This shows that the higher relation extension of A is monomial and hence that the tensor algebra is also monomial, and even has a quadratic relation ideal, because of Corollary 6.5.

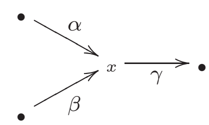
S3) Suppose we have the following subquiver



such that $\alpha\gamma$ and $\beta\gamma$ are not in the ideal I of the tensor algebra $T_A(\bigoplus_{i\geq 2} \operatorname{Ext}^i_A(DA, A))$. Then one of the three arrows is new. First assume that γ is a new arrow, then γ corresponds to an overlapping ω ending at x which implies that α or β must be a new arrow, say β which correspond to another overlapping ω' starting at x. But then the last arrow of ω and the first arrow of ω' are not bound by a relation and also α is not bound by a relation with the first arrow of ω'. This contradicts A being gentle.

Suppose now that α is a new arrow corresponding to an overlapping ω starting at x. Because of the first case, we may assume that γ is not new. Since β is not bound by a relation with the first arrow in ω, it must be with γ, contradicting the assumption $\beta\gamma \notin I$.

G1) Suppose we have a subquiver



such that $\alpha\gamma$ and $\beta\gamma$ are in the relation ideal of the tensor algebra $T_A(\bigoplus_{i\geq 2} \operatorname{Ext}^i_A(DA, A))$. If γ is a new arrow corresponding to an overlapping ω_γ ending at x then α or β must be new, say β, and corresponding to an overlapping ω_β as above, which is bound by no relation with α. It follows from our description of the bound quiver that the new arrow β is not bound by a relation with γ, because γ is not in the overlapping ω_β, and this is a contradiction. □

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