ON MULTIPLE SOLUTIONS OF A SEMILINEAR SCHRÖDINGER EQUATION WITH PERIODIC POTENTIAL

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ABSTRACT. This paper is concerned with the semilinear Schrödinger equation $(*) -\Delta u + V(x)u = f(x,u),\ \ u \in H^1(\mathbb{R}^N)$, where V and f are periodic in the $x-$variables, f is a superlinear and subcritical nonlinearity, and 0 lies in a spectral gap of $-\Delta u + V$. It is shown that, if f is odd in u then $(*)$ has infinitely many solutions. The proof relies on an infinite-dimensional Fountain Theorem without $(PS)-$type assumption, established in this paper.

1. Introduction

In this paper we consider the semilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1}$$

Our assumptions on f and V stated below imply that the solutions of (1) are critical points of a $\mathcal{C}^1$ functional defined on the Sobolev space $H^1(\mathbb{R}^N)$, with a strongly indefinite quadratic part. We are interested in multiplicity of solutions of (1) without the following superquadraticity condition due to Ambrosetti and Rabinowitz [1]:

$$\exists \mu > 2 \text{ such that } 0 < \mu F(x,u) \leqslant u f(x,u), \quad \forall u \in \mathbb{R}\backslash\{0\}, \ \ x \in \mathbb{R}^N, \tag{2}$$

where $F(x,u) = \int_0^u f(s,u)ds$. It is well known that the role of (2) is to insure the boundedness of all Palais-Smale sequences of the associated energy functional, and without (2) it becomes more complicated. However, there are many functions which are superlinear but do not satisfy (2) (see [2] and [6]).

In this paper, by combining the ideas in [2] and [5], we establish an infinite-dimensional Fountain Theorem without the usual Palais-Smale compactness condition. The main ingredients are the degree theory of Kryszewski and Szulkin [15], and the monotonicity method introduced by Struwe in [13] and [14], and developed by Jeanjean in [10]. By taking advantage of our abstract result, we study the existence of solutions of (1) under the following conditions:

- (f_1) The function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous and $1-$periodic with respect to each variable $x_j, j = 1, ..., N$.
- (f_2) There is a constant $c > 0$ such that $|f(x,u)| \leqslant c(1 + |u|^{p-1})$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$, where $p > 2$ for $N = 1, 2$ and $2 < p < 2^* := \frac{2N}{N-2}$ if $N \geqslant 3$.
- (f_3) $f(x,u) = \circ(u)$ uniformly with respect to x as $|u| \to 0$.
- (f_4) $\frac{F(x,u)}{|u|^\mu} \to \infty$ as $|u| \to \infty$, uniformly in x, where $\mu > 2$.

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(f_5) $u \mapsto \frac{f(x,u)}{|u|}$ is strictly increasing in $\mathbb{R}\backslash\{0\}$.

(f_6) The function $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and $1-$periodic with respect to each variable $x_j, j = 1, ..., N$, and 0 lies in a gap of the spectrum of $-\Delta + V$.

(f_7) For all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$, $f(x, -u) = -f(x, u)$.

The periodicity conditions on f and V imply that if u is a solution of (1), then so is $g * u$ for every $g \in \mathbb{Z}^N$, where $g * u(x) := u(x + g)$. Two solutions u and v are said to be geometrically distinct if $\{g * u/g \in \mathbb{Z}^N\} \cap \{g * v/g \in \mathbb{Z}^N\} = \varnothing$.

We have the following result:

Theorem 1. *Assume $(f_1) - (f_7)$. Then problem (1) has infinitely many geometrically distinct solutions.*

There is a number of papers dealing with problem (1) when 0 lies in a spectral gap of the operator $-\Delta + V$ (see for instance [3], [4], [5] or [9]). Theorem 1 was first proved by Szulkin and Weth in [4], by reducing the indefinite variational problem to a definite one. In [3] and [5] the authors obtained infinitely many geometrically distinct solutions, however the Ambrosetti-Rabinowitz condition (2) was used. In [8], by using the monotonicity method and a critical point theory developed in [12], and without the Ambrosetti-Rabinowitz condition, Zhao et al. obtained infinitely many solutions for the following system

$$-\Delta u + V(x)u = g(x, v), \quad -\Delta v + V(x)v = f(x, u), \quad u(x) \to 0, v(x) \to 0 \ \textit{as} \ |x| \to \infty,$$

provided f and g are odd nonlinearities.

The rest of paper is organized as follows. In section 2 we state and prove an infinite-dimension variant of the fountain theorem, and in section 3 we prove Theorem 1.

2. Abstract result

Let Y be a closed separable subspace of a Hilbert space $\big(X, (\cdot), ||\cdot||\big)$ such that $Z := Y^\perp = \overline{\oplus_{j=0}^\infty \mathbb{R}e^j}$ with $||e^j|| = 1$, and let $P : X \to Y$ and $Q : X \to Z$ be the orthogonal projections. Let $(e_j)_{j\geqslant 0}$ be an orthonormal basis of Y. On $X = Y \oplus Z$ we consider the norm

$$|||u||| := \max\Big(\sum_{j=0}^\infty \frac{1}{2^{j+1}}|(Pu, e_j)|, ||Qu||\Big),$$

and we denote by τ the topology and all topological notions related to the topology generated by $|||\cdot|||$ $\big($see [3], [15] or [5]$\big)$. Clearly $|||u||| \leqslant ||u||$ for every $u \in X$. Moreover, if (u_n) is a bounded sequence in X then

$$u_n \xrightarrow{\tau} u \Longleftrightarrow Pu_n \rightharpoonup Pu \ \textit{and} \ Qu_n \to Qu. \tag{3}$$

We use the following notations:

$$Y_k := Y \oplus (\oplus_{j=0}^k \mathbb{R}e^j), \qquad Z_k := \overline{\oplus_{j=k}^\infty \mathbb{R}e^j},$$

$$B_k = \{u \in Y_k : ||u|| \leqslant \rho_k\}, \quad N_k = \{u \in Z_k : ||u|| = r_k\} \ \textit{where} \ \ 0 < r_k < \rho_k, k \geqslant 1.$$

Consider the following family of $\mathcal{C}^1$-functionals $\Phi_\lambda : X \to \mathbb{R}$ defined by:

$$\Phi_\lambda(u) := L(u) - \lambda J(u), \qquad \lambda \in [1, 2],$$

For $k \geqslant 1$, let $\Gamma_k(\lambda)$ be the set of $\gamma : B_k \to X$ such that

(a) γ is odd and $\tau-$continuous, and $\gamma_{|_{\partial B_k}} = id$,

(b) every $u \in int(B_k)$ has a $\tau-$neighborhood N_u in Y_k such that $(id-\gamma)(N_u \cap int(B_k))$ is contained in a finite-dimensional subspace of X,

(c) $\Phi_\lambda(\gamma(u)) \leqslant \Phi_\lambda(u)\ \forall u \in B_k$.

Define

$$a_k(\lambda) := \sup_{\substack{u\in Y_k \\ \|u\|=\rho_k}} \Phi_\lambda(u), \quad b_k(\lambda) := \inf_{\substack{u\in Z_k \\ \|u\|=r_k}} \Phi_\lambda(u),$$

$$c_k(\lambda) := \inf_{\gamma\in\Gamma_k(\lambda)} \sup_{u\in B_k} \Phi_\lambda\big(\gamma(u)\big).$$

The following lemma was proved by Batkam and Colin in [5], by using the degree theory of Kryszewski and Szulkin.

Lemma 2 (Intersection lemma). *If γ satisfies conditions (a) and (b) above, then $\gamma(B_k) \cap N_k \neq \varnothing$.*

In the following we assume that:

(A_1) For every $\lambda \in [1,2]$ Φ_λ is τ-upper semicontinuous and $\nabla\Phi_\lambda$ is weakly sequentially continuous.

(A_2) $J(u) \geqslant 0$ for every $u \in X$; $L(u) \to \infty$ or $J(u) \to \infty$ as $||u|| \to \infty$.

(A_3) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for every $(\lambda, u) \in [1,2] \times X$.

The main result of this section is stated as follow:

Theorem 3. *Under assumptions $(A_1)-(A_3)$, if there are $0 < r_k < \rho_k$ such that $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1,2]$, then $c_k(\lambda) \geqslant b_k(\lambda)$ for all $\lambda \in [1,2]$. Moreover, for a.e $\lambda \in [1,2]$, there exists a sequence $(u_k^n(\lambda))_n$ such that*

$$\sup_n ||u_k^n(\lambda)|| < \infty, \quad \Phi_\lambda'(u_k^n(\lambda)) \to 0 \ \text{ and } \ \Phi_\lambda(u_k^n(\lambda)) \to c_k(\lambda).$$

The proof of this theorem follows the lines of the proof of Theorem 2.1 in [2], with some appropriate modifications. For completeness we write it in detail.

Proof. We already know by Lemma 2 that $c_k(\lambda) \geqslant b_k(\lambda)$ for every $\lambda \in [1,2]$. By (A_3) the map $\lambda \mapsto c_k(\lambda)$ is nonincreasing, consequently $c_k'(\lambda) = \frac{d}{d\lambda}(c_k(\lambda))$ exists for a.e $\lambda \in [1,2]$. We consider those $\lambda \in [1,2]$ such that $c_k'(\lambda)$ exists. Let (λ_n) be a sequence of $[1,2]$ such that $\lambda_n < \lambda$ and $\lambda_n \to \lambda$, then there is $n_0(\lambda)$ such that

$$-c_k'(\lambda) - 1 \leqslant \frac{c_k(\lambda_n) - c_k(\lambda)}{\lambda - \lambda_n} \leqslant -c_k'(\lambda) + 1, \quad \forall n \geqslant n_0(\lambda).$$

Step 1. Let $n \geqslant n_0(\lambda)$ and $\gamma_n \in \Gamma_k(\lambda)$ such that

$$\sup_{B_k} \Phi_{\lambda_n}(\gamma_n(u)) \leqslant c_k(\lambda_n) + (\lambda - \lambda_n).$$

We deduce from (A_3) that if $\Phi_\lambda(\gamma_n(u)) \geqslant c_k(\lambda) - (\lambda - \lambda_n)$ for some $u \in B_k$, then $J(\gamma_n(u)) \leqslant -c_k'(\lambda) + 3$ and $L(\gamma_n(u)) \leqslant c_k(\lambda) + \lambda(-c_k'(\lambda) + 3)$, hence there exists $m = m(c_k'(\lambda))$ such that $||\gamma_n(u)|| \leqslant m$. Now since $\lambda_n < \lambda$, for $u \in B_k$ we have

$$\Phi_\lambda(\gamma_n(u)) \leqslant \Phi_{\lambda_n}(\gamma_n(u)) \leqslant c_k(\lambda) + (2 - c_k'(\lambda))(\lambda - \lambda_n). \tag{4}$$

Step 2. Define for $\epsilon > 0$ small enough

$$F_\epsilon(\lambda, k) := \{u \in X \ \ ||u|| \leqslant m(c_k'(\lambda)) + 4, \ \ c_k(\lambda) - \epsilon \leqslant \Phi_\lambda(u) \leqslant c_k(\lambda) + \epsilon\}.$$

We claim that

$$\inf_{u\in F_\epsilon(\lambda,k)} ||\Phi_\lambda'(u)|| = 0.$$

In fact, assume by contradiction that there is $\epsilon_0 > 0$ such that $||\Phi_\lambda(u)|| \geqslant \epsilon_0$ for every $u \in F_{\epsilon_0}(\lambda, k)$. Choose γ_n such that $\lambda - \lambda_n < \epsilon_0$, $\lambda - \lambda_n < \frac{c_k(\lambda) - a_k(\lambda)}{2}$ and $(2 - c'_k(\lambda))(\lambda - \lambda_n) \leqslant \epsilon_0$, and define

$$F(\lambda, k) := \{u \in X \mid ||u|| \leqslant m(c'_k(\lambda)) + 4, \quad c_k(\lambda) - (\lambda - \lambda_n) \leqslant \Phi_\lambda(u) \leqslant c_k(\lambda) + \epsilon_0\}$$

and

$$w_\lambda(u) := 2||\nabla\Phi_\lambda(u)||^{-2}\nabla\Phi_\lambda(u), \quad \forall u \in F(\lambda, k).$$

Since $(\nabla\Phi_\lambda(u), w(u)) = 2 \ \forall u \in F(\lambda, k)$, we deduce from the weakly sequentially continuity of $\nabla\Phi_\lambda$ the existence of a symmetric $\tau-$neighborhood N_u of u such that $(\nabla\Phi_\lambda(v), w_\lambda(u)) > 1 \ \forall v \in N_u \cap F(\lambda, k)$. Now, since Φ_λ is $\tau-$upper semicontinuous, $\widetilde{N} := \Phi_\lambda^{-1}(]-\infty, c_\lambda(k) - (\lambda - \lambda_n)[)$ is $\tau-$open. Hence $\mathcal{N} := \widetilde{N} \cup \{N_u / u \in F(\lambda, k)\}$ is a $\tau-$open covering of

$$\widetilde{F}(\lambda, k) := \{u \in X \mid ||u|| \leqslant m(c'_k(\lambda)) + 4, \quad \Phi_\lambda(u) \leqslant c_k(\lambda) + \epsilon_0\}.$$

Since $\big(\widetilde{F}(\lambda, k), \tau\big)$ is metric, there is a τ-locally finite τ-open covering $\mathcal{M} := \{M_i : i \in I\}$ of $\widetilde{F}(\lambda, k)$ finer than $\mathcal{N}$. Let

$$V := \bigcup_{i \in I} M_i.$$

For every $i \in I$ one has either $M_i \subset N_v$ for some v or $M_i \subset \widetilde{N}$. In the first case we define $v_i := w_\lambda(v)$ and in the second case $v_i := 0$. Let $\{\lambda_i : i \in I\}$ be a τ-Lipschitz continuous partition of unity subordinated to $\mathcal{M}$ and define on V

$$h_\lambda(u) := \sum_{i \in I} \lambda_i(u) v_i,$$

and

$$\widetilde{h}_\lambda(u) := \frac{1}{2}(h(u) - h(-u)).$$

We know that the vector field $\widetilde{h}_\lambda$ satisfies the following properties (see [5]):

(a) $\widetilde{h}$ is τ-locally Lipschitz continuous and locally Lipschitz continuous,
(b) each point $u \in V$ has a τ-neighborhood V_u such that $\widetilde{h}_\lambda(V_u)$ is contained in a finite dimensional subspace of X,
(c) $(\nabla\Phi_\lambda(u), \widetilde{h}_\lambda(u)) \geqslant 0 \ \forall u \in V$,
(d) $(\nabla\Phi_\lambda(u), \widetilde{h}_\lambda(u)) > 1 \ \forall u \in F(\lambda, k)$.

Set $F^*(\lambda, k) := \Phi_\lambda^{-1}(]-\infty, c_k(\lambda) - (\lambda - \lambda_n)[)$ and define

$$\psi_\lambda(u) := dist_\tau(u, F^*(\lambda, k))\big[dist_\tau(u, F^*(\lambda, k)) + dist_\tau(u, F(\lambda, k))\big]^{-1} \quad \forall u \in V,$$

and

$$f_\lambda(u) := \psi_\lambda(u)\widetilde{h}_\lambda(u) \quad \forall u \in V.$$

It is easy to verify that f_λ is odd, $\tau-$locally Lipschitz, $\tau-$continuous, locally Lipschitz and continuous. Define

$$F^{**}(\lambda, k) := \{u \in X \mid ||u|| \leqslant m(c'_k(\lambda)), \quad c_k(\lambda) - (\lambda - \lambda_n) \leqslant \Phi_\lambda(u) \leqslant c_k(\lambda) + \epsilon_0\}.$$

Since $||\Phi'_\lambda(u)|| \geqslant \epsilon_0$ for $u \in F_{\epsilon_0}(\lambda, k)$, we have $||f_\lambda(u)|| \leqslant \frac{2}{\epsilon_0}$ for $u \in F_{\epsilon_0}(\lambda, k)$. Hence for every $u \in A(\lambda, k) := \Phi_\lambda^{-1}(]-\infty, c_k(\lambda) - (\lambda - \lambda_n)[) \cup F^{**}(\lambda, k)$, the Cauchy problem

$$\begin{cases} \frac{d}{dt}\sigma(t, u) = -f_\lambda(\sigma(t, u)) \\ \sigma(0, u) = u \end{cases}$$

has a unique solution $\sigma(\cdot, u)$. Moreover σ is continuous on $\mathbb{R}^+ \times A(\lambda, k)$, and since Φ_λ is odd, $\sigma(t, \cdot)$ is odd.
Obviously for every $u \in A(\lambda, k)$ the map $t \mapsto \Phi_\lambda(\sigma(t, u))$ is nonincreasing.

Claim 1: σ is $\tau-$continuous and every $(t, u) \in [0, 1] \times A(\lambda, k)$ has a $\tau-$neighborhood $N_{(t,u)}$ such that $\{v - \sigma(s, v)/(s, v) \in N_{(t,u)} \cap ([0, 1] \times A(\lambda, k))\}$ is contained in a finite-dimensional subspace of X.

In fact let $(t_0, u_0) \in [0, 2\epsilon_0] \times A(\lambda, k)$. The set $K := \mu([0, 2\epsilon_0] \times \{u_0\})$ is τ-compact. Thus there exists $r, p > 0$ such that

$$\begin{aligned} U : \quad &= \quad \{u \in X : \||u - K|\| < r\} \subset V \\ u, v \in U \quad &\Rightarrow \quad \||f_\lambda(u) - f_\lambda(v)|\| \leqslant p\||u - v|\| \end{aligned}$$

and $f_\lambda(U)$ is contained in a finite-dimensional subspace W of X.
If $\sigma(t, u) \in U$ for $0 \leqslant s \leqslant t \leqslant 2\epsilon_0$, then

$$\begin{aligned} \||\sigma(t, u) - \sigma(t, u_0)|\| \quad &\leqslant \quad \||u - u_0|\| + \int_0^t \||f_\lambda(\sigma(s, u)) - f_\lambda(\sigma(s, u_0))|\| ds \\ &\leqslant \quad \||u - u_0|\| + p\int_0^t \||\sigma(s, u) - \sigma(s, u_0)|\| ds. \end{aligned}$$

It then follows from Gronwall lemma that

$$\||\sigma(t, u) - \sigma(t, u_0)|\| \leqslant \||u - u_0|\| \exp(pt) \leqslant \||u - u_0|\| \exp(2\epsilon_0 p).$$

In particular for $\||u - u_0|\| < r \exp(-2p\epsilon_0)$ and $0 < t < 2\epsilon_0$ we obtain

$$u - \sigma(t, u) = \int_0^t f_\lambda(\sigma(s, u)) ds \in W.$$

Now let $0 < \nu < r \exp(-2p\epsilon_0)$. If $\||u - u_0|\| < \nu$, $|t - t_0| < \nu$ and $0 < t < 2\epsilon_0$, then we have

$$\begin{aligned} \||\sigma(t, u) - \sigma(t_0, u_0)|\| \quad &\leqslant \quad \||\sigma(t, u) - \sigma(t, u_0)|\| + \int_{t_0}^t \||f_\lambda(\sigma(s, u_0))|\| ds \\ &\leqslant \quad \Big(\exp(2p\epsilon_0) + \frac{\delta}{4\epsilon_0}\Big)\nu, \end{aligned}$$

which implies that σ is τ-continuous.

Claim 2: If we define $\beta_n(u) := \sigma(2\epsilon_0, \gamma_n(u))\ \forall u \in B_k$, then $\beta_n \in \Gamma_k(\lambda)$.

Obviously β_n is odd, $\tau-$continuous and $\Phi_\lambda(\beta_n(u)) \leqslant \Phi_\lambda(u)\ \forall u \in B_k$. Let $u \in \partial B_k$, then $\Phi_\lambda(u) \leqslant a_k(\lambda) \leqslant c_k(\lambda) - 2(\lambda - \lambda_n) < c_k(\lambda) - (\lambda - \lambda_n)$ which implies that $u \in F^*(\lambda, k)$. Since $f_\lambda = 0$ on $F^*(\lambda, k)$, $\sigma(\cdot, u)$ is a constant map on $F^*(\lambda, k)$ and we have $\sigma(t, u) = \sigma(0, u) = u$ for every $t \in [0, 1]$. It then follows that $\beta_n(u) = \sigma(2\epsilon_0, \gamma_n(u)) = \sigma(2\epsilon_0, u) = u$. Now let $u \in int(B_k)$, since $\gamma_n \in \Gamma_k(\lambda)$, u has a τ-neighborhood N_u in Y_k such that $(id - \gamma_n)(N_u \cap int(B_k)) \subset W_1$, where W_1 is a finite-dimensional subspace of X. By claim1 the point $(1, \gamma(u))$ has a τ-neighborhood $M_{(1,\gamma(u))} = M_1 \times \mathcal{M}_{\gamma(u)}$ such that $\{z - \eta(s, z)/(s, z) \in M_{(1,\gamma(u))} \cap ([0, 1] \times A(\lambda, k))\}$ is contained in a finite-dimensional subspace W_2 of X. Thus for every $v \in N_u \cap \gamma^{-1}(\mathcal{M}_{\gamma(u)}) \cap int(B_k)$, we have $(id - \beta)(v) = (id - \gamma)(v) + \gamma(v) - \eta(1, \gamma(v)) \in W_1 + W_2$ which is finite-dimensional.

Claim 3: For every $u \in B_k$, $\Phi_\lambda(\beta_n(u)) \leqslant c_k(\lambda) - (\lambda - \lambda_n)$.

Let $u \in B_k$. Since the map $t \mapsto \Phi_\lambda(\sigma(t,u))$ is nonincreasing, the claim is obvious if $\Phi_\lambda(\gamma_n(u)) < c_k(\lambda) - (\lambda - \lambda_n)$. Assume now that $\Phi_\lambda(\gamma_n(u)) \geqslant c_k(\lambda) - (\lambda - \lambda_n)$. It then follows from the choice of γ_n and (1) that $\gamma_n(u) \in F(\lambda, k)$. Since $||\sigma(t,\gamma_n(u)) - \gamma_n(u)|| \leqslant \int_0^t ||f_\lambda(\sigma(s,\gamma_(u)))||ds \leqslant \frac{2t}{\epsilon_0}$, we have $||\sigma(2\epsilon_0, \gamma_n(u))|| \leqslant 4 + ||\gamma_n(u)|| \leqslant m+4$. Assume that

$$\Phi_\lambda(\beta_n(u)) > c_k(\lambda) - (\lambda - \lambda_n), \tag{5}$$

then we have

$$\begin{aligned} c_k(\lambda) - (\lambda - \lambda_n) &< \Phi_\lambda(\sigma(2\epsilon_0, \gamma_n(u))) \\ &\leqslant \Phi_\lambda(\sigma(t, \gamma_n(u))) \\ &\leqslant \Phi_\lambda(\gamma_n(u)) \\ &\leqslant c_k(\lambda) + (2 - c_k'(\lambda))(\lambda - \lambda_n) \\ &\leqslant c_k(\lambda) + \epsilon_0, \end{aligned}$$

and $\sigma(t, \gamma_n(u)) \in F(\lambda, k)$ $\forall t \in [0, 2\epsilon_0]$. It follows that

$$\begin{aligned} \Phi_\lambda(\sigma(2\epsilon_0, \gamma_(u))) &= \Phi_\lambda(\gamma_n(u)) + \int_0^{2\epsilon_0} \frac{d}{ds}\Phi_\lambda(\sigma(s,\gamma_(u)))ds \\ &\leqslant \Phi_\lambda(\gamma_n(u)) - 2\epsilon_0. \end{aligned}$$

By (4) and the choice of γ_n we have $\Phi_\lambda(\gamma_n(u)) - 2\epsilon_0 \leqslant c_k(\lambda) - \epsilon_0$, which contradicts (5) since $\epsilon_0 > \lambda - \lambda_n$. Consequently, $\Phi_\lambda(\beta_n(u)) \leqslant c_k(\lambda) - (\lambda - \lambda_n)$ for every $u \in B_k$.

Now Claim 3 implies that

$$c_k(\lambda) \leqslant \sup_{u \in B_k} \Phi_\lambda(\beta_n(u)) \leqslant c_k(\lambda) - (\lambda - \lambda_n),$$

which is also a contradiction, and the proof is completed. □

3. Existence of solutions

In this section we assume that $(f_1) - (f_7)$ are satisfied.
We define the functional

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big)dx - \int_{\mathbb{R}^N} F(x,u)dx, \quad u \in H^1(\mathbb{R}^N). \tag{6}$$

It is well known that if Φ is of class $\mathcal{C}^1$ then its critical points correspond to weak solutions of (1). Let $L : H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ be the self-adjoint operator defined by $(Lu, v) := \int_{\mathbb{R}^N} \big(\nabla u \nabla v + V(x)uv\big)dx$. By (f_6), $X := H^1(\mathbb{R}^n)$ is the sum of two infinite $L-$invariant orthogonal subspaces Y and Z on which L is respectively negative definite and positive definite (see [3]). Let $P : X \to Y$ and $Q : X \to Z$ be the orthogonal projections, and consider the inner product on X defined by the formula

$$(u,v) := (L(Qu - Pu), v)_1, \quad u, v \in X$$

with the corresponding norm

$$\|u\| := \sqrt{(u,u)}.$$

The inner products $(\cdot)$ and $(\cdot)_1$ are equivalent, so Y and Z are also orthogonal with respect to $(\cdot)$.
One can verify easily that

$$\Phi(u) = \frac{1}{2}(\|Qu\|^2 - \|Pu\|^2) - \int_{\mathbb{R}^N} F(x,u)dx. \tag{7}$$

Lemma 4. $F(x,u) > 0$ *and* $\frac{1}{2}uf(x,u) > F(x,u)$ *for* $u \neq 0$.

We consider the family of functionals $\Phi_\lambda : X \to \mathbb{R}$

$$\Phi_\lambda(u) = \frac{1}{2}\|Qu\|^2 - \lambda\Big(\frac{1}{2}\|Pu\|^2 + \int_{\mathbb{R}^N} F(x,u)dx\Big), \quad \lambda \in [1,2].$$

The proof of the following lemma is standard (see for example [3]).

Lemma 5. *Φ_λ is of class $\mathcal{C}^1$ and $\tau-$upper semicontinuous, and $\nabla\Phi_\lambda$ is weakly sequentially continuous. Moreover Φ_λ maps bounded sets to bounded sets.*

Lemma 6. *For a.e. $\lambda \in [1,2]$ there exists $u_k(\lambda) \in X$ such that*

$$\Phi_\lambda(u_k(\lambda)) = c_k(\lambda) \quad and \quad \Phi'_\lambda(u_k(\lambda)) = 0 \quad for\ \ k\ \ big\ \ enough.$$

Proof. (f_4) and (f_5) imply that, for any $\delta > 0$ there is c_δ such that $F(x,u) \geqslant c_\delta|u|^\mu - \delta|u|^2$ for any u. Therefore it is easy to prove, for some ρ_k big enough, that $a_k(\lambda) \leqslant 0$, $\forall\lambda \in [1,2]$ (see [5]).

Now let $u \in Z_k$, then $Pu = 0$ and $Qu = u$. By (f_2) and (f_3) we have

$$\forall\epsilon > 0 \text{ small enough}, \exists c_\epsilon \text{ such that } |f(x,u)| \leqslant \epsilon|u| + c_\epsilon|u|^{p-1}. \tag{8}$$

This implies that

$$\begin{aligned}\Phi_\lambda(u) &\geqslant \frac{1}{2}\|u\|^2 - \frac{\lambda\epsilon}{2}|u|_2^2 - \frac{\lambda c_\epsilon}{p}|u|_p^p \\ &\geqslant \frac{1}{2}\|u\|^2 - \epsilon|u|_2^2 - \frac{2c_\epsilon}{p}|u|_p^p \quad (\text{since } \lambda \leqslant 2) \\ &\geqslant \frac{1}{2}\Big(\frac{1}{2}\|u\|^2 - c\beta_k^p\|u\|^p\Big),\end{aligned}$$

where

$$\beta_k := \sup_{\substack{v\in Z_k \\ \|v\|=1}} |v|_p.$$

If we choose $r_k := (cp\beta_k^p)^{\frac{1}{2-p}}$, then for $u \in Z_k$ with $||u|| = r_k$,

$$\Phi_\lambda(u) \geqslant \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{p}\Big)(cp\beta_k^p)^{\frac{2}{2-p}},$$

which implies that

$$b_k(\lambda) \geqslant \widetilde{b_k} := \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{p}\Big)(cp\beta_k^p)^{\frac{2}{2-p}}. \tag{9}$$

We know by [15] that $\sup_{\substack{v\in Z_k \\ \|v\|=1}} |v|_p \to 0$ as $k \to \infty$, which implies that $\widetilde{b_k} \to \infty$ as $k \to \infty$. Hence there is $k_0 > 0$ such that $b_k(\lambda) > 0 \geqslant a_k(\lambda)$ for $k > k_0$. Therefore, by Theorem 3, for a.e. $\lambda \in [1,2]$, there exists a sequence $(v_k^n(\lambda))_{n\geqslant 0}$ in X such that, for every $k > k_0$,

$$\sup_n ||v_k^n(\lambda)|| < \infty \ \ and\ \ \Phi_\lambda(v_k^n(\lambda)) \to c_k(\lambda), \ \ \Phi'_\lambda(v_k^n(\lambda)) \to 0 \quad as\ \ n \to \infty.$$

Let $k > k_0$. It is evident that no subsequence of $(v_k^n(\lambda))_n$ converges to 0. By Lemma 1.7 of [3], there is a sequence $(a_n) \subset \mathbb{R}^N$ and numbers $R, \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B(a_n,R)} |v_k^n(\lambda)|^2 dx \geqslant \delta.$$

Taking a subsequence if necessary we may suppose that

$$||v_k^n(\lambda)||_{L^2(B(a_n,r))} \geqslant \frac{\delta}{2} \quad \forall n. \tag{10}$$

By standard argument, there is $g_n \in \mathbb{Z}^N$ such that

$$||u_k^n(\lambda)||_{L^2(B(0,r+\frac{1}{2}\sqrt{N}))} \geqslant \frac{\delta}{2}, \quad \forall n, \tag{11}$$

where

$$u_k^n(\lambda) := g_n * v_k^n(\lambda). \tag{12}$$

Clearly we have $||u_k^n(\lambda)|| = ||v_k^n(\lambda)||$, $\Phi_\lambda(u_k^n(\lambda)) = \Phi_\lambda(v_k^n(\lambda))$ and $||\nabla\Phi_\lambda(u_k^n(\lambda))|| = ||\nabla\Phi_\lambda(v_k^n(\lambda))||$, hence

$$\Phi_\lambda(u_k^n(\lambda)) \to c_k(\lambda), \qquad \Phi'_\lambda(u_k^n(\lambda)) \to 0 \quad \textit{as } n \to \infty.$$

Since the sequence $(u_k^n(\lambda))_n$ is bounded, we deduce that, up to a subsequence

$$u_k^n(\lambda) \rightharpoonup u_k(\lambda) \ \textit{in } X, \ \textit{as } n \to \infty, \tag{13}$$

$$u_k^n(\lambda) \to u_k(\lambda) \ \textit{in } L^2_{loc}(\mathbb{R}^N), \ \textit{as } n \to \infty, \tag{14}$$

$$u_k^n(\lambda) \to u_k(\lambda) \ \textit{a.e on } \mathbb{R}^N, \ \textit{as } n \to \infty. \tag{15}$$

By (11) $u_k(\lambda) \neq 0$, and in view of the weak continuity of $\nabla\Phi_\lambda$ we get that $u_k(\lambda)$ is a critical point of Φ_λ. We proceed as in the proof of Theorem 20 of [5] to show that $\Phi_\lambda(u_k(\lambda)) = c_k(\lambda)$ and $\Phi'_\lambda(u_k(\lambda)) = 0$. □

As the consequence of the preceding lemma we have:

Lemma 7. *There is a sequence* $(\lambda_n) \subset [1,2]$ *and* $(z_k^n) \subset X\backslash\{0\}$ *such that* $\lambda_n \to 1$, $\Phi'_{\lambda_n}(z_k^n) = 0$ *and* $\Phi_{\lambda_n}(z_k^n) = c_k(\lambda_n)$.

Lemma 8. *Let* $\lambda \in [1,2]$. *If* $z_\lambda \neq 0$ *and* $\Phi'_\lambda(z_\lambda) = 0$, *then* $\Phi_\lambda(z_\lambda + w) \leqslant \Phi_\lambda(z_\lambda)$ *for every* $w \in \mathcal{Z}_\lambda := \{rz_\lambda + v/r \geqslant -1, v \in Y\}$.

Proof. Let $w = rz_\lambda + v \in \mathcal{Z}_\lambda$. Since $z_\lambda + w = (1+r)z_\lambda + v$, $\Phi'_\lambda(z_\lambda) = 0$ and $(Qz_\lambda, v) = 0$, one can easily verify that

$$\begin{aligned}\Phi_\lambda(z_\lambda + w) - \Phi_\lambda(z_\lambda) &= -\frac{\lambda}{2}||v||^2 + \lambda\int_{\mathbb{R}^N} \big[f(x,z_\lambda)\big(r(\frac{r}{2}+1)z_\lambda + \\ &\quad (1+r)v\big) + F(x,z_\lambda) - F(x,z_\lambda+w)\big]dx.\end{aligned}$$

And the conclusion follows from [4] Lemma 2.2. □

Lemma 9. *The sequence* (z_k^n) *is bounded.*

Proof. To show that (z_k^n) is bounded we proceed by contradiction. Suppose that $||z_k^n|| \to \infty$ as $n \to \infty$, and define $w_k^n = z_k^n/||z_k^n||$. Following the approach developed by Jeanjean [10], we will find the contradiction by showing that neither (Qw_k^n) vanishes nor it does not vanish.

(1) Assume that (Qw_k^n) is vanishing. Then by [15] Lemma 1.21, $Qw_k^n \to 0$ as $n \to \infty$ in $L^p(\mathbb{R}^N)$. By $(f_1) - (f_3)$, for every $\epsilon > 0$ there is c_ϵ such that $|F(x,u)| \leqslant \epsilon|u|^2 + c_\epsilon|u|^p$. We then deduce that for every $r \geqslant 0$, $\int_{\mathbb{R}^N} F(x, rQw_k^n)dx \to 0$ as $n \to \infty$. Now since $||Qw_k^n||^2 \geqslant \frac{1}{2}$, it follows from Lemma 8 that

$$\Phi_{\lambda_n}(z_k^n) \geqslant \Phi_{\lambda_n}(rQw_k^n) \geqslant \frac{r^2}{4} - \lambda_n \int_{\mathbb{R}^N} F(x, rQw_k^n)dx.$$

Thus we have $c_k(1) \geqslant \frac{r^2}{4}$, which is a contradiction if we take r big enough.

(2) Assume that (Qw_k^n) is not vanishing. Then there are $R, \delta > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{y_n + B_R} |Qw_k^n|^2 dx \geqslant \delta.$$

Up to a translation and a subsequence we may suppose that

$$\liminf_{n\to\infty} \int_{B(0, R+\frac{1}{2}\sqrt{n})} |Qw_k^n|^2 dx \geqslant \frac{\delta}{2}. \tag{16}$$

Setting $w_k^n \rightharpoonup w_k$ as $n \to \infty$, then (16) implies, since $Qw_k^n \to Qw_k$ in $L^2_{loc}(\mathbb{R}^N)$, that $Qw_k \neq 0$. And this implies that $|z_k^n| \to \infty$ as $n \to \infty$. It then follows from (f_4) and Fatou's Lemma that

$$0 \leqslant \frac{\Phi_{\lambda_n}(z_k^n)}{||z_k^n||^2} \to -\infty,$$

which is also a contradiction.

Consequently, (z_k^n) is bounded. □

Proof Theorem 1. Consider the sequence (z_k^n) above. From the relations

$$\Phi(z_k^n) = \Phi_{\lambda_n}(z_k^n) + (\lambda_n - 1)||Pz_k^n||^2 + (\lambda_n - 1)\int_{\mathbb{R}^N} F(x, z_k^n)dx,$$

and

$$\left\langle \Phi'(z_k^n) - \Phi'_{\lambda_n}(z_k^n), v \right\rangle = (\lambda_n - 1)\big[(Pz_k^n, v) + \int_{\mathbb{R}^N} v f(x, z_k^n)dx\big],$$

we deduce that (z_k^n) is a $(PS)-$sequence for Φ at level $c_k(1)$. By repeating the argument of Lemma 6 we obtain, up to a translation and a subsequence, the existence of $z_k \in X$ such that $\Phi'(z_k) = 0$ and $\Phi(z_k) \geqslant \widetilde{b}_k$, and the conclusion follows since $\widetilde{b}_k \to \infty$ as $k \to \infty$. □

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