

On the first Hochschild cohomology group of a cluster-tilted algebra

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Abstract

Given a cluster-tilted algebra B , we study its first Hochschild cohomology group $\mathrm{HH}^1(B)$ with coefficients in the B - B -bimodule B . If C is a tilted algebra such that B is the relation extension of C , then we show that if C is constrained, or else if B is tame, then $\mathrm{HH}^1(B)$ is isomorphic, as a k -vector space, to the direct sum of $\mathrm{HH}^1(C)$ with $k^{n_{B,C}}$, where $n_{B,C}$ is an invariant linking the bound quivers of B and C . In the representation-finite case, $\mathrm{HH}^1(B)$ can be read off simply by looking at the quiver of B .

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1 Introduction

Our objective here is, for a cluster-tilted algebra B , to study its first Hochschild cohomology group $\mathrm{HH}^1(B)$ with coefficients in the B - B -bimodule B , see [CE].

Cluster-tilted algebras were defined in [BMR] and in [CCS] for the type \mathbb{A} , as a by-product of the extensive theory of cluster algebras of Fomin and Zelevinsky [FZ]. Now, it has been shown in [ABS1] that every cluster-tilted algebra B is the relation extension of a tilted algebra C . Our goal is to relate the Hochschild cohomologies of the two algebras B and C . The main step in our argument consists in defining an equivalence relation between the arrows in the quiver of B which are not in the quiver of C . The number of equivalence classes is then denoted by $n_{B,C}$. The first two authors

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have shown in [AR] that, if the cluster-tilted algebra B is schurian, then there is a short exact sequence of vector spaces

$$0 \rightarrow k^{n_{B,C}} \rightarrow \mathrm{HH}^1(B) \rightarrow \mathrm{HH}^1(C) \rightarrow 0.$$

This holds true, for instance, when B is representation-finite. In the present paper, we start by proving that this result also holds true in case the tilted algebra C is constrained in the sense of [BM], that is, for every arrow $i \rightarrow j$ in the quiver of C , we have $\dim_k(e_i C e_j) \leq 1$, where e_i, e_j are the primitive idempotents corresponding to the points i, j respectively. Our first theorem can be stated as follows.

THEOREM 1.1 *Let k be an algebraically closed field and B a cluster-tilted k -algebra. If C is a constrained tilted algebra such that B is the relation extension of C , then there is a short exact sequence of vector spaces*

$$0 \rightarrow k^{n_{B,C}} \rightarrow \mathrm{HH}^1(B) \rightarrow \mathrm{HH}^1(C) \rightarrow 0.$$

We next show that, for any cluster-tilted algebra B , we have $\mathrm{HH}^1(B) = 0$ if and only if B is hereditary and its quiver is a tree, that is, B is simply connected. This answers positively for all cluster-tilted algebras Skowroński's question in [S, Problem 1].

We then consider tame cluster-tilted algebras. Because of [BMR], a cluster-tilted algebra is tame if and only if it is of euclidean or Dynkin type. Our second theorem says that in this case, one can delete the assumption that the tilted algebra is constrained.

THEOREM 1.2 *Let k be an algebraically closed field and B a tame cluster-tilted k -algebra. If C is a tilted algebra such that B is the relation extension of C , then there is a short exact sequence of vector spaces*

$$0 \rightarrow k^{n_{B,C}} \rightarrow \mathrm{HH}^1(B) \rightarrow \mathrm{HH}^1(C) \rightarrow 0.$$

Finally, we consider the case where the cluster-tilted algebra B is representation-finite and show that the k -dimension of $\mathrm{HH}^1(B)$ can be computed simply by looking at the quiver of B : indeed, in this case, for any tilted algebra C such that B is a relation extension of C , we have $\mathrm{HH}^1(C) = 0$ and moreover the invariant $n_{B,C}$ does not depend on the particular choice of C (and thus is denoted simply by n_B). Recalling that an arrow in the quiver of B is called inner if it belongs to two chordless cycles, our theorem may be stated as follows.

THEOREM 1.3 *Let B be a representation-finite cluster-tilted algebra. Then the dimension n_B of $\mathrm{HH}^1(B)$ equals the number of chordless cycles minus the number of inner arrows in the quiver of B .*

The paper is organized as follows. In section 2, after briefly setting the notation and recalling the necessary notions, we present results on systems of relations in cluster-tilted algebras. We then introduce the arrow equivalence relation in section 3, and section 4 is devoted to the proof of Theorem 1.1. Sections 5 and 6 contain the proofs of Theorems 1.2 and 1.3, respectively.

2 Systems of relations.

Let k be an algebraically closed field, then it is well-known that any basic and connected finite dimensional k -algebra C can be written in the form $C = kQ/I$, where Q is a connected quiver, kQ its path algebra and I an admissible ideal of kQ . The pair (Q, I) is then called a *bound quiver*. We recall that finitely generated C -modules can be identified with representations of the bound quiver (Q, I) , thus any such module M can be written as $M = (M(x), M(\alpha))_{x \in Q_0, \alpha \in Q_1}$ (see, for instance, [ASS]).

A *relation* from $x \in Q_0$ to $y \in Q_0$ is a linear combination $\rho = \sum_{i=1}^m a_i w_i$ where each w_i is a path of length at least two from x to y and $a_i \in k$ for each i . If $m = 1$ then ρ is *monomial*. The relation ρ is *minimal* if each scalar a_i is non-zero and $\sum_J a_i w_i \notin I$ for any non-empty proper subset J of the set $\{1, \dots, m\}$, and it is *strongly minimal* if each scalar a_i is non-zero and $\sum_J b_i w_i \notin I$ for any non-empty proper subset J of the set $\{1, \dots, m\}$, where each b_i is a non-zero scalar.

We sometimes consider an algebra C as a category, in which the object class C_0 is a complete set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents of C and $C(x, y) = e_x C e_y$ is the set of morphisms from e_x to e_y . An algebra C is *constrained* if, for any arrow from x to y in Q_1 , we have $\dim_k e_x C e_y = 1$.

For a general background on the cluster category and cluster-tilting, we refer the reader to [BMRRT]. It is shown in [ABS1] that, if T is a tilting module over a hereditary algebra A , so that $C = \text{End}_A(T)$ is a tilted algebra, then the trivial extension $\tilde{C} = C \times \text{Ext}_C^2(DC, C)$ (the *relation-extension* of C) is cluster-tilted and, conversely, any cluster-tilted algebra is of this form (but in general, not uniquely: see [ABS2]). As a consequence, we have a description of the quiver of \tilde{C} . Let R be a *system of relations* for the tilted algebra $C = kQ/I$, that is, R is a subset of $\cup_{x,y \in Q_0} e_x I e_y$ such that

R , but no proper subset of R , generates I as an ideal of kQ . It is shown in [ABS1] that the quiver \tilde{Q} of \tilde{C} is as follows:

- (a) $\tilde{Q}_0 = Q_0$;
- (b) For $x, y \in Q_0$, the set of arrows in \tilde{Q} from x to y equals the set of arrows in Q from x to y (which we call *old arrows*) plus $|R \cap I(y, x)|$ additional arrows (which we call *new arrows*).

The relations in \tilde{I} are given by the partial derivatives of the potential $W = \sum_{\rho \in R} \alpha_\rho \rho$, with α_ρ the new arrow associated to the relation ρ , see [Ke].

Now we show that R can be chosen as a set of strongly minimal relations.

LEMMA 2.1 *If $\rho = \sum_{i=1}^m \lambda_i w_i \in R$, with $\lambda_i \neq 0$, is not strongly minimal, there exists $\rho' = \sum_{i=1}^m \mu_i w_i \in I$ with $\mu_1 = \lambda_1$ which is strongly minimal.*

Proof. We proceed by induction on m . If $m = 2$ and $\rho = \lambda_1 w_1 + \lambda_2 w_2$ is not strongly minimal, then it is clear that w_1, w_2 are relations in I and hence we may take $\rho' = \lambda_1 w_1$. Assume now $m > 2$ and ρ is not strongly minimal. Since m is finite, it is clear that there exists a strongly minimal relation $\rho_1 = \sum_J \beta_j w_j \in I$, with J a proper non empty subset of $\{1, \dots, m\}$, $\beta_j \neq 0$. If $1 \in J$, we take $\rho' = \frac{\lambda_1}{\beta_1} \rho_1$ and we are done. If $1 \notin J$, let s be the first element in J . We apply the inductive hypothesis to the relation $\rho - \frac{\lambda_s}{\beta_s} \rho_1$. \square

LEMMA 2.2 *Any system of relations $R = \{\rho_1, \dots, \rho_t\}$ can be replaced by a system of strongly minimal relations $R' = \{\rho'_1, \dots, \rho'_t\}$*

Proof. We proceed by induction on t . If $t = 1$ then $\rho = \sum_{i=1}^m \lambda_i w_i$ is already strongly minimal, since if it is not, then, by the previous lemma we get a relation $\rho' = \sum_{i=1}^m \mu_i w_i \in I$ with $\mu_1 = \lambda_1$. Without loss of generality we may assume that w_1 has maximal length, and hence the relation $\rho - \rho' = \sum_{i=2}^m (\lambda_i - \mu_i) w_i$ belonging to the ideal generated by ρ yields a contradiction: in its expression as an element in $\langle \rho \rangle$, there should be a summand of the form $\mu u_1 w_1 u_2$, with μ a non-zero scalar, u_1, u_2 paths in Q , and then $u_1 w_1 u_2$ is w_1 or a path of greater length, so this term cannot appear in $\rho - \rho'$.

Let $t > 1$, let $\{w_1, \dots, w_s\}$ be a complete set of paths appearing in the relations ρ_i , that is,

$$\rho_i = \lambda_{1i} w_1 + \dots + \lambda_{si} w_s.$$

Without loss of generality, we may assume that w_1 has maximal length and that $\lambda_{11} \neq 0$. Now, the ideal generated by the set $\{\rho_1, \dots, \rho_t\}$ is equal to the ideal generated by the set

$$\{\rho_1, \tilde{\rho}_2, \dots, \tilde{\rho}_t\}$$

with

$$\tilde{\rho}_j = \rho_j - \frac{\lambda_{1j}}{\lambda_{11}} \rho_1.$$

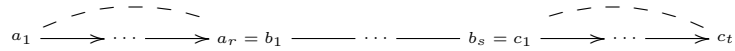
If we apply the previous lemma to ρ_1 we get a strongly minimal relation ρ'_1 with λ_{11} as the first coefficient. Following an argument similar to what we did in the case $t = 1$, using the maximality of w_1 we get that the relation $\rho_1 - \rho'_1$ belongs to the

ideal $\langle \tilde{\rho}_2, \dots, \tilde{\rho}_t \rangle$, and so we get a system of relations $\{\rho'_1, \tilde{\rho}_2, \dots, \tilde{\rho}_t\}$, with ρ'_1 strongly minimal. Now we proceed by induction on the set $\{\tilde{\rho}_2, \dots, \tilde{\rho}_t\}$, and we get a system of relations $\{\rho'_2, \dots, \rho'_t\}$ which are strongly minimal with respect to the ideal $I' = \langle \rho'_2, \dots, \rho'_t \rangle$. Assume that one of these relations is not strongly minimal with respect to I , say $\rho'_i = \sum_{i=2}^s \beta_i w_i$ and $\rho'' = \sum_J \mu_i w_i \in I$, where J is a proper subset of $\{2, \dots, s\}$. So $\rho'' \notin I'$ says that if we write it as an element in I , the relation ρ'_1 should appear. Again we get a contradiction when considering the summands that contain w_1 as a subpath. \square

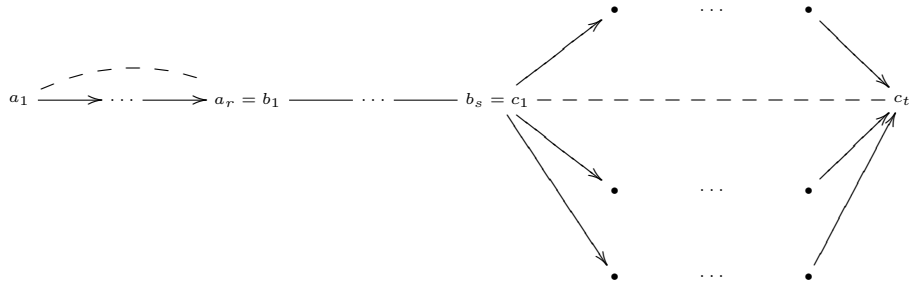
LEMMA 2.3 *Let $C = kQ/I$ be a tilted algebra. Then its relation-extension \tilde{C} contains no walk of the form $w = \alpha w' \beta$, where α, β are new arrows, and w' is a walk consisting entirely of old arrows, no subpath of which is antiparallel to a new arrow.*

Proof. The proof is similar to the proof in the schurian case (see [AR, Lemma 2.1]). We insert it for the convenience of the reader. Suppose there exists such a walk, and assume without loss of generality that the length of w' is minimal. Since new arrows correspond to relations in \tilde{C} , and the quiver Q of C is acyclic, then the existence of such a walk in the quiver \tilde{Q} of \tilde{C} implies that \tilde{C} contains a subquiver, maybe not full, of one of the following forms, where w' is the walk from b_1 to b_s :

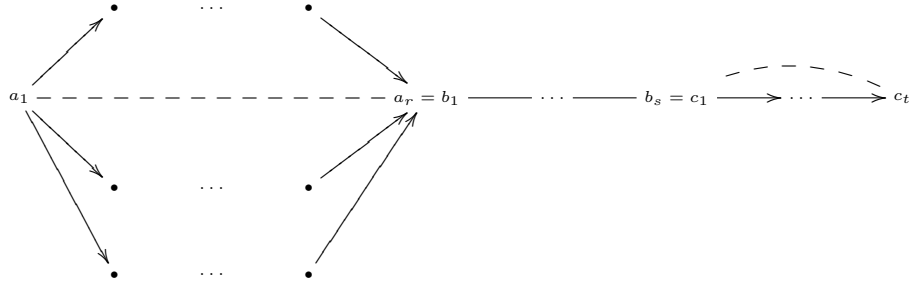
(a)



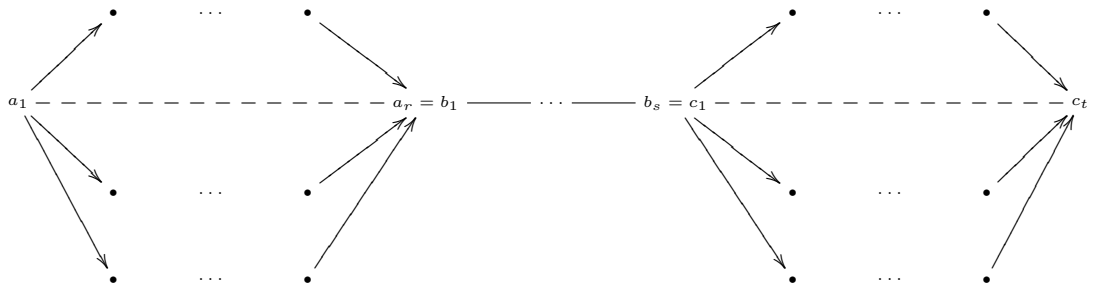
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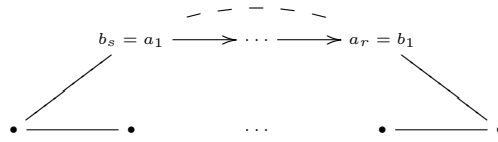
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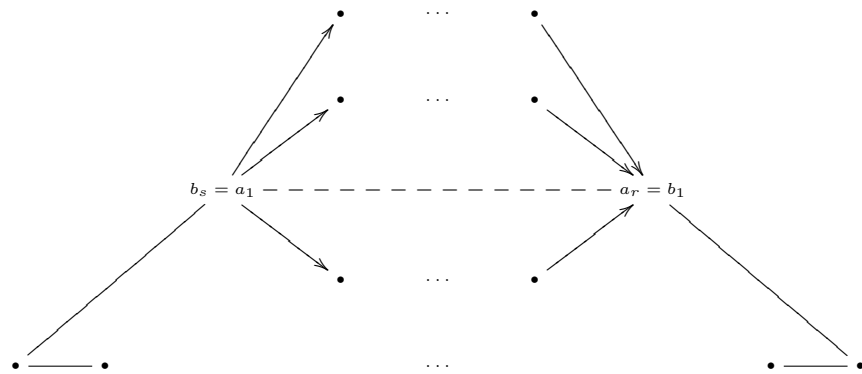
(d)



(e)



(f)



where we have represented relations by dotted lines. The last two cases occur when $\alpha = \beta$. The hypothesis that w' has no subpath antiparallel to a new arrow means that there is no relation in C having both of its endpoints among the b_j 's. Let C' be the full subcategory of C generated by all the points a_i, b_j, c_k . By [Ha1, III.6.5, p. 146], C' is a tilted algebra. Since w' is of minimal length, then there is no additional arrow between two b_j 's. In each of these cases above, let M be the C' -module defined as a representation by

$$M(x) = \begin{cases} k, & \text{if } x \in \{b_1, \dots, b_s\}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$M(\alpha) = \begin{cases} id, & \text{if } s(\alpha), t(\alpha) \in \{b_1, \dots, b_s\}, \\ 0, & \text{otherwise,} \end{cases}$$

for every point x and arrow α in the quiver of C' . Since there is no relation having its two endpoints among the b_j 's, then M is indeed a module. It is clearly indecomposable and it can be easily seen that both of its projective and its injective dimensions equal two, a contradiction because C' is tilted. \square

3 Arrow equivalence

The following lemma is an easy consequence of the main result in [ACT]. For the benefit of the reader, we give an independent proof.

Recall from [DWZ] that for a given arrow β , the *cyclic partial derivative* ∂_β in β is defined on each cyclic path $\beta_1\beta_2 \cdots \beta_s$ by $\partial_\beta(\beta_1\beta_2 \cdots \beta_s) = \sum_{i:\beta=\beta_i} \beta_{i+1} \cdots \beta_s\beta_1 \cdots \beta_{i-1}$. Note that $\partial_\beta(\beta_1\beta_2 \cdots \beta_s) = \partial_\beta(\beta_j \cdots \beta_s\beta_1 \cdots \beta_{j-1})$, in other words, the cyclic derivative is invariant under cyclic permutations.

LEMMA 3.1 *Let $B = k\tilde{Q}/\tilde{I}$ be a cluster-tilted algebra, and $C = kQ/I$ a tilted algebra such that $B = \tilde{C}$. Let $\rho = \sum_{i=1}^m a_i w_i$ be a minimal relation in \tilde{I} . Then either ρ is a relation in I , or there exist exactly m new arrows $\alpha_1, \dots, \alpha_m$ such that $w_i = u_i \alpha_i v_i$ (with u_i, v_i paths consisting entirely of old arrows).*

Proof. Let ρ_1, \dots, ρ_s be a system of minimal relations for the tilted algebra C . Then each relation ρ_i induces a new arrow α_i and the product $\rho_i \alpha_i$ is a linear combination of cyclic paths in the quiver of the cluster tilted algebra B . The potential of B can be given as $W = \sum_{i=1}^s \rho_i \alpha_i$ and the ideal of B is generated by all partial derivatives $\partial_\beta W$ of the potential W with respect to the arrows β . If β is one of the new arrows α_i then $\partial_\beta W$ is just the ‘‘old’’ relation $\rho_i \in I$.

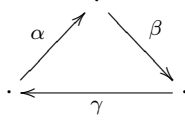
If β is an old arrow then $\partial_\beta W = \sum_{i=1}^s (\partial_\beta \rho_i) \alpha_i$ and each term on the right hand side contains exactly one new arrow α_i . \square

Lemma 3.1 above brings us to our main definition. Let $B = k\tilde{Q}/\tilde{I}$ be a cluster-tilted algebra and $C = kQ/I$ a tilted algebra such that $B = \tilde{C}$. We define a relation \sim on the set $\tilde{Q}_1 \setminus Q_1$ of new arrows as follows. For every $\alpha \in \tilde{Q}_1 \setminus Q_1$, we set $\alpha \sim \alpha$. If $\rho = \sum_{i=1}^m a_i w_i$ is a strongly minimal relation in \tilde{I} and α_i are as in Lemma 3.1 above, then we set $\alpha_i \sim \alpha_j$ for any i, j such that $1 \leq i, j \leq m$.

By Lemma 2.3, the relation \sim is unambiguously defined. It is clearly reflexive and symmetric. We let \approx be the least equivalence relation defined on the set $\tilde{Q}_1 \setminus Q_1$ such that $\alpha \sim \beta$ implies $\alpha \approx \beta$ (that is, \approx is the transitive closure of \sim).

We define the *relation invariant* of B to be the number $n_{B,C}$ of equivalence classes under the relation \approx .

The following two lemmata will be useful in section 5. They use essentially the fact that cluster-tilted algebras of type $\tilde{\mathbb{A}}$ are gentle (because of [ABCP, Lemma 2.5]) and in particular all relations are monomial of length 2 contained inside 3-cycles that is, cycles of the form



bound by $\alpha\beta = \beta\gamma = \gamma\alpha = 0$.

LEMMA 3.2 *Let B be a cluster-tilted algebra of type $\tilde{\mathbb{A}}$ and let C_1, C_2 be tilted algebras such that $B = \tilde{C}_1 = \tilde{C}_2$. Let R_1, R_2 be systems of relations for C_1, C_2 respectively. Then $|R_1| = |R_2|$.*

Proof. Indeed, in order to obtain C_1 and C_2 from B , we have to delete exactly one arrow from each chordless cycle (for the notion of chordless cycle, see [BGZ] or section 6 below). Because B is of type $\tilde{\mathbb{A}}$, then the chordless cycles are 3-cycles, and no arrow belongs to two distinct 3-cycles. Deleting exactly one arrow from each 3-cycle leaves a path of length 2. The system of relations for the tilted algebra consists in exactly these paths of length 2. This implies the statement. \square

LEMMA 3.3 *Let $B = \tilde{C}$, where C is a tilted algebra of type $\tilde{\mathbb{A}}$. Let R be a system of relations for C . Then $n_{B,C} = |R|$. In particular, $n_{B,C}$ does not depend on the choice of C .*

Proof. Let α_i, α_j be two equivalent new arrows, then there exists a sequence of new arrows

$$\alpha_i = \beta_1 \sim \beta_2 \sim \cdots \sim \beta_t = \alpha_j$$

where $\beta_\ell, \beta_{\ell+1}$ appear in the same strongly minimal relation in (\tilde{Q}, \tilde{I}) . Now, B is gentle. Hence strongly minimal relations contain just one monomial. Therefore $\beta_\ell = \beta_{\ell+1}$ for each ℓ , and $\alpha_i = \alpha_j$. This shows that the relation invariant $n_{B,C}$ is equal to the number of new arrows, and the latter is equal to $|R|$ because of [ABS1, Theorem 2.6]. \square

4 Hochschild cohomology of cluster-tilted algebras

For the Hochschild cohomology groups, we refer the reader to [CE, Ha2]. We recall that, if B is a finite dimensional k -algebra, then the first Hochschild cohomology group of B is given by the quotient $\mathrm{HH}^1(B) = \mathrm{Der} B / \mathrm{Int} B$, where $\mathrm{Der} B$ is the k -vector space of derivations and $\mathrm{Int} B$ the k -vector space of inner derivations on B . Moreover, we have

$\mathrm{HH}^1(B) = \mathrm{Der}_0 B / \mathrm{Int}_0 B$, where $\mathrm{Der}_0 B$ is the k -vector space of the *normalized derivations*, that is, of the derivations δ such that $\delta(e_x) = 0$ for every primitive idempotent e_x , and $\mathrm{Int}_0 B$ is the subspace of the interior normalized derivations, that is, derivations $\delta = \delta_c$ given by $\delta_c(b) = cb - bc$, where c is a fixed element in B (see, for instance, [Ha2, 3.1]).

We start by presenting a result for split algebras concerning the first Hochschild cohomology group. A *split algebra* B is a k -algebra with a subalgebra C and a two-sided ideal M such that $B = C \oplus M$. In other words B consists of the following data: a k -algebra C and a multiplicative C -bimodule M with a product, that is, an associative C -bimodule map $M \otimes_C M \rightarrow M$. The algebra structure in $C \oplus M$ is given by

$$(c + m)(c' + m') = cc' + cm' + mc' + mm'.$$

In particular, the trivial extension of C by M is a split algebra with $M^2 = 0$.

LEMMA 4.1 *Let $B = C \oplus M$ be a split algebra. Then there exists a k -linear map $\mathrm{HH}^1(B) \rightarrow \mathrm{HH}^1(C)$.*

Proof. We show first that we can define a k -linear map $\phi : \mathrm{Der} B \rightarrow \mathrm{Der} C$. Let $\delta \in \mathrm{Der} B$ and let $\phi(\delta) = p\delta i : C \rightarrow C$, with $i : C \rightarrow C \oplus M$ and $p : C \oplus M \rightarrow C$ the canonical inclusion and projection maps respectively. Then $\phi(\delta) \in \mathrm{Der} C$ since

$$\begin{aligned} \phi(\delta)(ab) &= p\delta i(ab) \\ &= p\delta[(a+0).(b+0)] = p[(a+0)\delta(b+0) + \delta(a+0)(b+0)] \\ &= ap\delta(b+0) + p\delta(a+0)b = a\phi(\delta)(b) + \phi(\delta)(b)a. \end{aligned}$$

On the other hand, for any $c + m \in B$, we have that

$$\phi(\delta_{c+m})(a) = p\delta_{c+m}(a+0) = p[(c+m)(a+0) - (a+0)(c+m)] = ca - ac$$

and hence $\phi(\mathrm{Int} B) \subseteq \mathrm{Int} C$. □

THEOREM 4.2 *Let B be a cluster-tilted algebra, and C a constrained tilted algebra such that $B = \tilde{C}$. Then there exists a short exact sequence of k -vector spaces*

$$0 \rightarrow k^{n_{B,C}} \rightarrow \mathrm{HH}^1(B) \rightarrow \mathrm{HH}^1(C) \rightarrow 0.$$

Proof. As usual, we let $B = k\tilde{Q}/\tilde{I}$ and $C = kQ/I$. Because of Lemma 2.2, we can choose a system of relations $R = \{\rho_1, \dots, \rho_m\}$ with ρ_i a strongly minimal relation for all i . We show first that there exists a short exact sequence of k -vector spaces

$$0 \rightarrow k^{n_{B,C}} \xrightarrow{\zeta} \text{Der}_0 B \xrightarrow{\phi} \text{Der}_0 C \rightarrow 0$$

where $\phi(\delta) = p\delta i$ as in the previous lemma. Note that ϕ is well-defined on the normalized derivations because, if $\delta \in \text{Der}_0 B$, then $\phi(\delta)(e_x) = p\delta(e_x) = 0$, whence $\phi(\delta) \in \text{Der}_0 C$.

Let $\delta \in \text{Der}_0 C$ then, for every arrow $\alpha : x \rightarrow y$ in Q_1 , we have $\alpha = e_x \alpha e_y$, hence

$$\delta(\alpha) = \delta(e_x \alpha e_y) = e_x \delta(\alpha) e_y \in e_x C e_y.$$

Since C is constrained, there exists a scalar λ_α such that $\delta(\alpha) = \lambda_\alpha \alpha$. Let then $w = \alpha_1 \alpha_2 \dots \alpha_t$ be a path in Q , then

$$\delta(w) = \delta(\alpha_1 \alpha_2 \dots \alpha_t) = (\lambda_{\alpha_1} + \dots + \lambda_{\alpha_t}) \alpha_1 \alpha_2 \dots \alpha_t.$$

We denote $\lambda_w = \lambda_{\alpha_1} + \dots + \lambda_{\alpha_t}$. In particular, δ is uniquely determined by its value on the arrows.

Let $\rho = \sum_{i=1}^m a_i w_i$, with $a_i \in k^*$, be a strongly minimal relation, then $\rho \in I$ implies

$$\sum_{i=1}^m a_i \delta(w_i) = \sum_{i=1}^m a_i \lambda_{w_i} w_i \in I$$

which yields

$$\sum_{i=1}^m a_i \lambda_{w_i} w_i - \lambda_{w_1} \sum_{i=1}^m a_i w_i = \sum_{i=2}^m a_i (\lambda_{w_i} - \lambda_{w_1}) w_i \in I$$

and since ρ is strongly minimal we get that $\lambda_{w_i} = \lambda_{w_1}$ for any i . We denote $\lambda_\rho = \lambda_{w_i}$. We use this observation to prove that ϕ is surjective. Moreover, we define a section $\varphi : \text{Der}_0 C \rightarrow \text{Der}_0 B$ for ϕ .

Let $\delta \in \text{Der}_0 C$, then we let $\varphi(\delta) = \tilde{\delta}$ be defined by its action on the arrows according to

$$\tilde{\delta}(\alpha) = \begin{cases} \delta(\alpha), & \text{if } \alpha \in Q_1, \\ -\lambda_\rho \alpha, & \text{if } \alpha = \alpha_\rho \in \tilde{Q}_1 \setminus Q_1, \end{cases}$$

where ρ is the strongly minimal relation associated to the new arrow $\alpha = \alpha_\rho$.

In order to show that $\tilde{\delta}(\alpha)$ is well-defined, we prove that $\tilde{\delta}\partial_\beta(W) = \tilde{\delta}\partial_\beta(\sum_\rho \alpha_\rho \rho) \in \tilde{I}$, for any arrow $\beta \in \tilde{Q}_1$: the assertion is clear if β is a new arrow since

$$\tilde{\delta}\partial_\beta(W) = \tilde{\delta}(\rho_\beta) = \delta(\rho_\beta).$$

If β is an old arrow, let $\mu u_0 \beta u_1 \cdots \beta u_m$ be any summand in ρ where the arrow β appears at least once, that is, $m \geq 1$. Then $\lambda_\rho = \sum_{i=0}^m \lambda_{u_i} + m\lambda_\beta$ and

$$\begin{aligned} \tilde{\delta}(\partial_\beta(\alpha_\rho u_0 \beta u_1 \cdots \beta u_m)) &= \sum_{i=1}^m \tilde{\delta}(u_i \beta \cdots \beta u_m \alpha_\rho u_0 \beta \cdots u_{i-1}) \\ &= \left(\sum_{i=0}^m \lambda_{u_i} + (m-1)\lambda_\beta - \lambda_\rho \right) \sum_{i=1}^m u_i \beta \cdots \beta u_m \alpha_\rho u_0 \beta \cdots u_{i-1} \\ &= -\lambda_\beta \sum_{i=1}^m u_i \beta \cdots \beta u_m \alpha_\rho u_0 \beta \cdots u_{i-1} \\ &= -\lambda_\beta \partial_\beta(\alpha_\rho u_0 \beta u_1 \cdots \beta u_m). \end{aligned}$$

So $\tilde{\delta} \partial_\beta(W) = -\lambda_\beta \partial_\beta(W)$. Thus $\tilde{\delta}(\alpha)$, and hence φ , are well defined. It is also clear that $\phi\varphi = id$.

Now we define the map

$$\zeta : k^{n_{B,C}} \rightarrow \text{Der}_0 B$$

as follows. Let $(\mu_\alpha)_{\alpha \in S} \in k^{n_{B,C}}$ where S is a complete set of representatives of the equivalence classes of the new arrows under the relation \approx . We set $\zeta(\mu_\alpha)_{\alpha \in S} = \delta$ where we define $\delta \in \text{Der}_0 B$ by its value on the arrows as follows

$$\delta(\alpha) = \begin{cases} 0 & \text{if } \alpha \in Q_1 \\ \mu_{\alpha'} \alpha & \text{if } \alpha \in \tilde{Q}_1 \setminus Q_1 \text{ and } \alpha \sim \alpha', \alpha' \in S. \end{cases}$$

This is clearly a derivation, ζ is injective and $\phi\zeta = 0$.

Now we want to prove that $\text{Ker } \phi = k^{n_{B,C}}$. Let $\delta \in \text{Der}_0 B$ be such that $\phi(\delta) = 0$. This implies that $\delta(\alpha) = \lambda_\alpha \alpha = 0$ for any old arrow $\alpha \in Q_1$, and moreover $\delta(u) = 0$ for any path u in C . On the other hand, for every new arrow $\alpha \in S$ there is an element $(\lambda_\alpha)_{\alpha \in S} \in k^{n_{B,C}}$ such that $\delta(\alpha) = \lambda_\alpha \alpha + w$ with w a linear combination of paths in C parallel to α . This element does not depend on the particular representative chosen in the class of α . Assume thus that $\alpha_1 \sim \alpha_2$. Then there exists a strongly minimal relation $\sum_{i=1}^s a_i u_i \alpha_i v_i$, with $a_i \in k^*$ and u_i, v_i paths in C . Then we have

$$\delta(u_i \alpha_i v_i) = u_i \delta(\alpha_i) v_i = \lambda_{\alpha_i} u_i \alpha_i v_i + u_i w_i v_i$$

with $w_i \in C$, and hence

$$\delta\left(\sum_{i=1}^s a_i u_i \alpha_i v_i\right) = \sum_{i=1}^s (\lambda_{\alpha_i} a_i u_i \alpha_i v_i + a_i u_i w_i v_i)$$

is an element in \tilde{I} . Let $\sum_J \lambda_{\alpha_i} a_i u_i \alpha_i v_i + \sum_{J'} a_i u_i w_i v_i$ be a minimal relation in \tilde{I} , with J and J' subsets of the set $\{1, \dots, s\}$, and $\sum_J \lambda_{\alpha_i} a_i u_i \alpha_i v_i \neq 0$. Using Lemma 3.1 in

this minimal relation, we get that $\sum_J \lambda_{\alpha_i} a_i u_i \alpha_i v_i \in \tilde{I}$. But then J must be the whole set $\{1, \dots, s\}$ since the relation $\sum_{i=1}^s a_i u_i \alpha_i v_i$ is strongly minimal. Now

$$\lambda_{\alpha_1} \sum_{i=1}^s a_i u_i \alpha_i v_i - \sum_{i=1}^s \lambda_{\alpha_i} a_i u_i \alpha_i v_i = \sum_{i=2}^s (\lambda_{\alpha_1} - \lambda_{\alpha_i}) a_i u_i \alpha_i v_i \in \tilde{I}$$

and since the relation is strongly minimal we get that $\lambda_{\alpha_1} = \lambda_{\alpha_2}$.

From the previous lemma we know that $\phi(\text{Int } B) \subseteq \text{Int } C$, and this map preserves normalized derivations. Hence $\phi(\text{Int}_0 B) \subseteq \text{Int}_0 C$. We now claim that the equality holds. Let $\delta_a \in \text{Int}_0 C$, then there exists $a \in C$ such that $\delta_a(c) = ac - ca$ (for every $c \in C$). Since δ_a is normalized, we have, for every primitive idempotent e_x ,

$$0 = \delta_a(e_x) = ae_x - e_x a.$$

Hence

$$a = a.1 = a \sum e_x = \sum ae_x = \sum (ae_x)e_x = \sum (e_x a)e_x = \sum a_x e_x$$

with $a_x \in k$ for every x . This indeed follows from the fact that $e_x a e_x \in e_x C e_x$ and C is constrained.

We now prove that $\tilde{\delta}_a$ is an inner derivation. For an arrow $\alpha \in Q_1$ having source $s(\alpha)$ and target $t(\alpha)$, we have

$$\tilde{\delta}_a(\alpha) = \delta_a(\alpha) = a\alpha - \alpha a = (a_{s(\alpha)} - a_{t(\alpha)})\alpha.$$

Let now $\alpha \in \tilde{Q}_1 \setminus Q_1$ and $\beta_1 \cdots \beta_r$ be a path appearing in the relation defining α . Note that

$$\delta_a(\beta_i) = \lambda_{\beta_i} \beta_i = (a_{s(\beta_i)} - a_{t(\beta_i)})\beta_i$$

for every i such that $1 \leq i \leq r$. Therefore

$$\sum_{i=1}^r \lambda_{\beta_i} = \sum_{i=1}^r (a_{s(\beta_i)} - a_{t(\beta_i)}) = -a_{s(\alpha)} + a_{t(\alpha)}$$

so we have

$$\tilde{\delta}_a(\alpha) = -\left(\sum_{i=1}^r \lambda_{\beta_i}\right)\alpha = (a_{s(\alpha)} - a_{t(\alpha)})\alpha.$$

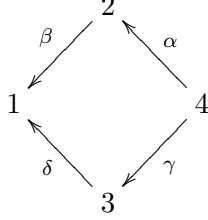
This shows that $\tilde{\delta}_a \in \text{Int}_0 B$, so $\varphi(\text{Int}_0 C) \subseteq \text{Int}_0 B$. Hence $\text{Int}_0 C = \phi\varphi(\text{Int}_0 C) \subseteq \phi(\text{Int}_0 B)$ and we get the desired equality.

We have thus shown that there exists a short exact sequence of k -vector spaces

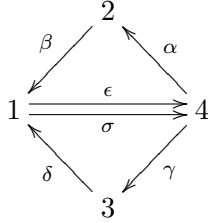
$$0 \rightarrow k^{n_{B,C}} \xrightarrow{\zeta} \text{Der}_0 B \xrightarrow{\phi} \text{Der}_0 C \rightarrow 0.$$

Since $\phi(\text{Int}_0 B) = \text{Int}_0 C$, the statement follows at once. \square

EXAMPLE 4.3 Let C be the tilted algebra of euclidean type $\tilde{\mathbb{A}}_{2,2}$ given by the quiver



bound by the relations $\alpha\beta = 0$ and $\gamma\delta = 0$. This algebra is constrained, so Theorem 4.2 applies. The corresponding cluster-tilted algebra B is given by the quiver



bound by the relations $\alpha\beta = \beta\epsilon = \epsilon\alpha = 0$ and $\gamma\delta = \delta\sigma = \sigma\gamma = 0$. Note that B is not schurian so the results from [AR] cannot be used. The arrow equivalence class in this example consists of the two new arrows ϵ and σ , and therefore the relation invariant $n_{B,C}$ is equal to 2. Now Theorem 4.2 implies that $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(C) \oplus k^2 \cong k^3$.

The following result has been proved for schurian cluster-tilted algebras in [AR, Corollary 3.4]. The statement is inspired from Skowroński's famous question [S, Problem 1]: For which algebras is simple connectedness equivalent to the vanishing of the first Hochschild cohomology group?

THEOREM 4.4 Let $B = k\tilde{Q}/\tilde{I}$ be a cluster-tilted algebra. Then $\mathrm{HH}^1(B) = 0$ if and only if B is hereditary with ordinary quiver a tree.

Proof. By [ABS3], the cluster repetitive algebra is a Galois covering of B with infinite cyclic group \mathbb{Z} . Moreover it is connected if and only if B is not hereditary (because of [ABS3, 1.4, Lemma 5]). Assume thus that B is not hereditary. Because of the universal property of the Galois covering, there exists a group epimorphism

$$\pi_1(\tilde{Q}, \tilde{I}) \rightarrow \mathbb{Z}.$$

Let k^+ denote the additive group of the field k . The previous epimorphism induces a monomorphism of abelian groups

$$\mathrm{Hom}(\mathbb{Z}, k^+) \rightarrow \mathrm{Hom}(\pi_1(\tilde{Q}, \tilde{I}), k^+)$$

which, composed with the canonical monomorphism $\text{Hom}(\pi_1(\tilde{Q}, \tilde{I}), k^+) \rightarrow HH^1(B)$ of [PS, Corollary 3], yields a monomorphism $\text{Hom}(\mathbb{Z}, k^+) \rightarrow HH^1(B)$.

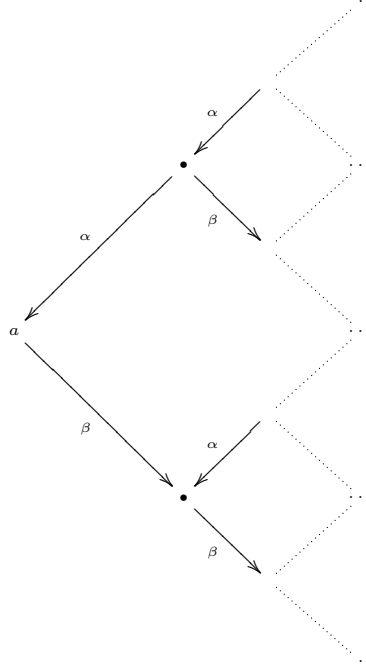
Therefore, if B is not hereditary, we have $HH^1(B) \neq 0$. On the other hand, if B is hereditary, then, because of [Ha2, 1.6], we have $HH^1(B) = 0$ if and only if the quiver \tilde{Q} of B is a tree. \square

5 The tame case

Our objective in this section is to show that our main theorem 4.2 can be used to compute the first Hochschild cohomology group of any tame cluster-tilted algebra. Because of [BMR, Theorem A], the tame cluster-tilted algebras are just the cluster-tilted algebras of euclidean type, that is, the relation extensions of the tilted algebras of euclidean type. Since representation-finite tilted algebras are schurian and thus constrained, we can assume that we are dealing with relation extensions of representation-infinite tilted algebras of euclidean type.

An algebra K is *tame concealed* if there exists a hereditary algebra A and a postprojective tilting A -module T such that $K = \text{End}_A(T)$. Then $\Gamma(\text{mod}K)$ consists of a postprojective component \mathcal{P}_K , a preinjective component \mathcal{Q}_K and a family $\mathcal{T}_K = (\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(k)}$ of stable tubes separating \mathcal{P}_K from \mathcal{Q}_K , see [Ri, 4.3].

We now define *tubular extensions* of a tame concealed algebra. A *branch* L with a *root* a is a finite connected full bound subquiver, containing a , of the following infinite tree, bound by all possible relations of the form $\alpha\beta = 0$.



Let now K be a tame concealed algebra, and $(E_i)_{i=1}^n$ be a family of simple regular K -modules. For each i , let L_i be a branch with root a_i . The tubular extension $B = K[E_i, L_i]_{i=1}^n$ has as objects those of K, L_1, \dots, L_n and as morphism spaces

$$B(x, y) = \begin{cases} K(x, y) & \text{if } x, y \in K_0 \\ L_i(x, y) & \text{if } x, y \in (L_i)_0 \\ L_i(x, a_i) \otimes_k E_i(y) & \text{if } x \in (L_i)_0, y \in K_0 \\ 0 & \text{otherwise.} \end{cases}$$

The *tubular coextension* ${}^n_{i=1}[E_i, L_i]K$ is defined dually.

For each $\lambda \in \mathbb{P}_1(k)$, let r_λ denote the rank of the stable tube \mathcal{T}_λ of $\Gamma(\text{mod}K)$. The *tubular type* $n_B = (n_\lambda)_{\lambda \in \mathbb{P}_1(k)}$ of B is defined by

$$n_\lambda = r_\lambda + \sum_{E_i \in \mathcal{T}_\lambda} |(L_i)_0|.$$

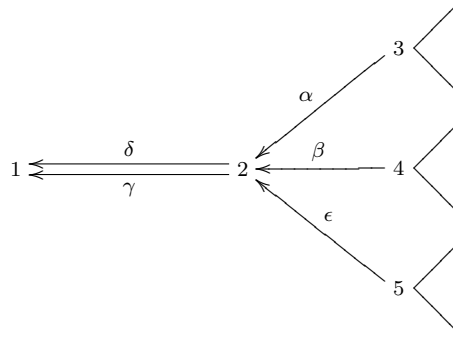
Since all but at most finitely many n_λ equal 1, we write for n_B the finite sequence containing at least two n_λ , including all those larger than 1, in non-decreasing order. We say that n_B is *domestic* if it is one of the forms $(p, q), (2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5)$. The following structure theorem is due to Ringel, see [Ri, Theorem 4.9, p. 241].

THEOREM 5.1 *Let C be a representation-infinite tilted algebra of euclidean type. Then C contains a unique tame concealed full convex subcategory K and C is a domestic tubular extension or a domestic tubular coextension of K .*

As a consequence of Ringel's theorem, we obtain the following.

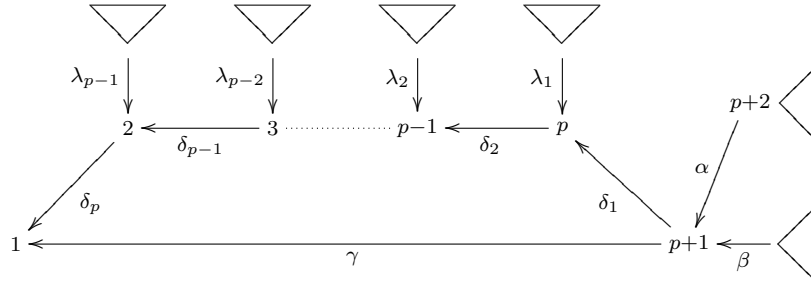
LEMMA 5.2 *Let C be a tilted algebra of euclidean type which is not constrained. Then C is given by one of the following two bound quivers, or their duals.*

(1)



where the triangles are branches, possibly empty, bound by $\alpha\delta = 0, \beta\delta = \beta\gamma, \epsilon\gamma = 0$, and the branch relations.

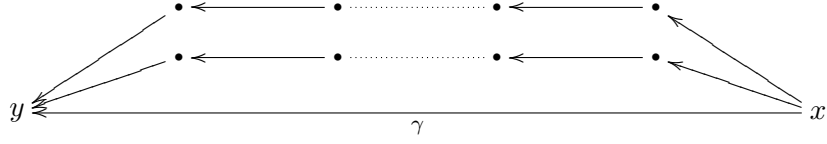
(2)



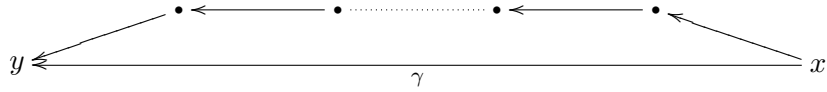
where the triangles are branches, possibly empty, bound by $\alpha\delta_1 \cdots \delta_p = \alpha\gamma, \beta\gamma = 0, \lambda_i\delta_{i+1} = 0$ for all i such that $1 \leq i < p$, and the branch relations.

Proof. Assume C is a tilted algebra of euclidean type which is not constrained. Then there exists an arrow $\gamma : x \rightarrow y$ such that $\dim_k C(x, y) \geq 2$. Since C is tame, we actually have $\dim_k C(x, y) = 2$. In particular, C is representation-infinite. Applying Ringel's theorem, we get that C is, up to duality, a domestic tubular extension of a

unique tame concealed full convex subcategory K of C . On the other hand, let K' be the convex envelope of the points x, y in C . Then K' is of the form



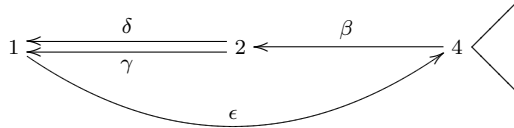
with $\dim_k K'(x, y) = 2$. Note that K' is a full convex subcategory of C , hence it is tilted (because of [Ha1, III.6.5 p.146]). Applying Lemma 2.3 to K' , we deduce that K' is of the form



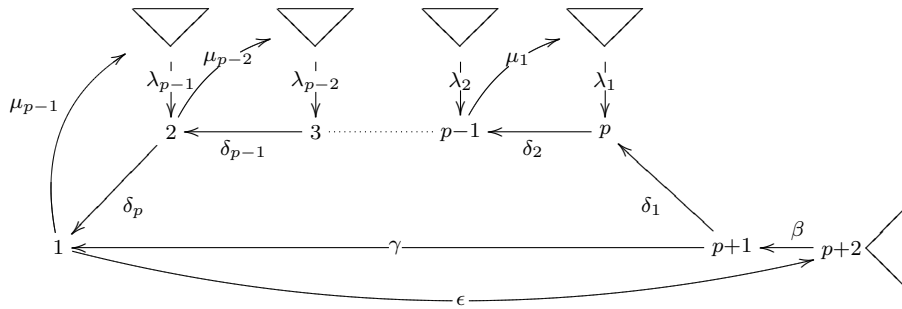
Since K' is hereditary, we get that $K' = K$. The statement now follows by considering the possible branch extensions of K . \square

LEMMA 5.3 *Let B be a cluster-tilted algebra of euclidean type. Assume that there exists no constrained tilted algebra C such that $B = \tilde{C}$. Then B is a cluster-tilted algebra of type $\tilde{\mathbb{A}}$ of one of the following forms or their duals:*

(i)



(ii)



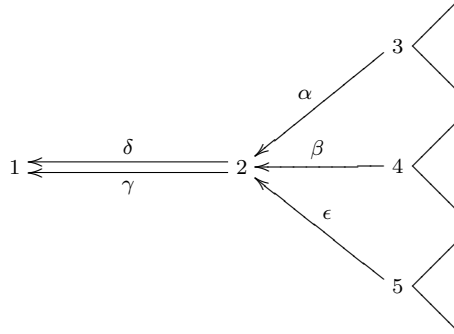
where the triangles are cluster-tilted algebras of type \mathbb{A} , possibly empty, bound by $\beta\gamma = 0$, $\gamma\epsilon = 0$, $\epsilon\beta = 0$, and, in the case (ii), by the additional relations $\lambda_i\delta_{i+1} = 0$, $\delta_{i+1}\mu_i = 0$, $\mu_i\lambda_i = 0$.

Proof. Let B be cluster-tilted of euclidean type. Because of [ABS1], there exists a tilted algebra C such that

$$B = \tilde{C} = C \ltimes \text{Ext}^2(DC, C).$$

The hypothesis says that C is not constrained. Because of Lemma 5.2, C is given by one of the bound quivers in (1) or (2) above. We examine these cases separately.

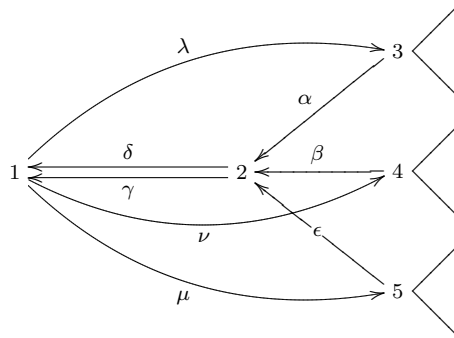
(1) Assume C is given by the quiver



where the triangles are branches, possibly empty, bound by $\alpha\delta = 0$, $\beta\delta = \beta\gamma$, $\epsilon\gamma = 0$ and the branch relations. Observe that, if one of the branches is empty, then it has no root and consequently, the arrow from that root to the point 2 does not exist.

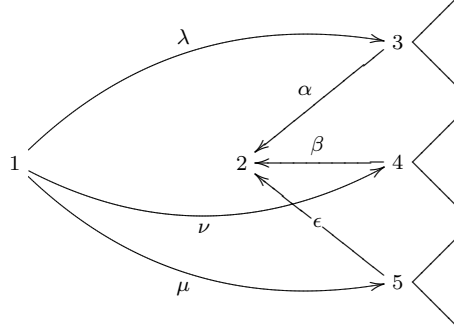
We consider the following subcases:

(1a) Assume none of the branches rooted at 3,4,5 is empty. In this case, we refer to C as C_1 . Then the corresponding cluster-tilted algebra B is of the form



where the triangles are cluster-tilted algebras of type \mathbb{A} , and there are, additionally, the relations of C_1 and the relations $\lambda\alpha = -\nu\beta$, $\nu\beta = \mu\epsilon$, $\delta\lambda = 0$, $\delta\nu = \gamma\nu$ and $\gamma\mu = 0$.

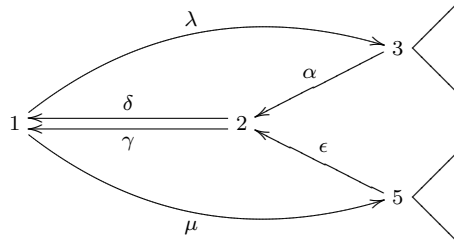
Now, this algebra B can be written as $B = \tilde{C}'_1$, where C'_1 is given by the quiver



where the triangles are again branches, bound by relations $\lambda\alpha = -\nu\beta$, $\nu\beta = \mu\epsilon$.

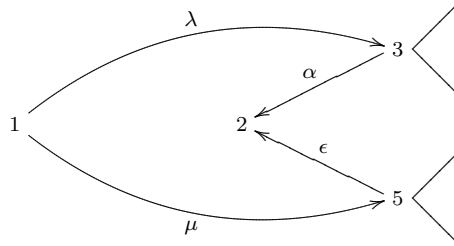
This is easily seen to be a representation-finite tilted algebra of type $\tilde{\mathbb{D}}$ (indeed, one can simply construct the Auslander-Reiten quiver of the algebra and identify a complete slice). In particular, C'_1 is constrained, a contradiction.

(1b) Assume that the branch rooted, say at 4, is empty while the other two are not. In this case, we refer to C as C_2 . Then the cluster-tilted algebra B is of the form



where the triangles are cluster-tilted algebras of type \mathbb{A} , bound by the relations of C_2 and the additional relations $\lambda\alpha = 0$, $\delta\lambda = 0$, $\gamma\mu = 0$, $\mu\epsilon = 0$.

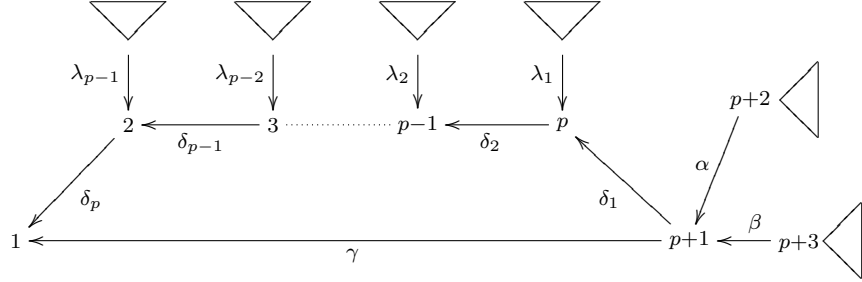
Again, the algebra B can be written as $B = \tilde{C}'_2$, where C'_2 is given by the quiver



where the triangles are branches, bound by $\lambda\alpha = 0$, $\mu\epsilon = 0$ and the branch relations. This is easily seen to be a representation-finite tilted algebra of type $\tilde{\mathbb{A}}$ (see, for instance, [AS]), thus C'_2 is constrained, another contradiction.

(1c) If at least two of the branches, say at 4 and 5, are empty, then we are left with the quiver (i) of the statement.

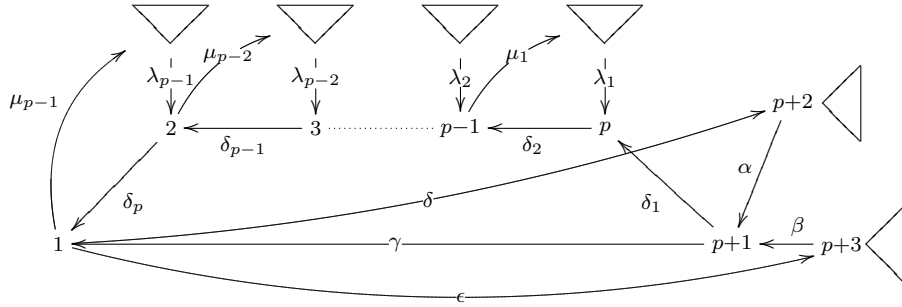
(2) Assume C is given by the quiver



where the triangles are branches, possibly empty, bound by $\alpha\delta_1 \cdots \delta_p = \alpha\gamma$, $\beta\gamma = 0$, $\lambda_i\delta_{i+1} = 0$ for all $1 \leq i < p$, and the branch relations.

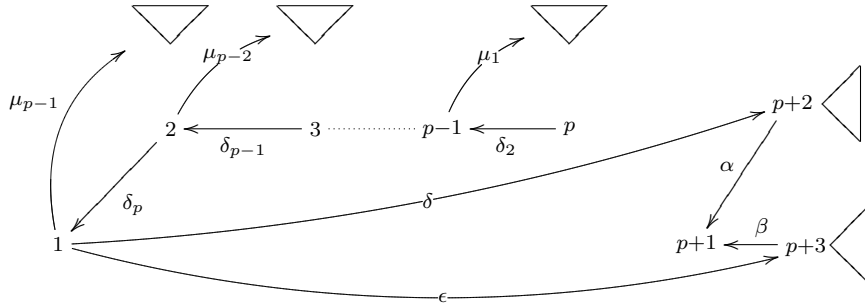
We consider the following subcases.

(2a) Assume that none of the branches rooted at $p+1, p+2$ is empty. In this case, we refer to C as C_3 . Then the corresponding cluster-tilted algebra is of the form



where the triangles are cluster-tilted algebras of type \mathbb{A} , bound by the relations of C_3 and the additional relations $\epsilon\beta = \delta\alpha$, $\gamma\epsilon = 0$, $\delta_1 \cdots \delta_p\delta = \gamma\delta$ and $\mu_i\lambda_{i+\delta_{i+2}} \cdots \delta_p\delta\alpha\delta_1 \cdots \delta_i = 0$, $\delta_{i+1}\mu_i = 0$, for all i .

Now, this algebra can be written as $B = \tilde{C}'_3$, where C'_3 is given by the quiver



with the inherited relations. This is again seen to be a representation-finite tilted algebra of type $\tilde{\mathbb{D}}$. In particular, it is constrained, a contradiction.

(2b) If at least one of the branches, say at $p + 2$ is empty, then we are left with the quiver (ii) of the statement. \square

Observe that in the proof of Lemma 5.3, in each of the cases (1a), (1b) and (2a), we have replaced the original non-constrained tilted algebra C_1, C_2 and C_3 by a constrained one C'_1, C'_2 and C'_3 , respectively.

LEMMA 5.4 *With the above notation, for each $i \in \{1, 2, 3\}$, we have $n_{B, C_i} = n_{B, C'_i}$ and $\mathrm{HH}^1(C_i) \cong \mathrm{HH}^1(C'_i)$.*

Proof. The first statement follows immediately from the description of the relations in the respective algebras. Thus $n_{B, C_1} = n_{B, C'_1} = 1$, $n_{B, C_2} = n_{B, C'_2} = 2$ and $n_{B, C_3} = n_{B, C'_3} = 1$.

It suffices to show the second statement. We consider each of the cases as in the proof of Lemma 5.3.

(1a) Let D_1 be the full convex subcategory of C_1 (and C'_1) generated by all points except the point 1. Then D_1 is a representation-finite tilted algebra and C_1 (or C'_1) is a one-point coextension (or extension, respectively) of D_1 by an indecomposable module. This module being a rigid brick, we deduce immediately from Happel's sequence [Ha2, 5.3] that

$$\mathrm{HH}^1(C_1) \cong \mathrm{HH}^1(D_1) \cong \mathrm{HH}^1(C'_1).$$

(1b) Let D_2 be the full convex subcategory of C_2 (and C'_2) generated by all points except the point 1. Then D_2 is a representation-finite tilted algebra and C_2 (or C'_2) is a one-point coextension (or extension, respectively) of D_2 by the direct sum of two Hom-orthogonal, rigid bricks X, Y such that $\mathrm{Ext}_{D_2}^1(X, Y) = 0$ and $\mathrm{Ext}_{D_2}^1(Y, X) = 0$. Again Happel's sequence yields

$$\mathrm{HH}^1(C_2) \cong \mathrm{HH}^1(D_2) \cong \mathrm{HH}^1(C'_2).$$

(2a) Let D_3 be the full convex subcategory of C_3 (and C'_3) generated by all points except the points $1, 2, \dots, p$. Then there is a sequence

$$C_3 = E_0 \supsetneq E_1 \supsetneq \dots \supsetneq E_p = D_3,$$

where E_i is a one-point coextension of E_{i+1} . Moreover, each E_i is a direct product of representation-finite tilted algebras and the coextension module is a direct sum of rigid bricks with supports in distinct connected components of E_i . Similarly, there is a sequence

$$C'_3 = F_p \supsetneq F_{p-1} \supsetneq \dots \supsetneq F_0 = D_3,$$

where F_{i+1} is a one-point extension of F_i . Moreover, each F_i is a direct product of representation-finite tilted algebras and the extension module is a direct sum of rigid bricks with supports in distinct connected components of F_i . Therefore easy inductions yield

$$\mathrm{HH}^1(C_3) \cong \mathrm{HH}^1(D_3) \cong \mathrm{HH}^1(C'_3).$$

□

LEMMA 5.5 *Let $B = \tilde{C}$ be a non-hereditary cluster-tilted algebra of type $\tilde{\mathbb{A}}$ of one of the forms of Lemma 5.3 and R a system of relations for C . Then*

(i) *If B is of the form (i), then $\mathrm{HH}^1(B) = k^{|R|+2}$*

(ii) *If B is of the form (ii), then $\mathrm{HH}^1(B) = k^{|R|+1}$*

Proof. (i) We use the formula of [CS], as applied to our special situation in [AR, Proposition 5.1]

$$\dim_k \mathrm{HH}^1(B) = \dim_k Z(B) - |\tilde{Q}_0 // N| + |\tilde{Q}_1 // N| - |(\tilde{Q}_1 // N)_e| - \dim_k \mathrm{Im} R_g.$$

Here, $Z(B)$ is the centre of B , so $\dim_k Z(B) = 1$. Next, $\tilde{Q}_0 // N$ is the set of non-zero oriented cycles in (\tilde{Q}, \tilde{I}) (where, as usual, $B = k\tilde{Q}/\tilde{I}$) including points. Then

$$|\tilde{Q}_0 // N| = |\tilde{Q}_0| = |Q_0|.$$

Thirdly, $\tilde{Q}_1 // N$ is the set of pairs (α, w) , where $\alpha \in \tilde{Q}_1$ and w is a non-zero path (of length ≥ 0) parallel to α . This consists of all pairs (α, α) , with $\alpha \in \tilde{Q}_1$ and the two pairs (δ, γ) , (γ, δ) arising from the double arrow

$$1 \begin{array}{c} \xleftarrow{\delta} \\ \xrightarrow{\gamma} \end{array} 2.$$

Thus, $|\tilde{Q}_1 // N| = |\tilde{Q}_1| + 2$.

Since it is shown in [AR, Proof of Proposition 5.1] that $R_g = 0$, there remains to compute

$$(\tilde{Q}_1 // N)_e = (\tilde{Q}_1 // N) \setminus \left((\tilde{Q}_1 // N)_g \cup (\tilde{Q}_1 // N)_a \right).$$

Here:

1. $(\tilde{Q}_1 // N)_g$ is the set of all pairs $(\alpha, w) \in \tilde{Q}_1 // N$ where w is either a point or a path starting or ending with the arrow α . Therefore

$$(\tilde{Q}_1 // N)_g = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}_1\}.$$

2. $(\tilde{Q}_1 // N)_a$ is the set of all pairs $(\alpha, w) \in \tilde{Q}_1 // N$ where, in each relation where α appears, replacing α by w yields a zero path. Therefore

$$(\tilde{Q}_1 // N)_a = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}_1\} \cup \{(\delta, \gamma)\}.$$

This implies that

$$|(\tilde{Q}_1 // N)| - |(\tilde{Q}_1 // N)_e| = |(\tilde{Q}_1 // N)_g \cup (\tilde{Q}_1 // N)_a| = |\tilde{Q}_1| + 1.$$

Therefore

$$\begin{aligned} \dim_k \mathrm{HH}^1(B) &= 1 - |\tilde{Q}_0| + |\tilde{Q}_1| + 1 \\ &= 1 - |Q_0| + |Q_1| + |R| + 1 \\ &= |R| + 2, \end{aligned}$$

because $|\tilde{Q}_1| = |Q_1| + |R|$ and $|Q_0| = |Q_1|$.

- (ii) For this case again $\dim_k Z(B) = 1$ and $|(\tilde{Q}_0 // N)| = |\tilde{Q}_0| = |Q_0|$. Here

$$\tilde{Q}_1 // N = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}\} \cup \{(\gamma, \delta_1 \cdots \delta_p)\}.$$

Now, as before

$$(\tilde{Q}_1 // N)_g = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}_1\},$$

while

$$(\tilde{Q}_1 // N)_a = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}_1\},$$

so that

$$|(\tilde{Q}_1 // N)| - |(\tilde{Q}_1 // N)_e| = |(\tilde{Q}_1 // N)_g \cup (\tilde{Q}_1 // N)_a| = |\tilde{Q}_1|.$$

Therefore

$$\begin{aligned} \dim_k \mathrm{HH}^1(B) &= 1 - |\tilde{Q}_0| + |\tilde{Q}_1| \\ &= 1 - |Q_0| + |Q_1| + |R| \\ &= |R| + 1, \end{aligned}$$

because $|\tilde{Q}_1| = |Q_1| + |R|$ and $|Q_0| = |Q_1|$. □

THEOREM 5.6 *Let $B = \tilde{C}$ be a tame cluster-tilted algebra with C tilted. Then there exists a short exact sequence of k -vector spaces*

$$0 \longrightarrow k^{n_{B,C}} \longrightarrow \mathrm{HH}^1(B) \longrightarrow \mathrm{HH}^1(C) \longrightarrow 0.$$

Proof. If C is hereditary, then $B = C$, $n_{B,C} = 0$ and $\mathrm{HH}^1(B) = \mathrm{HH}^1(C)$. If not, assume first that C is constrained. Then the result follows from Theorem 4.2. Otherwise, C is, up to duality, of one of the forms (i) (ii) of Lemma 5.2. As observed in the proof of Lemma 5.3, we have two distinct cases:

(a) Either one can replace the non-constrained algebra C by a constrained algebra C' such that $n_{B,C} = n_{B,C'}$ and $\mathrm{HH}^1(C) \cong \mathrm{HH}^1(C')$ because of Lemma 5.4. The statement then follows from Theorem 4.2 applied to B and C' .

(b) Otherwise B is, up to duality, of one of the forms (i) (ii) of Lemma 5.3. Note that there exist several tilted algebras C having B as a relation-extension. However, because of Lemma 3.2, the cardinality $|R|$ of a system of relations R for each such tilted algebra C is independent of the choice of C . Moreover, in this case, $n_{B,C} = |R|$, by Lemma 3.3.

Using Lemma 5.5, it suffices to prove that, if C is of the form (i), then $\mathrm{HH}^1(C) = k^2$ and, if C is of the form (ii), then $\mathrm{HH}^1(C) = k$. This follows from another straightforward application of Happel's sequence. \square

6 The representation-finite case

Throughout this section, let B be a representation-finite cluster-tilted algebra. We present easy methods to compute the relation invariant $n_{B,C}$ and thus $\mathrm{HH}^1(B)$ in this case. Let \tilde{Q} be the quiver of B and let n be the number of points in \tilde{Q} .

Choose a tilted algebra C such that $B = C \rtimes \mathrm{Ext}_C^2(DC, C)$. The *number of relations* in C is the dimension of $\mathrm{Ext}_C^2(S_C, S_C)$, where S_C is the sum of a complete set of representatives of the isomorphism classes of simple C -modules. We say that a relation r in B is a *new relation* if it is not a relation in C . It has been shown in [AR, Corollary 3.3] that in this case $n_{B,C}$ is equal to the number of relations in C minus the number of new commutativity relations in B , and, moreover,

$$\mathrm{HH}^1(B) = k^{n_{B,C}}.$$

In particular, the integer $n_{B,C}$ does not depend on the choice of the tilted algebra C , and therefore we shall denote it in the rest of this section by n_B . The objective of this section is to show that one can read off the integer n_B from the quiver \tilde{Q} of B .

Recall from [BGZ] that a *chordless cycle* in \tilde{Q} is a full subquiver induced by a set of points $\{x_1, x_2, \dots, x_p\}$ which is topologically a cycle, that is, the edges in the chordless cycle are precisely the edges $x_i - x_{i+1}$.

LEMMA 6.1 *The number of chordless cycles in \tilde{Q} is equal to the number of zero relations in C plus twice the number of commutativity relations in C .*

Proof. Consider the map $\{\text{relations in } C\} \rightarrow \{\text{new arrows in } B\}$ that associates to a relation $\rho \in \text{Ext}_C^2(S_i, S_j)$ the new arrow $\alpha(\rho) : j \rightarrow i$. By [BR, Corollary 3.7], every chordless cycle contains exactly one new arrow, and therefore it suffices to show that if ρ is a commutativity relation, then $\alpha(\rho)$ lies in precisely two chordless cycles in \tilde{Q} , and if ρ is a zero relation, then $\alpha(\rho)$ lies in precisely one chordless cycle in \tilde{Q} .

If ρ is a commutativity relation, say $\rho = c_1 - c_2$ where c_1, c_2 are paths from i to j in Q , then the concatenations $\alpha(\rho)c_1$ and $\alpha(\rho)c_2$ are two chordless cycles. Then it follows from the fact that \tilde{Q} is a planar quiver (see [CCS2, Theorem A1]), that $\alpha(\rho)$ lies in precisely two chordless cycles.

Otherwise, ρ is a zero relation in C , and $\alpha(\rho)\rho$ is a chordless cycle in \tilde{Q} . We have to show that $\alpha(\rho)$ does not lie in two chordless cycles. Suppose the contrary. Because of [FZ2, Proposition 9.7], every chordless cycle in \tilde{Q} is oriented. Therefore there exists another path ρ' from i to j in \tilde{Q} such that $\alpha(\rho)\rho'$ is a chordless cycle. If ρ' is also a path in Q , then ρ and ρ' are two parallel paths whose difference $\rho - \rho'$ is not a relation in C . This implies that the fundamental group of C is non-trivial, and this contradicts the well-known fact that tilted algebras of Dynkin type are simply connected (see, for instance, [L]). On the other hand, if ρ' is a path in \tilde{Q} but not in Q then it must contain at least one new arrow. But then the chordless cycle $\alpha(\rho)\rho'$ contains two new arrows, a contradiction to [BR, Corollary 3.7]. □

An arrow in \tilde{Q} is called *inner arrow* if it is contained in two chordless cycles. Arrows which are not inner arrows are called *outer arrows*.

LEMMA 6.2 *The number of new inner arrows in B is equal to the number of commutativity relations in C .*

Proof. Each commutativity relation in C gives a new inner arrow in B . Conversely, suppose that α is a new inner arrow in B and let ρ, ρ' be the two paths in \tilde{Q} such that $\alpha\rho$ and $\alpha\rho'$ are the chordless cycles. By [BR, Corollary 3.7], ρ and ρ' contain no new arrows, and hence ρ and ρ' are paths in Q . Since the algebra C is simply connected, it follows that $\rho - \rho'$ is a relation in C . □

LEMMA 6.3 *The number of old inner arrows in B is equal to the number of new commutativity relations in B .*

Proof. We recall from [CCS2, BMR2] the description of B as a bound quiver algebra: For any arrow α in \tilde{Q} let \mathcal{S}_α be the set of paths ρ in \tilde{Q} such that $\rho\alpha$ is a chordless cycle and define

$$\rho_\alpha = \begin{cases} \rho & \text{if } \mathcal{S}_\alpha = \{\rho\} \\ \rho - \rho' & \text{if } \mathcal{S}_\alpha = \{\rho, \rho'\}. \end{cases}$$

Let I be the ideal in $k\tilde{Q}$ generated by the relations $\cup_{\alpha \in (\tilde{Q})_1} \{\rho_\alpha\}$. Then

$$B = k\tilde{Q}/I.$$

Because of the previous remarks, commutativity relations are in bijection with inner arrows. If the relation is new, then the arrow is old and if the arrow is new than the relation is old. \square

We are now able to prove the main theorem of this section.

THEOREM 6.4 *Let B be a representation finite cluster-tilted algebra and \tilde{Q} the quiver of B . Then n_B equals the number of chordless cycles in \tilde{Q} minus the number of inner arrows in \tilde{Q} .*

Proof. By definition, n_B is the number of relations in C minus the number of new commutativity relations in B . By Lemmata 6.1 and 6.2, the number of relations in C is equal to the number of chordless cycles minus the number of new inner arrows in \tilde{Q} . On the other hand, the number of new commutativity relations in B is equal to the number of new inner arrows in \tilde{Q} , because of Lemma 6.3. Therefore

$$n_B = \text{number of chordless cycles in } \tilde{Q} - \text{number of inner arrows in } \tilde{Q}.$$

\square

COROLLARY 6.5 *If \tilde{Q} is connected then*

$$n_B = 1 + \text{number of outer arrows in } \tilde{Q} - n.$$

Proof. Because of [CCS2, Theorem A1] the quiver \tilde{Q} is planar. In particular, every arrow lies in at most two chordless cycles. Hence one can associate a simplicial complex on the 2-dimensional sphere to the quiver \tilde{Q} , in such a way that Q_0 is the set of points, Q_1 the set of edges and the set of chordless cycles is the set of faces of the simplicial complex except the face coming from the “outside” of the quiver (the unbounded component of the complement when embedded in the plane). Using Euler’s formula, we see that the number of chordless cycles in \tilde{Q} is equal to $1 + |(\tilde{Q})_1| - |(\tilde{Q})_0|$, and then Theorem 6.4 yields

$$n_B = 1 + (|(\tilde{Q})_1| - \text{number of inner arrows in } \tilde{Q}) - |(\tilde{Q})_0|,$$

and the statement follows. \square

REMARK 6.6 If \tilde{Q} is not connected then

$$n_B = \text{number of connected components of } \tilde{Q} \\ + \text{number of outer arrows in } \tilde{Q} - n.$$

As an application, we show the following corollary on deleting points. Let $x \in (\tilde{Q})_0$, and $e_x \in B$ the associated idempotent. Then B/Be_xB is cluster-tilted and the quiver of B/Be_xB is obtained from \tilde{Q} by deleting the point x and all arrows adjacent to x , see [BMR3, Section 2]. Define the *Hochschild degree* of x to be the integer

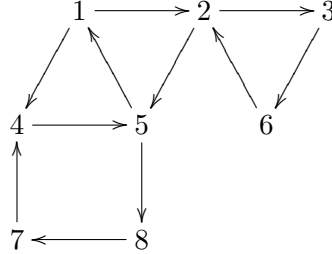
$$\text{deg}_{\text{HH}}(x) = n_B - n_{B/Be_xB}$$

COROLLARY 6.7

$$\text{deg}_{\text{HH}}(x) = \text{number of chordless cycles going through } x \\ - \text{number of inner arrows on the chordless cycles going through } x$$

Proof. Using Theorem 6.4, we get that $\text{deg}_{\text{HH}}(x)$ is equal to the number of chordless cycles that are adjacent to x minus the number of inner arrows in \tilde{Q} plus the number of inner arrows in Q_{B/Be_xB} . Now α is an inner arrow in \tilde{Q} which is not an inner arrow in Q_{B/Be_xB} , precisely if α lies on two chordless cycles in \tilde{Q} at least one of which goes through x . \square

EXAMPLE 6.8 The following quiver is the quiver of a cluster-tilted algebra of type \mathbb{E}_8 .



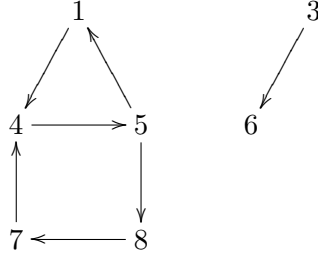
The quiver has 4 chordless cycles and 2 inner arrows, so Theorem 6.4 yields

$$\text{HH}^1(B) = k^{4-2} = k^2.$$

On the other hand, the quiver has 9 outer arrows, so, using Corollary 6.5, we also get

$$\text{HH}^1(B) = k^{1+9-8} = k^2.$$

The point 2 has Hochschild degree $2 - 1 = 1$, by Corollary 6.7. So $\text{HH}^1(B/Be_2B) = k$. The quiver of B/Be_2B is the following.



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