Mutating seeds: types A and $\tilde{A}$.

Ibrahim Assem[1] Christophe Reutenauer[2]

[1]Département de mathématiques, Université de Sherbrooke
Sherbrooke (Québec) J1K2R1, Canada

[2]Laboratoire de combinatoire et d'informatique mathématique,
Université du Québec à Montréal
Case postale 8888, succursale Centre-ville
Montréal (Québec) H3C 3P8, Canada

June 3, 2011

Abstract

In the cases A and $\tilde{A}$, we describe all seeds obtained by sequences of mutations from an initial seed. In the $\tilde{A}$ case, we deduce a linear representation of the group of mutations which contains as entries all cluster variables obtained after an arbitrary sequence of mutations (this sequence is an element of the group). Nontransjective variables correspond to certain subgroups of finite index. A noncommutative rational series is constructed, which contains all this information.

1 Introduction

Recall that a *representative function* f on a group G is a function from G into some field K which is the composition of a linear representation of $G \to K^{n\times n}$ followed by a linear form on $K^{n\times n}$. Equivalently, the set of translates $f.g$, $g \in G$ (with the natural right action of G: $(f.g)(g_1) = f(gg_1)$ for any g, g_1 in G), spans a finite dimensional subspace of the vector space of functions on G. See [27] I.1, [1, 17].

As an example, take the additive group $G = \mathbb{Z}$. A representative function on this group is a sequence indexed by $\mathbb{Z}$ which satisfies a linear recursion which works in both directions (for example the Fibonacci sequence extended to negative integers).

The interest of representative functions is illustrated for example by: the theorem of Peter-Weyl, which asserts that the representative functions on a compact group are dense in the space of continuous functions on this group (see [29], where representative functions are called *matrix elements*); their role in the theory of affine algebraic groups, see [27, 1]; the theorem of Kleene-Schützenberger which asserts that representative functions on a free monoid coincide with noncommutative rational series, see [8]. Moreover, representative functions on an algebra (a generalization of that on groups) are the elements of the dual coalgebra of this algebra, or Sweedler dual, see [28, 1, 17].

Since we need a slightly more general notion of representative functions (with values in a ring instead of a field), we have developed this in an Appendix.

We give some definitions and results from the theory of cluster algebras of Fomin and Zelevinsky [22]. Let Q_0 be a quiver with set of vertices $\{1, \ldots, n\}$, with a variable x_i associated to each vertex. This data is called the *initial seed*, denoted by S_0. A *seed* is a couple $S = (Q, (y_1, \ldots, y_n))$, where Q is a quiver with set of vertices $\{1, \ldots, n\}$ and where $y_1, \ldots, y_n$ generate the field $\mathbb{Q}(x_1, \ldots, x_n)$ (in particular, they are algebraically independent over $\mathbb{Q}$, since the transcendence degree of $\mathbb{Q}(x_1, \ldots, x_n)$ is n); y_i is called the i-th *cluster variable* of S, or the *variable at vertex* i, and denoted by $y_i(S)$. Such a seed is obtained from the initial seed by a sequence of operations called mutations.

Mutation at vertex i on a seed has the following property: it replaces the quiver Q by another quiver with the same vertex set, and with a new cluster variable at vertex i (that is, y_i), the others being unchanged. Mutation at vertex i is involutive, which means that if one performs it twice then one recovers the original seed. Denote by μ_i the mutation at vertex i.

Consider the group generated by the set $\{\mu_1, \ldots, \mu_n\}$, subject to the relation that the generators are involutive. We call this group the *group of mutations*, denoted by M. It acts naturally on seeds. If m is an element of this group, and S a seed, we denote by S^m the seed obtained by applying the sequence of mutations determined by m to the seed S.

Suppose that the initial quiver Q is of type $\tilde{A}_{n-1}$ with an acyclic orientation (note that $\tilde{A}_{n-1}$ has n vertices). Fix i. We shall show that the function from M into $\mathbb{Q}(x_1, \ldots, x_n)$, which associates to m the i-th cluster variable of S_0^m, is a representative function of the mutation group. Moreover, we show that if y is a fixed nontransjective cluster variable, then the set of $m \in M$ such that $y_i(S_0^m) = y$ is a finite union of cosets of a normal subgroup of finite index of the mutation group.

Our objective is to describe precisely the mutated seeds in type $\tilde{A}$. Note that the mutated quivers of type $\tilde{A}$ are known, see [6] (see for example Figure

2 in that article) or [2]. What we add to these descriptions is the explicit computation of the cluster variables associated to each vertex. This is done, on one hand, by embedding the *cyclic part* of the mutated quiver into an SL_2-tiling of the plane, which contains all transjective cluster variables; and on the other hand, by describing the remaining parts of the quiver in terms of *continuant trees*, which are tree-like graphs whose vertices are indexed by *signed continuant polynomials*; the latter give directly the nontransjective cluster variables. The *signed continuant polynomials* are a variant of the continuant polynomials. They have been considered implicitly by Coxeter [16] Eq. (7.5), in [7], and explicitly by Grégoire Dupont, in [18] where he uses them to study the regular cluster variables. He also gives a version with coefficients in [19] and applies them to the study of positivity of the regular cluster variables in [20].

As a byproduct of the concept of continuant trees, which are shown to correspond to triangulations of an $n+3$-gon, we obtain a description of the mutated seeds in type A_n. This may be of some interest, since it presents some novelty, and is completely elementary. The mutation formula turns out to be a consequence of a formula on continuants polynomials, which goes back to Euler. Mutated quivers in type A_n are known, see [11].

2 Preliminaries

2.1 Signed continuants polynomials

The ordinary continuant polynomials, already considered by Euler, are defined, for any elements $a_1, \ldots, a_n$ of a ring R by the recursion $p_n(a_1, \ldots, a_n) = p_{n-1}(a_1, \ldots, a_{n-1})a_n + p_{n-2}(a_1, \ldots, a_{n-2})$, with initial conditions $p_{-1} := 0$ and $p_0 := 1$. See [30] p.133, [14] p.116, [25] p.302, [8] p.186. The terminology comes from their link with continued fractions as shows the following identity, valid if R is commutative and if the inversions are defined in R:

$$\frac{p(a_1, \ldots, a_n)}{p(a_2, \ldots, a_n)} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} \,. \tag{1}$$

The *signed continuant polynomials* are a variant of the continuant polynomials. They are defined as follows. Let $a_1, \ldots, a_n$ be as above. Define for $n \geq 1$,

$$q_n(a_1, \ldots, a_n) = q_{n-1}(a_1, \ldots, a_{n-1})a_n - q_{n-2}(a_1, \ldots, a_{n-2}), \tag{2}$$

setting $q_{-1} := 0$ and $q_0 := 1$. We omit indices when possible, writing simply $q(x_1, \ldots, x_n)$ for $q_n(x_1, \ldots, x_n)$. Let us now consider the particular SL_2 matrices

$$Q(a) := \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}.$$

One has the following result, see [7] 8.1.

Lemma 2.1.

$$Q(a_1)Q(a_2)\cdots Q(a_n) = \begin{pmatrix} -q(a_2, \ldots, a_{n-1}), & -q(a_2, \ldots, a_n) \\ q(a_1, \ldots, a_{n-1}), & q(a_1, \ldots, a_n) \end{pmatrix}. \tag{3}$$

It follows from this matrix equation that one has also

$$q(a_1, \ldots, a_n) = a_1 q(a_2, \ldots, a_n) - q(a_3, \ldots, a_n). \tag{4}$$

The q's satisfy the following identity (a consequence of [16] Eq.(7.4)), which is a variant of Eq.(1) and which holds under the same assumptions:

$$\frac{q(a_1, \ldots, a_n)}{q(a_2, \ldots, a_n)} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}\ .$$

Indeed, this holds for $n = 1$. Assume that it holds for n. Then the continued fraction for $n + 1$ is equal by induction to

$$a_1 - \frac{1}{\frac{q(a_2, \ldots, a_{n+1})}{q(a_3, \ldots, a_{n+1})}} = a_1 - \frac{q(a_3, \ldots, a_{n+1})}{q(a_2, \ldots, a_{n+1})} = \frac{a_1 q(a_2, \ldots, a_{n+1}) - q(a_3, \ldots, a_{n+1})}{q(a_2, \ldots, a_{n+1})}.$$

Thus the statement follows from Eq.(4).

It will be useful to adopt the following notation, which avoids the use of indices: let w be a finite sequence of elements of the ring R (a *word* on R). For example, $w = a_1 \cdots a_n$ (not the product in the ring); then we write $q(w)$ for $q(a_1, \ldots, a_n)$. If u, v are two such sequences, we denote by uv their concatenation. Then we have the following result, valid when R is *commutative*, which is assumed from now on.

Lemma 2.2. *Let u, v, w be words on R and a, b be in R. Then*

$$q(uav)q(vbw) = q(u)q(w) + q(uavbw)q(v).$$

In the case of ordinary continuants polynomials, there is an analogue of this identity, due to Euler and given in [25] Eq.(6.134) p.303.

Proof. We prove first this equation when $w = 1$ (*empty word*), that is $q(uav)q(vb) = q(u) + q(uavb)q(v)$. We adopt the notation $Q(u)$ for the matrix product corresponding to u. Then we have by Eq.(3):

$$Q(uavb) = \begin{pmatrix} * & * \\ q(uav) & q(uavb) \end{pmatrix}, Q(vb) = \begin{pmatrix} * & * \\ q(v) & q(vb) \end{pmatrix}, Q(ua) = \begin{pmatrix} * & * \\ q(u) & * \end{pmatrix}.$$

Thus

$$Q(ua) = Q(uavb)Q(vb)^{-1} = \begin{pmatrix} * & * \\ q(uav) & q(uavb) \end{pmatrix} \begin{pmatrix} q(vb) & * \\ -q(v) & * \end{pmatrix}.$$

Thus $q(u) = q(uav)q(vb) - q(uabv)q(v)$ which proves the lemma when $w = 1$.

Suppose now that w is of length 1, that is $w = c$, $c \in R$. Then the left-hand side of the equation in the lemma is by Eq.(2) equal to

$$q(uav)q(vbc) = q(uav)((q(vb)c - q(v)) = q(uav)q(vb)c - q(uav)q(v).$$

By the $w = 1$ case and by Eq.(2), this is equal to

$$\begin{aligned} q(u)c + q(uavb)q(v)c - q(uav)q(v) &= q(u)c + (q(uavb)c - q(uav))q(v) \\ &= q(u)c + q(uavbc)q(v), \end{aligned}$$

which proves the $w = c$ case.

Otherwise, we may write $w = w'cd$ for c, d in R. Then by Eq.(2)

$$\begin{aligned} q(uav)q(vbw) &= q(uav)q(vbw'cd) = q(uav)(q(vbw'c)d - q(vbw')) \\ &= q(uav)q(vbw'c)d - q(uav)q(vbw') \end{aligned}$$

By induction (cases $w = w'$ and $w = w'c$), this is equal to

$$\begin{aligned} &q(u)q(w'c)d + q(uavbw'c)q(v)d - q(u)q(w') - q(uavbw')q(v) \\ &= q(u)(q(w'c)d - q(w')) + (q(uavbw'c)d - q(uavbw'))q(v) \\ &= q(u)q(w'cd) + q(uavbw'cd)q(v) = q(u)q(w) + q(uavbw)q(v). \end{aligned}$$

□

Lemma 2.3. *Suppose that*

$$Q(a_1)Q(a_2)\cdots Q(a_{n+3}) = -1.$$

Then for any i with $1 \leq i \leq n+3$, we have $q(a_1, \ldots, a_{i-1}) = q(a_{i+1}, \ldots, a_{n+2})$.

Proof. By hypothesis, we have $(Q(a_1)\cdots Q(a_{i-1}))^{-1} = -Q(a_i)\cdots Q(a_{n+3})$. Using Eq. (3) and the fact that the matrices have determinant 1, we obtain

$$\begin{pmatrix} q(a_1, \ldots, a_{i-1}) & * \\ * & * \end{pmatrix} = -\begin{pmatrix} -q(a_{i+1}, \ldots, a_{n+2}) & * \\ * & * \end{pmatrix}.$$

□

2.2 Continuant trees

We call *pre-continuant tree* a planar quiver T which is constructed from a planar binary, not necessarily complete, tree τ as follows:

- An edge between a node and its left son is oriented towards this node;
- an edge between a node and its right son is oriented towards this son;
- If a node has two sons, then an arrow from the right son towards its left son is created.

See Figure 1 for an example, disregarding the labels. These quivers are already known: they are described in [11] and [2] (Th.2.7). Their tree-like shape comes from the classification of the tilted algebras of the linearly oriented quivers A_n, see [26]. The additional arrows from right to left sons arise because of [3, 10].

Note that since the original tree τ is planar, we may use for T the terminology of planar binary trees: subtree at a vertex (which is a pre-continuant tree), left and right son, father...

Now we call *continuant tree* a pre-continuant tree T together with a labelling of its vertices by words on R, as follows:

- the length of the label of a vertex is the number of vertices of the subtree having this vertex as root;
- if a vertex is a left (resp. right) son, then its label is a prefix (resp. suffix) of its father;
- finally, to each vertex labelled u is associated the signed continuant polynomial $q(u)$.

See Figure 1 for an example. Note that a continuant tree is completely determined by the underlying pre-continuant tree together with the label w (of length equal to the number of vertices) of the root. For later use, we call this continuant tree a w-continuant tree, or a $(a_1, \ldots, a_n)$-continuant tree, if $w = a_1 \ldots a_n$.

2.3 Mutation of a continuant tree outside the root

Recall the definition of the mutation of a quiver; to each vertex k of this quiver is associated an element $y_k \in R$, called *variable at* k. We assume that the quiver has no cyclic path of length 1 or 2. The mutation at a vertex k is performed as follows (see [22] or e.g. [24] 3.2.):

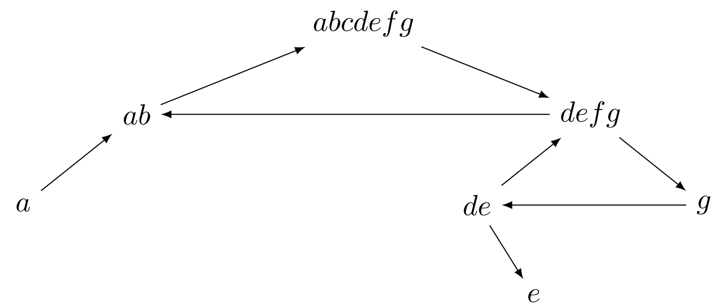

Figure 1: A continuant tree

- Reverse all arrows incident with k.
- For each pair of arrows $i \to k \to j$, add an arrow $i \to j$.
- Remove pairs of opposite arrows, until no such pair exists.

Now, the variables of the new quiver are the same at each vertex, except at vertex k where the new variable y'_k (the *mutated variable*) must satisfy the *exchange relation*

$$y_k y'_k = \prod_{i \to k} y_i + \prod_{k \to j} y_j,$$

where the products are taken over all arrows ending or starting at k, respectively. Note that uniqueness of y'_k is ensured if R has no zero divisors, an hypothesis which will be made in the sequel.

Assume now that T is a continuant tree. The variables at the vertices of T are the corresponding continuant polynomials, that is, if k is a vertex with associated word u, then the variable at k is $q(u)$.

Lemma 2.4. *Mutation of a continuant tree at a vertex k which is not the root gives another continuant tree.*

Proof. The lemma is illustrated in Figure 2, where only the vertices involved in the mutation are represented. The vertex k is the one with uav on the left and the one with vbw on the right. By inspection and by definition of mutation, it is seen that the quiver on the right is mutated from the quiver at vertex k. The mutation formula for the labels is a consequence of Lemma 2.2, which ensures the existence of the mutated variable. Note the limiting cases where some vertices among u, v, w are missing; that is,

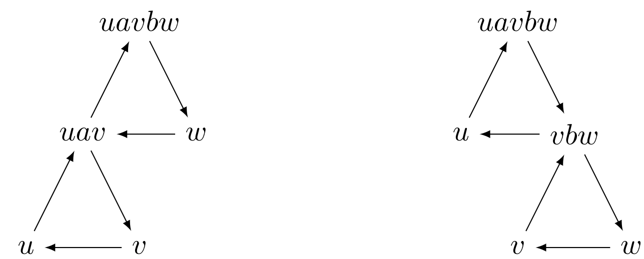

Figure 2: mutation of a continuant tree at a vertex different from the root

the corresponding word is empty: they are still covered by this proof since the continuant polynomial $q(x)$ equals 1 if x is the empty word. Moreover, the mutation from right to left in Figure 2 follows from the the fact that mutation is an involution. $\square$

2.4 SL_2-tilings of the plane

Following [5], we call *SL_2-tiling of the plane* a mapping $t : \mathbb{Z}^2 \mapsto K$, for some field K, such that, for any x, y in $\mathbb{Z}$,

$$\begin{vmatrix} t(x,y) & t(x+1,y) \\ t(x,y+1) & t(x+1,y+1) \end{vmatrix} = 1.$$

Here we represent the discrete plane $\mathbb{Z}^2$ with matrix-like coordinates, so that the y-axis points downwards, and the x-axis points to the right. Note that the x-coordinates therefore represent the column indices, and the y coordinates represent the row indices.

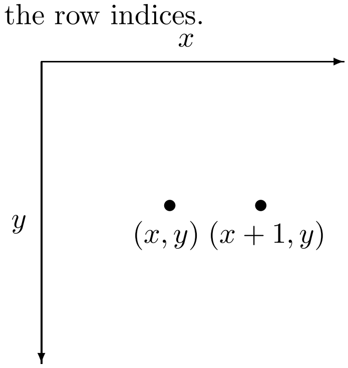

Figure 3: Coordinates conventions

An example is given below, with $K = \mathbb{Q}$.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & & & 1 & 1 & 1 & 1 & \\
 & & & & & & & & & 1 & 1 & 2 & 3 & 4 & \\
 & & & & & & & & & 1 & 2 & 5 & 8 & 11 & \\
 & & & & & & & & & 1 & 3 & 8 & 13 & 18 & \\
 & & & & \dots & & \dots & & 1 & 1 & 4 & 11 & 18 & 25 & \dots \\
 & & & & & & & & 1 & 2 & 9 & 25 & 41 & 57 & \\
 & & & & & & & & 1 & 3 & 14 & 39 & 64 & 89 & \\
 & & & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 19 & 53 & 87 & 121 & \\
1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 25 & 119 & 332 & 545 & 758 & \\
1 & 2 & 3 & 4 & 9 & 14 & 19 & 24 & 29 & 121 & 576 & 1607 & 2368 & 3669 & \dots \\
 & & & & & \dots & & \dots & & & & \dots & & &
\end{array}
$$

Here is another example, with K the field of fractions over $\mathbb{Q}$ in the variables $a, b, c, d, e, f, \dots$.

$$
\begin{array}{cccccc}
 & & & \cdots & & \\
 & \frac{d+b+bce}{cd} & \frac{1+ce}{d} & e & f & \\
\cdots & b & c & d & \frac{1+df}{e} & \cdots \\
 & a & \frac{1+ac}{b} & \frac{b+d+acd}{bc} & \frac{bce+b+bdf+dac+d^2acf+d+d^2f}{bcde} & \\
 & & & \cdots & &
\end{array}
$$

2.5 Tameness

Note that an SL_2-tiling of the plane, viewed as an infinite matrix, has necessarily rank at least 2. Following [7], we say that the tiling is *tame* if its rank is 2.

Given three *successive* columns C_0, C_1, C_2 of a tame SL_2-tiling t, there is a unique coefficient α such that

$$C_0 + C_2 = \alpha C_1. \tag{5}$$

This follows from the fact that the adjacent 2 by 2 minors are equal to 1. We call α the *linearization coefficient* of column C_1. Similarly for rows.

The following result extends Eq.(5) (obtained for $n = 1$).

Lemma 2.5. *Let t be a tame SL_2-tiling of the plane and $C_0, \dots, C_{n+1}$ successive columns of t, with linearization coefficients $\alpha_0, \dots, \alpha_{n+1}$. Then for any i in $\{1, \dots, n\}$*

$$q(\alpha_{i+1}, \dots, \alpha_n)C_0 + q(\alpha_1, \dots, \alpha_{i-1})C_{n+1} = q(\alpha_1, \dots, \alpha_n)C_i.$$

Proof. We use the identity $C_j = -q(\alpha_2, \ldots, \alpha_{j-1})C_0 + q(\alpha_1, \ldots, \alpha_{j-1})C_1$, which follows from Eq.($*$) in the proof of Proposition 5.4 in the Appendix, and from Eq.(3).

Suppose first that $i = 1$. Then

$$q(\alpha_2, \ldots, \alpha_n)C_0 + C_{n+1} = q(\alpha_2, \ldots, \alpha_n)C_0 - q(\alpha_2, \ldots, \alpha_n)C_0 + q(\alpha_1, \ldots, \alpha_n)C_1$$

$$= q(\alpha_1, \ldots, \alpha_n)C_1,$$

which proves the identity for $i = 1$. Suppose now that $i > 1$. Then we have by Lemma 2.2 (with u equal to the empty word, $a = \alpha_1$, $v = \alpha_2 \cdots \alpha_{i-1}$, $b = \alpha_i$, $w = \alpha_{i+1} \cdots \alpha_n$):

$$q(\alpha_1, \ldots, \alpha_{i-1})q(\alpha_2, \ldots, \alpha_n) = q(\alpha_{i+1}, \ldots, \alpha_n) + q(\alpha_1, \ldots, \alpha_n)q(\alpha_2, \ldots, \alpha_{i-1}).$$

Thus we obtain

$$q(\alpha_{i+1}, \ldots, \alpha_n)C_0 + q(\alpha_1, \ldots, \alpha_{i-1})C_{n+1}$$

$$= q(\alpha_{i+1}, \ldots, \alpha_n)C_0 + q(\alpha_1, \ldots, \alpha_{i-1})(-q(\alpha_2, \ldots, \alpha_n)C_0 + q(\alpha_1, \ldots, \alpha_n)C_1)$$

$$= (q(\alpha_{i+1}, \ldots, \alpha_n) - q(\alpha_1, \ldots, \alpha_{i-1})q(\alpha_2, \ldots, \alpha_n))C_0$$

$$+ q(\alpha_1, \ldots, \alpha_{i-1})q(\alpha_1, \ldots, \alpha_n)C_1$$

$$= -q(\alpha_1, \ldots, \alpha_n)q(\alpha_2, \ldots, \alpha_{i-1})C_0 + q(\alpha_1, \ldots, \alpha_{i-1})q(\alpha_1, \ldots, \alpha_n)C_1$$

$$= q(\alpha_1, \ldots, \alpha_n)(-q(\alpha_2, \ldots, \alpha_{i-1})C_0 + q(\alpha_1, \ldots, \alpha_{i-1})C_1)$$

$$= q(\alpha_1, \ldots, \alpha_n)C_i.$$

□

It has been shown in [7] (in the more general case of SL_k-tilings) that tameness of SL_2-tilings is characterized by the fact that the infinite matrix of 2 by 2 minors is of rank 1. This is important for the proof of the next result.

Corollary 2.6. *For $n, m \geq 0$, let $(a_{ij})_{0 \leq i \leq n+1, 0 \leq j \leq m+1}$ be a connected submatrix of a tame SL_2-tiling t. Let $\beta_0, \ldots, \beta_{n+1}$ denote the linearization coefficients of the corresponding rows of t and $\alpha_0, \ldots, \alpha_{m+1}$ denote those of the corresponding columns. Then*

$$\det(\begin{pmatrix} a_{00} & a_{0,m+1} \\ a_{n+1,0} & a_{n+1,m+1} \end{pmatrix}) = q(\beta_1, \ldots, \beta_n)q(\alpha_1, \ldots, \alpha_m).$$

Proof. We suppose first that $n = 0$. Denote

$$C_j = \begin{pmatrix} a_{0,j} \\ a_{1,j} \end{pmatrix}.$$

Then we have by Lemma 2.5, $q(\alpha_2, \ldots, \alpha_m)C_0 + C_{m+1} = q(\alpha_1, \ldots, \alpha_m)C_1$. Thus

$$\det(C_0, C_{m+1}) = q(\alpha_1, \ldots, \alpha_m) \det(C_0, C_1) = q(\alpha_1, \ldots, \alpha_m),$$

since t is an SL_2-tiling.

Now, in the general case, it follows from [7], Prop.4, that the determinant of the corollary is equal to the product

$$\det(\begin{pmatrix} a_{00} & a_{0,m+1} \\ a_{1,0} & a_{1,m+1} \end{pmatrix}) \det(\begin{pmatrix} a_{00} & a_{0,1} \\ a_{n+1,0} & a_{n+1,1} \end{pmatrix}),$$

which proves the corollary, by the first part. $\square$

2.6 Frontier

We call *frontier* a bi-infinite sequence

$$\ldots \xi_{-2}x_{-2}\xi_{-1}x_{-1}\xi_0 x_0 \xi_1 x_1 \xi_2 x_2 \xi_3 x_3 \ldots \tag{6}$$

where $\xi_i \in \{x, y\}$ and x_i are elements of K^*, for any $i \in \mathbf{Z}$. It is called *admissible* if there are arbitrarily large and arbitrarily small i's such that $\xi_i = x$, and similarly for y; in other words, none of the two sequences $(\xi_n)_{n \geq 0}$ and $(\xi_n)_{n \leq 0}$ is ultimately constant. The x_i's are called the *variables* of the frontier.

Each frontier may be embedded into the plane: the variables label points in the plane, and the x (resp. y) determine a bi-infinite discrete path, in such a way that x (resp. y) corresponds to a segment of the form $[(a, b), (a + 1, b)]$ (resp $[(a, b), (a, b - 1)]$); recall the coordinate conventions, see Figure 2.4. For example, the path corresponding to the frontier $\ldots x_{-4}xx_{-3}yx_{-2}yx_{-1}yx_0xx_1xx_2yx_3xx_4xx_5 \ldots$ is given in Figure 4.

We need the following notation of [5]. Let

$$M(a, x, b) = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \text{ and } M(a, y, b) = \begin{pmatrix} b & 0 \\ 1 & a \end{pmatrix}.$$

Given an admissible frontier, embedded in the plane as explained previously, let $P \in \mathbf{Z}^2$. Then we obtain a finite word, which is a factor of the

$$
\begin{array}{ccccccc}
 & & & & & & \cdot \\
 & & & & & \cdot & \\
 & & & & & & \\
 & & & x_3 & x_4 & x_5 & \\
 & x_0 & x_1 & x_2 & \vdots & & \\
 & x_{-1} & & & \vdots & & \\
 & x_{-2} & & & \vdots & & \\
x_{-4} & x_{-3} & \cdots & \cdots & P & & \\
\cdot & & & & & & \\
\cdot & & & & & &
\end{array}
$$

Figure 4: A frontier

frontier, by projecting the point P horizontally and vertically onto the frontier. We call this word the *word* of P. It is illustrated in Figure 4, where the word of the point P is $x_{-3}yx_{-2}yx_{-1}yx_0xx_1xx_2yx_3xx_4$. We define the word of a point only for points below the frontier; for points above, the situation is symmetric and we omit this case.

Theorem 2.7. *Given an admissible frontier, there exists a unique tame SL_2-tiling t of the plane over K, extending the embedding of the frontier into the plane. It is defined, for any point P below the frontier, with associated word $x_0\xi_1x_1\xi_2...\xi_{n+1}x_{n+1}$, where $n \geq 1$, $x_i \in K^*$ and $\xi_i \in \{x, y\}$, by the formula*

$$t(P) = \frac{1}{x_1x_2...x_n}(1, x_0)M(x_1, \xi_2, x_2)\cdots M(x_{n-1}, \xi_n, x_n)(1, x_{n+1})^T. \quad (7)$$

The existence of the tiling, together with the formula, is from [5]. The uniqueness and the tameness follow from [7] (uniqueness was proved in [5] under some extra assumption on K). Note that $\xi_1 = y$ and $\xi_{n+1} = y$, by definition of the word associated to P.

For later use, we introduce the notation

$$M(x_1\xi_2x_2\cdots x_{n-1}\xi_nx_n) = M(x_1, \xi_2, x_2)\cdots M(x_{n-1}, \xi_n, x_n).$$

As a particular case of the previous construction, consider a frontier having period n; this means that it can be written in the form ${}^\infty(x_1\xi_1 \dots x_n\xi_n)^\infty$. Then the associated tiling has the period determined by the vector $(p, -q)$, where p (resp. q) is the number of x's (resp. of y's) among $\xi_1, \dots, \xi_n$. Note that $p+q=n$. Moreover, the sequence of the linearization coefficients of the columns of the tiling has the period p, and the sequence of the linearization coefficients of the rows has the period q. This follows from the fact that the tiling is invariant under translation by the vector $(p,-q)$ since the frontier has this property.

3 Case $\tilde{A}$

We call $\tilde{A}_{n-1}$-*quiver* an acyclic quiver of type $\tilde{A}$ with n vertices; that is, an acyclic quiver such that the underlying undirected graph is an n-gon.

3.1 The mutated quivers

The description of the mutated quivers of type $\tilde{A}$ are known, see [6] (see for example Figure 2) or [2]. Our description below is equivalent to it.

A *decorated* $\tilde{A}$-*quiver* is a *planar* quiver G with set of vertices $\{1,\ldots,n\}$, obtained from its subgraph Q, which is a $\tilde{A}_r$-quiver, $1 \le r \le n-1$, called the *cyclic part of* G, as follows: there is a subset of the set of arrows of Q such that to each arrow $x \to y$ in this subset is associated a pre-continuant tree, which is connected to Q by its root r via the two arrows $y \to r \to x$.

See Figure 5 for an example of a decorated $\tilde{A}$-quiver: the arrows of Q are boldfaced.

We verify that the mutation at vertex j of G yields also a decorated $\tilde{A}$-quiver; we denote it by G', with its cyclic part Q'.

a) Suppose first that j is a vertex in Q. Let i,k be the vertices adjacent to j in Q. If in Q these three vertices form a path of length 2, $i \to j \to k$ say, then Q' is obtained by suppressing j in Q and replacing these two arrows by an arrow $i \to k$; the pre-continuant tree corresponding to the new arrow $i \to k$ of Q' is obtained by taking the new root j, and putting the tree of $i \to j$ as the left subtree and the tree of $j \to k$ as the right subtree;

b) If in Q one has $i \to j \leftarrow k$ or $i \leftarrow j \to k$, then these two arrows are reversed in Q'; the corresponding pre-continuant trees are exchanged: more precisely, if there is a pre-continuant tree with root l such that $i \leftarrow l \leftarrow j$ (or $i \to l \to j$), then, after mutation, it becomes so that we have $j \leftarrow l \leftarrow k$ (or $j \to l \to k$);

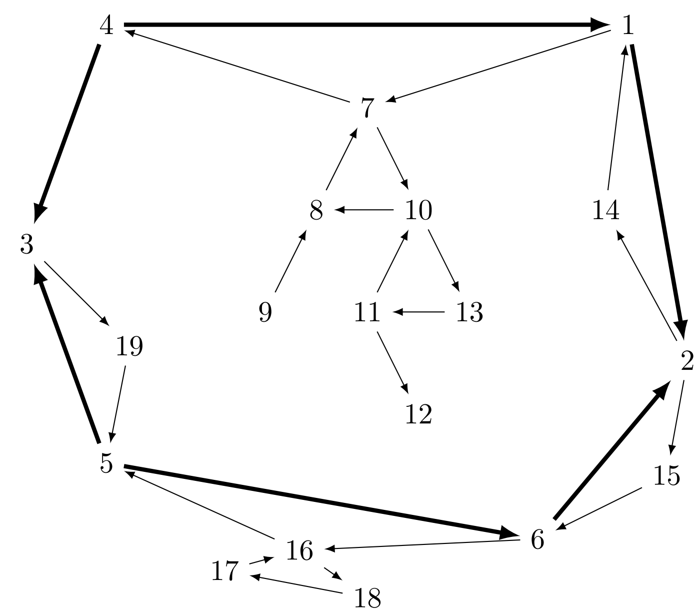

Figure 5: A decorated $\tilde{A}$-quiver with $n = 19$ vertices

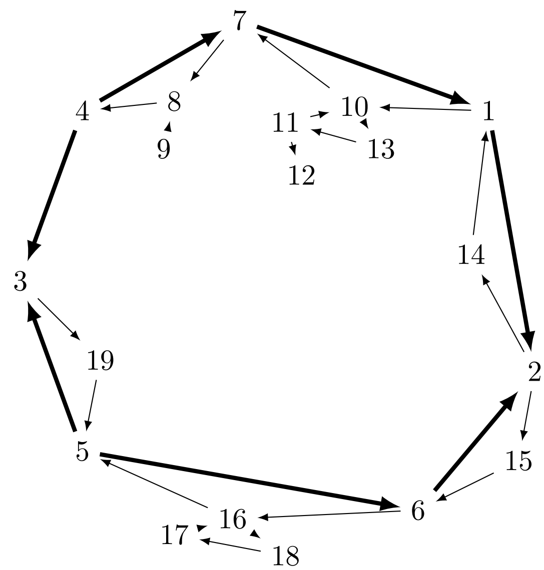

Figure 6: Mutation at 7

c) Suppose now that j is a vertex in $G \setminus Q$; that is, j is a vertex on one of the pre-continuant trees, T say. If j is not the root of T, we apply the construction of Subsection 2.3;

d) If j is a root, then denote by i, k the adjacent vertices in Q, with $i \leftarrow j \leftarrow k$. Note that we have $i \to k$ in Q. Then Q is replaced by Q', which has the new vertex j, with new arrows $i \to j \to k$, and by suppressing the arrow $i \to k$. The pre-continuant tree corresponding to the arrow $i \to j$ (resp. $j \to k$) in Q' is the left (resp. right) subtree of T.

Note that the mutations of type a) are inverse of those of type d), and that the inverses of mutations of type b) (or c)) are of the same type. As an example of case d), see Figure 6 which is obtained by mutation at vertex 7 of Figure 5. Case a) is obtained by reversing this mutation. An example of case b) is seen in Figure 7, which is obtained by mutation at vertex 5 of Figure 5.

3.2 Elementary properties of decorated $\tilde{A}$-quivers

Consider a decorated $\tilde{A}$-quiver with n vertices and cyclic part Q. We associate to each arrow of Q a natural positive number that we call its *length*:

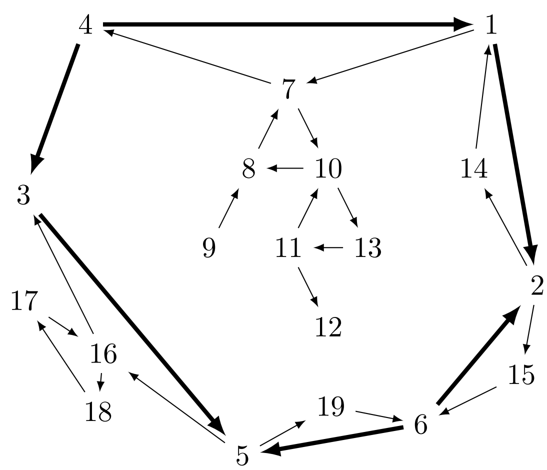

Figure 7: Mutation at 5

it is 1 if this arrow has no associated pre-continuant tree, and it is l if this arrow has a pre-continuant tree with $l-1$ vertices. For example, in Figure 5, the arrow $4 \to 3$ has length 1 and the arrow $4 \to 1$ has length 8.

For later use, we need to consider other points than the vertices in G; we call them *points*: a point is either a vertex of Q, or a point on an arrow of Q such that its distance to the vertices of this arrow is an integer, considering the previous definition of length. For example the arrow $4 \to 1$ on the Figure 5 has 7 points on it, together with its two vertices.

We denote by p (resp. q) the sum of the lengths of the clockwise oriented (counter-clockwise) arrows. Then $p+q=n$. It is straightforward to verify that p and q are invariant under mutation. We call p, q the *parameters* of G.

Note also that, as it was observed in Subsection 2.2, that each pre-continuant tree of G will become naturally a continuant tree once we associate to the corresponding arrow of Q, of length l, a sequence of length $l-1$ on the ring R. This will be done in the next subsection.

For this we fix a planar $\tilde{A}_{n-1}$-quiver, denoted by G_0 and called the *initial quiver*, with set of vertices $\{1, \ldots, n\}$ and arrows of the form $i \to i+1$ (in which case we let $\xi_i = x$) or $i \leftarrow i+1$ (in which case we let $\xi_i = y$), with $i+1$ taken $\mod n$; moreover, there are p clockwise oriented arrows and q

counterclockwise oriented arrows, $p+q=n$. Note that this is a particular decorated $\tilde{A}$-quiver, with no attached pre-continuant trees, and therefore with all arrows of length 1; its parameters are p,q.

We associate to G_0 the SL_2-tiling t as in [5]. In other words we consider the frontier $^{\infty}(x_1\xi_1\dots x_n\xi_n)^{\infty}$ where x_i is the *initial variable* attached to vertex i.

Note that this tiling t is periodic modulo the vector $(p,-q)$ and that the sequence of the linearization coefficients of the columns (resp. rows) of t is periodic of period p (resp. q). See Subsection 2.6.

Given two points on the same horizontal line in $\mathbb{Z}^2$, call *linearization sequence between them* the word (the finite sequence) formed by the column linearization coefficients of t for the columns lying strictly between them, scanning in increasing order of the x-coordinate. Similarly, for two points lying on the same vertical line.

3.3 Embedding of a decorated $\tilde{A}$-quiver into $\mathbb{Z}^2$

An *embedding* of a decorated $\tilde{A}$-quiver G into $\mathbb{Z}^2$ is a universal covering of its cyclic part Q, contained in the Euclidean plane, which respects the length of arrows, and which respects the orientation; that is, in such a way that clockwise (resp. counter-clockwise) oriented arrows of length l of Q correspond to horizontal (resp. vertical) segments of the form $[(u,v),(u+l,v)]$ (resp. $[(u,v),(u,v+l)]$) with u,v integers.

We denote by $\pi(i)$ the set of points in $\mathbb{Z}^2$ which correspond to the vertex $i\in G$. This set of points is by construction of the form $A+\mathbb{Z}(p,-q)$, for some $A\in\mathbb{Z}^2$.

For example, Figure 8 shows an embedding of the quiver of Figure 5. In this figure, we have represented the pre-continuant trees of G, which however are not formally part of the embedding and which are represented for a better understanding.

Now, we see that in an embedded decorated $\tilde{A}$-quiver G, all pre-continuant trees of G become continuant trees: indeed, to each clockwise (resp. counter-clockwise) arrow, associate to it the sequence of column (resp. row) linearization coefficients between A and B, where $A\to B$ is one of the corresponding arrows in the embedding. There are infinitely many arrows, but by periodicity, this is well-defined. We call this sequence the *linearization sequence of the arrow*; it depends on the embedding. For later use, note that this dependence is modulo the subgroup of $\mathbb{Z}^2$ generated by the vectors $(p,0)$ and $(0,q)$; indeed, the column (resp. row) linearization coefficients have period p (resp. q). Hence, if we translate correspondingly the embedding, the

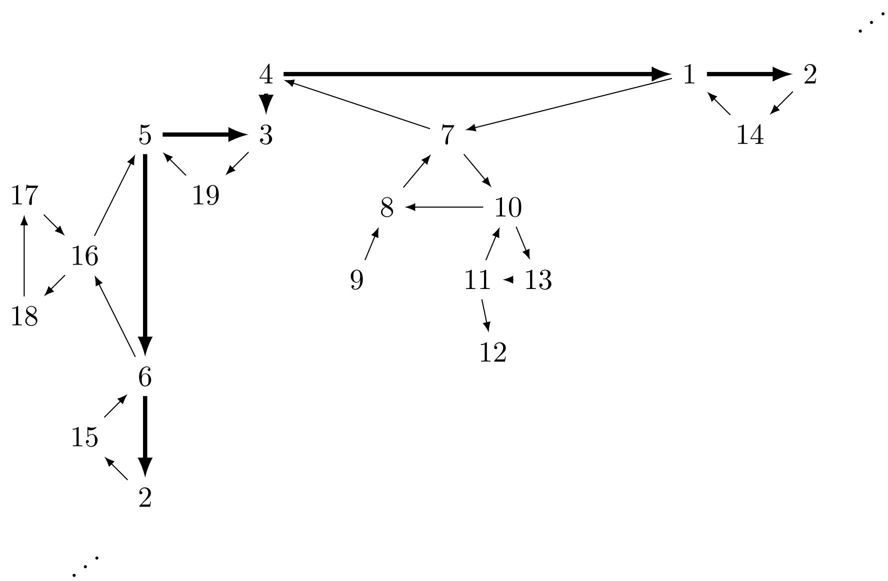

Figure 8: Embedding of Figure 5

linearization sequences of arrows do not change.

Recall the definition of points of G given in Subsection 3.2. Clearly, the mapping π may be extended to the arrows of Q, hence to the points of G, by respecting the length.

An embedding π of G being given there is a unique point $P = P(G, \pi)$ in $\mathbb{Z}^2$ defined as follows: it is the intersection of the bi-infinite path defined by π and the x-axis, and which has the smallest x-coordinate (the intersection may be a segment). We denote by $\xi(G, \pi)$ the x-coordinate of P. Note that P corresponds to a unique point on the cyclic part Q of G, that we denote by $u(G, \pi)$.

Lemma 3.1. *Let u be a point on G. Then for each vertex k on the cyclic part of G,there is a vector $(i, j) \in \mathbb{Z}^2$ such that $\pi(k) = \pi(u)+(i, j)+\mathbb{Z}(p, -q)$ for any embedding π of G.*

Proof. This follows because the covering respects the lengths of arrows and their orientations. $\square$

We now describe how the embeddings are modified by mutations. We refer to the cases a) to d) in Subsection 3.1:

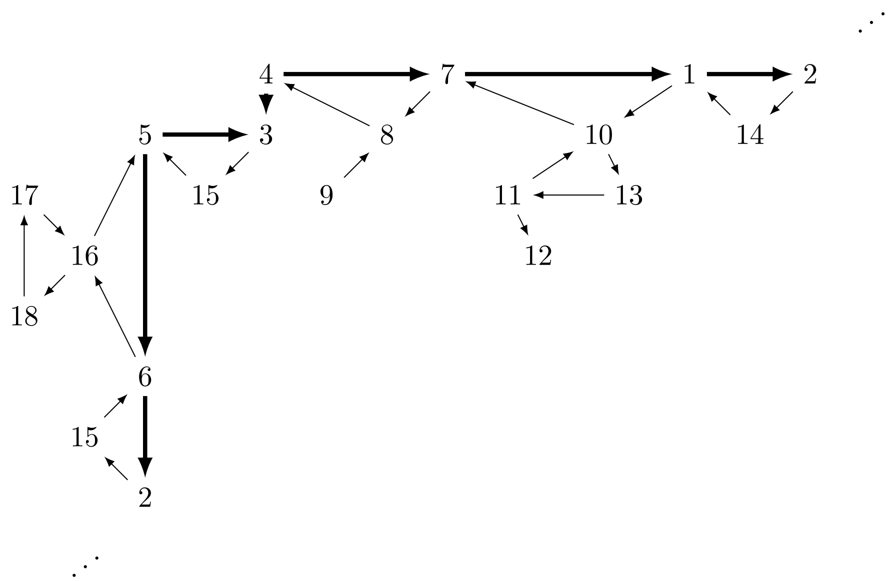

Figure 9: Mutation at 7: embedding

a) the new embedding is obtained by suppressing the vertices corresponding to j in the covering and by gluing the arrows $i \to j$ and $j \to k$ into a unique one $i \to k$;

b) in the embedding of G, the points I, J, K corresponding to i, j, k form three consecutive vertices of a rectangle; then J is replaced by the fourth point;

c) the new embedding is the same as the old one;

d) this is the reversal of case a).

As examples, see Figure 9, which is the embedding of Figure 6 and which is obtained from Figure 8 by a type d) mutation on vertex 7; and Figure 10, which is the embedding of Figure 7 and which is obtained from Figure 8 by a type b) mutation on vertex 5.

Lemma 3.2. *Given a decorated $\tilde{A}$-quiver with a distinguished point u, one may define the mutation at vertex j of the couple (G, u) in such a way that the mutated couple (G', u') satisfies: for each embedding π of G with $u(G, \pi) = u$, mutated at vertex j into (G', π'), one has $u(G', \pi') = u'$.*

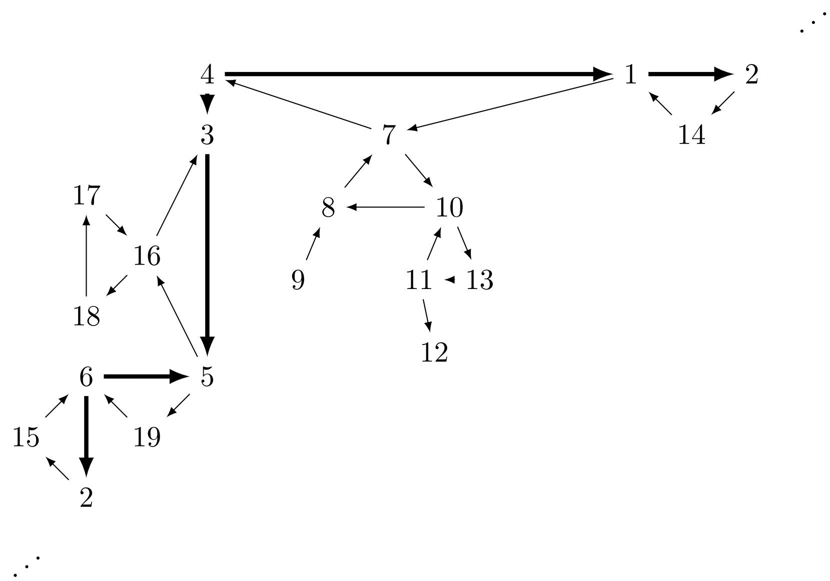

Figure 10: Mutation at 5: embedding

In other words, the distinguished point of G corresponding to an embedding π is mutated independently of π itself.

Proof. This is verified as follows: the point $u(G, \pi)$ is invariant except if the mutation is of type b) and if the point $P = P(G, \pi)$ is on the counterclockwise arrow a of Q (the cyclic part of G) incident to j; in this case the x-coordinate $\xi(G, \pi)$ of P is increased or decreased by the length of the clockwise arrow incident to j, depending if j is the tail or the head of these two arrows; moreover, if u is at distance l_1 of j, with $l = l_1 + l_2$ equal to the length of the arrow a, then u' is on the counter-clockwise arrow incident to j in the mutated graph G', at distance l_2 of j. $\square$

See for example Figure 8 and Figure 10 (mutation at 5 of Figure 8), with P on the arrow $5 \to 6$ in Figure 8 (thus the x-axis intersects this arrow); then after mutation, P is in Figure 10 on the arrow $3 \to 5$.

3.4 The mutated seeds in type $\tilde{A}$

Using the notations of Subsection 3.2, we start with the *initial quiver* G_0 and we let t be the corresponding SL_2-tiling. Recall that G_0 has p clockwise arrows and q counter-clockwise arrows, with $p + q = n$, which is the number of vertices of G_0. We let S_0 denote the *initial seed*, that is, $S_0 = (G_0, \{x_1, \dots, x_n\})$.

Theorem 3.3. *Each seed $S = (G, \{y_1, \dots, y_n\})$ in the mutation class of S_0 is obtained as follows: for some decorated $\tilde{A}$-quiver G with parameters p, q and some embedding π of G, the i-th cluster variable y_i of S is:*

- *if i is on the cyclic part of G, then y_i is equal to $t(\pi(i))$, where t is the SL_2-tiling associated to S_0;*
- *if i is a vertex of one of the continuant trees of G, then y_i is equal to the variable associated to this vertex.*

In order to understand this result, note that the tiling t is periodic modulo the vector $(p, -q)$, since so is its frontier; and the set $\pi(i)$ is of the form $A + \mathbb{Z}(p, -q)$ for some point A, by definition of an embedding; hence $t(\pi(i))$ is well-defined. Moreover, as seen in Subsection 3.2, each pre-continuant tree of G becomes naturally a continuant tree, once an embedding of G is given.

To illustrate the theorem, consider Figure 8: the vertices of G that appear as points in the covering (that is, 2,6,5,3,4,1) have as variables the images under t of these points. The other ones correspond to the second case in the theorem: for example, let a, b, c, d, e, f, g be the sequence of column linearization coefficients corresponding to the seven columns lying strictly between the columns of 4 and that of 1; then the vertex 7 gets the variable $q(abcdefg)$, the vertex 10 gets $q(defg)$, 8 gets $q(ab)$ and 12 gets $q(e)$ (compare with Figure 1).

Proof. It is enough to prove that: (i) the initial seed is obtained in this way and that: (ii) the statement is compatible with mutations. For (i), this follows by choosing the initial embedding π_0 in such a way that the corresponding covering of $G_0 = Q_0$ (G_0 is equal to its cyclic part; there are no pre-continuant trees in G_0) is the frontier of the tiling. The latter has been constructed in such a way that this is possible.

We prove now (ii). Suppose that the mutation at j is of type d). By symmetry, we may assume that $i \to k$ is a clockwise oriented arrow of Q. Let A, C denote two points in the embedding π on the same horizontal which correspond to i and k respectively (the reader may use Figure 8 with $i = 4$, $j = 7$, $k = 1$, and A, C corresponding to 4 and 1). Let $\alpha_1, \dots, \alpha_r, \beta, \gamma_1, \dots, \gamma_s$ be the column linearization coefficients for the columns stricly lying between A and C, with r (resp. s) equal to the number of vertices of the left (resp. right) subtree of the continuant tree corresponding to this arrow (in the figure $r = 2$, $s = 4$); denote by a (resp. b) the roots of these subtrees (in the figure, a, b correspond to 8,10). Then the variables at i, k, j, a, b are respectively: $t(A)$, $t(C)$, $q(\alpha_1, \ldots, \alpha_r, \beta, \gamma_1, \ldots, \gamma_s)$, $q(\alpha_1, \ldots, \alpha_r)$, $q(\gamma_1, \ldots, \gamma_s)$. Moreover the arrows incident to j are $j \to i$, $j \to b$, $k \to j$, $a \to j$. After mutation, j is on the cyclic part Q' part of the mutated quiver G' and corresponds to a point B in the new embedding located between A and C at distance r of A (see Figure 8, with B corresponding to 7); hence the mutated variable at j is $t(B)$. Thus we must verify that

$$\begin{aligned} & t(B)q(\alpha_1, \ldots, \alpha_r, \beta, \gamma_1, \ldots, \gamma_s) \\ &= t(A)q(\gamma_1, \ldots, \gamma_s) + t(C)q(\alpha_1, \ldots, \alpha_r). \end{aligned}$$

This is Lemma 2.5.

Suppose now that the mutation is of type b). We may by symmetry assume that the arrows are of the form $j \to k$ and $j \to i$. Let A, B, C be three consecutive points in the embedding π corresponding to i, j, k. See Figure 8 with i, j, k equal to $6, 5, 3$ and A, B, C the corresponding points. Denote by $\alpha_1, \ldots, \alpha_r$ (resp. $\beta_1, \ldots, \beta_s$) the linearization coefficients of the columns (resp. rows) stricly lying between B and C (resp. B and A). Let D be the fourth point of the rectangle on A, B, C (in Figure 10, D corresponds to 5). Let a (resp. b) be the root of the tree of the arrow $j \to k$) (resp. $j \to i$) (in Figure 8, a, b correspond to 19, 16). Then we have the arrows $a \to j$ and $b \to j$. The variables in G at i, j, k, a, b are respectively $t(A), t(B), t(C)$, $q(\alpha_1, \ldots, \alpha_r)$, $q(\beta_1, \ldots, \beta_r)$; after mutation, the variable at j becomes $t(D)$. Thus we have to verify that

$$t(B)t(D) = t(A)t(C) + q(\alpha_1, \ldots, \alpha_r)q(\beta_1, \ldots, \beta_s).$$

This is Corollary 2.6.

A type a) mutation is the reverse of a type d) mutation and is treated similarly. The case of a type c) mutation follows from Lemma 2.4. $\square$

3.5 Transjective/Nontransjective variables

It follows from the previous theorem that the cluster variables either appear as elements of the SL_2-tiling, or as continuant polynomials of the linearization coefficients of the tiling; actually, only finitely many of them are of the latter form, since the pre-continuant trees appearing on decorated $\tilde{A}$-quivers are in finite number and since the sequences of linearization coefficients of the tiling are periodic. We shall see below that these two cases are mutually exclusive.

Let Q be a finite acyclic quiver and K an algebraically closed field. We denote by KQ the path algebra of Q, by $modKQ$ the category of finitely generated right KQ-modules and by $\mathcal{D}^b\left(modKQ\right)$ the bounded derived category over $modKQ$. Let τ denote the Auslander-Reiten translation and [1] the shift in $\mathcal{D}^b\left(modKQ\right)$. The *cluster category* $\mathcal{C}_Q$ of Q is defined to be the orbit category of $\mathcal{D}^b\left(modKQ\right)$ under the action of the automorphism $\tau^{-1}[1]$, see [9]. The Auslander-Reiten quiver $\Gamma\left(\mathcal{C}_Q\right)$ of $\mathcal{C}_Q$ has a unique component containing all the objects in $KQ[1]$, that is, the shifts of the indecomposable projective $KQ-$modules. This component is the *transjective component* Γ_{tr} of $\Gamma\left(\mathcal{C}_Q\right)$ and its objects are called *transjective*. If Q is a Dynkin quiver, then $\Gamma\left(\mathcal{C}_Q\right) = \Gamma_{tr}$. Otherwise, Γ_{tr} is isomorphic to the repetition quiver $\mathbb{Z}Q$ of Q, and $\Gamma\left(\mathcal{C}_Q\right)$ has infinitely many additional, so-called *regular*, components which are either stable tubes (if Q is euclidean), or of type $\mathbb{Z}\mathbb{A}_\infty$ (if Q is wild). Now, it is shown in [13] that there exists a bijection $X_?$ (called *canonical cluster character*) between the isomorphism classes of indecomposable objects M in $\mathcal{C}_Q$ which have no self-extensions and the cluster variables X_M. A cluster variable which is the image of a transjective object in $\mathcal{C}_Q$ under the canonical cluster character is called *transjective*. The others will be called *nontransjective*.

Lemma 3.4. *Let Q be a quiver of type $\tilde{A}$ with p clockwise oriented arrows and q counterclockwise oriented arrows. Then there are exactly $p(p-1)+q(q-1)$ nontransjective cluster variables.*

Proof. According to the description of the cluster category $\mathcal{C}_Q$, the nontransjective cluster variables are in bijection with the regular indecomposable objects in $\mathcal{C}_Q$ without self-extensions lying in the stable tubes. Now such an indecomposable lies necessarily in one of the two exceptional tubes of ranks p and q. Using the fact that the tubes are standard, it is easily seen that the tube of rank p (resp. q) contains exactly $p(p-1)$ (resp. $q(q-1)$) objects without self-extensions. $\square$

Theorem 3.5. *The transjective cluster variables are exactly those appearing on the SL_2-tiling. The nontransjective variables are exactly those appearing on the continuant trees of the decorated $\tilde{A}$-graphs.*

Proof. It is known [4] that the transjective variables are exactly those that are obtained by mutating only on sources or on sinks of the quivers. Now, as already observed in [5], the variables appearing on the SL_2-tiling are obtained by this kind of mutations, hence are all transjective. Thus the nontransjective variables all appear on the continuant trees. Hence it suffices to show that the number of variables on the continuant trees is at most $p(p-1)+q(q-1)$. Now, the sequence of column linearization coefficients is periodic of period p; and the variables on the continuant trees corresponding to columns are of the form $q(\alpha_1,\ldots,\alpha_k)$ with $1 \leq k \leq p-1$, since p is the maximum length of a clockwise oriented arrow in the cyclic part of a decorated $\tilde{A}$-quiver in the mutation class of G_0, because it has necessarily parameters p,q; hence there are at most $p(p-1)$ such variables. This ends the proof, since the case of rows is symmetric. □

As in the Introduction, let M denote the group generated by the set $\{\mu_1,\ldots,\mu_n\}$ of mutations, subject to the relations $\mu_i^2=1$.

Theorem 3.6. *Let $i \in \{1,\ldots,n\}$. Let y be a nontransjective cluster variable. The set of $m \in M$ such that $y_i(S_0^m)=y$ is a union of cosets of a subgroup of finite index of M.*

Proof. Consider the set of decorated $\tilde{A}$-quivers in the mutation class of G_0, with embedding considered modulo the subgroup H of $\mathbb{Z}^2$ generated by the vectors $(p,0)$ and $(0,q)$. This set is finite. By a remark made in Subsection 3.3, the continuant trees attached to G using π depend only on $\pi \mod H$. Now M acts on this finite set. Moreover, $y_i(S_0^m)=y$ is equivalent to the fact that $G_0.m=G$ satisfies: i is a vertex of one of the continuant trees of G and to this vertex is associated the variable y. □

Observe that the transjective variables are given, following [5], by Formula (7) in Theorem 2.7; this formula gives at the same time positivity and the Laurent phenomenon. We conclude this subsection by giving a similar formula for nontransjective variables. We limit ourselves to the case where these variables are obtained as continuant polynomials of column linearization coefficients, the case of rows being symmetric.

Given a finite set of consecutive columns of the SL_2-tiling associated to a frontier, we call *word* of this set the word that codes the intersection of the frontier with this set of columns, augmented with the first step to its left and the first to its right; note that these steps are both horizontal. For example the word of the set of columns containing the variables from x_0 to x_4 in Figure 4 is

$$x_{-4}xx_{-3}yx_{-2}yx_{-1}yx_0xx_1xx_2yx_3xx_4xx_5.$$

Theorem 3.7. *Consider an SL_2-tiling t associated to some frontier with variables in K. Let $C_1, \dots, C_k$ be k successive columns of t, with linearization coefficients $\alpha_1, \dots, \alpha_k$. Let $w = x_0\xi_1 \cdots \xi_n x_{n+1}$ be the word associated to this set of columns. Then the continuant polynomial $q(\alpha_1, \dots, \alpha_k)$ is equal to*

$$\frac{1}{x_1 x_2 \dots x_n}(x_0, 1)M(x_1, \xi_2, x_2) \cdots M(x_{n-1}, \xi_{n-1}, x_n)(1, x_{n+1})^T.$$

In order to prove this theorem, we need the following lemma.

Lemma 3.8. *Let $x_1, \dots, x_n, y_1, \dots, y_p$ be nonzero elements of K. Then*

$$\frac{1}{x_1 \cdots x_n y_1 \cdots y_p} M(x_1 y x_2 \cdots y x_n x y_1 y y_2 \cdots y y_p)$$

$$= \frac{1}{x_1 \cdots x_n} M(x_1 y x_2 \cdots y x_n) \begin{pmatrix} 1 \\ y_1 \end{pmatrix} \frac{1}{y_1 \cdots y_p}(x_n, 1) M(y_1 y y_2 \cdots y y_p) - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Proof. We have

$$M(x_1 y x_2 \cdots y x_n x y_1 y y_2 \cdots y y_p) = M(x_1 y x_2 \cdots y x_n) M(x_n x y_1) M(y_1 y y_2 \cdots y y_p).$$

Now

$$M(x_n x y_1) = \begin{pmatrix} x_n & 1 \\ 0 & y_1 \end{pmatrix} = \begin{pmatrix} x_n & 1 \\ x_n y_1 & y_1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ x_n y_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ y_1 \end{pmatrix} (x_n, 1) - x_n y_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Moreover

$$M(x_1 y x_2 \cdots y x_n) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M(y_1 y y_2 \cdots y y_p)$$

$$= \begin{pmatrix} x_2 & 0 \\ 1 & x_1 \end{pmatrix} \cdots \begin{pmatrix} x_n & 0 \\ 1 & x_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_2 & 0 \\ 1 & y_1 \end{pmatrix} \cdots \begin{pmatrix} y_p & 0 \\ 1 & y_{p-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ x_1 \cdots x_{n-1} y_2 \cdots y_p & 0 \end{pmatrix}.$$

Putting all this together, we obtain the lemma. □

Proof. (Theorem 3.7) Consider first the case $k = 1$, that is, there is only the column C_1. Then

$$w = x_0 x x_1 y x_2 \dots y x_n x x_{n+1}.$$

Note that x_1 is in column C_1. Let P be the point in the discrete plane at the right of the point on the frontier which is labelled by the variable x_1. Then the linearization coefficient α_1 of column C_1 is equal to $\frac{x_0+t(P)}{x_1}$. Now, the word associated to P (as in Subsection 2.6) is $x_1yx_2\dots yx_nxx_{n+1}$. Thus by Th. 2.7, we have

$$t(P) = \frac{1}{x_2\cdots x_n}(1, x_1)M(x_2, y, x_3)\cdots M(x_{n-1}, y, x_n)(1, x_{n+1})^T.$$

Let

$$A = \begin{pmatrix} b \\ d \end{pmatrix} = M(x_1, y, x_2)\cdots M(x_{n-1}, y, x_n)(1, x_{n+1})^T.$$

Then, since the second row of $M(x_1, y, x_2)$ is $(1, x_1)$, we have $t(P) = \frac{d}{x_2\cdots x_n}$. Since $M(x_1, y, x_2)\cdots M(x_{n-1}, y, x_n)$ is a product of lower triangular matrices, its (1,1)-entry is by definition of $M(a, y, b)$ equal to $x_2\cdots x_n$. Thus $b = x_2\cdots x_n$. Now $(x_0, 1)A$ is equal to $x_0b + d$. Divided by $x_1\cdots x_n$, this gives

$$\frac{x_0b + d}{x_1\cdots x_n} = \frac{x_0x_2\cdots x_n + d}{x_1\cdots x_n} = \frac{x_0 + \frac{d}{x_2\cdots x_n}}{x_1} = \alpha_1.$$

This proves the result for $k = 1$.

We now verify that the expression $r_k = r(\alpha_1, \dots, \alpha_k)$ in the theorem satisfies the recursion (2), first for $k = 2$ then for $k \geq 3$. This will end the proof.

Let C_1, C_2 be two successive columns, with α_1, α_2 as respective linearization coefficients. Then the words associated to the sets of columns $\{C_1\}$, $\{C_2\}$ and $\{C_1, C_2\}$ are respectively

$$x_0xx_1yx_2\cdots yx_nxy_1, x_nxy_1yy_2\cdots yy_pxz, x_0xx_1yx_2\cdots yx_nxy_1yy_2\cdots yy_pxz$$

for some integers n, p. Thus $r_1 = (1/x_1\cdots x_n)(x_0, 1)M(x_1yx_2\cdots x_n)(1, y_1)^T$ and $r_2 = (1/x_1\cdots x_ny_1\cdots y_p)(x_0, 1)M(x_1yx_2\cdots x_nxy_1\cdots y_p)(1, z)^T$. Now multiply the identity of the lemma at the left by $(x_0, 1)$ and at the right by $(1, z)^T$. We obtain r_2 at the left-hand side. On the right-hand side we have $r_1\alpha_2 - 1$, since by the first part of the proof

$$\alpha_2 = \frac{1}{y_1\cdots y_p}(x_n, 1)M(y_1yy_2\cdots yy_p)(1, z)^T$$

and since

$$(x_0, 1)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}(1, z)^T = 1.$$

Thus $r_2 = r_1\alpha_2 - 1$.

Let $C_1, C_2, \ldots, C_k$ be k successive columns. Then the words associated to the three sets of columns $\{C_1, \ldots, C_{k-2}\}$, $\{C_1, \ldots, C_{k-1}\}$, $\{C_1, \ldots, C_k\}$ are respectively of the form

$$wxx_1, wxx_1yx_2 \cdots yx_nxy_1, wxx_1yx_2 \cdots yx_nxy_1yy_2 \cdots yy_pxz,$$

for some word w and some natural integers n, p. Moreover, the word associated to $\{C_k\}$ is

$$x_nxy_1yy_2 \cdots yy_pxz.$$

Thus

$$r_{k-2} = (1/X)(x_0, 1)M(w)(1, x_1)^T,$$
$$r_{k-1} = (1/Xx_1 \cdots x_n)(x_0, 1)M(wx_1yx_2 \cdots x_n)(1, y_1)^T$$

and

$$r_k = (1/Xx_1 \cdots x_ny_1 \cdots y_p)(x_0, 1)M(wx_1yx_2 \cdots x_nxy_1 \cdots y_p)(1, z)^T,$$

where x_0 is the first letter of w and where X is the product of all the variables in w, except x_0. We multiply the identity of the lemma at the left by $(1/X)(x_0, 1)M(wxx_1)$ and at the right by $(1, z)^T$. We obtain an identity whose left-hand side is r_k; its right-hand side is equal to $r_{k-1}\alpha_k - r_{k-2}$, since by the first part of the proof

$$\alpha_k = \frac{1}{y_1 \cdots y_p}(x_n, 1)M(y_1yy_2 \cdots yy_p)(1, z)^T,$$

because $M(wxx_1) = M(w)M(z_0xx_1)$ (where z_0 is the last letter of w), and since

$$\begin{pmatrix} z_0 & 1 \\ 0 & x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (1, z)^T = (1, x_1)^T.$$

□

3.6 A linear representation of the mutation group

Recall that M denotes the group of mutations. We have a right action of M on the finite set $\mathcal{G}$ of decorated $\tilde{A}$-graphs with n vertices and parameters p, q; this action is defined on the generators (the mutations) in Subsection 3.1. We denote by $G.m$ this action, for $G \in \mathcal{G}$ and $m \in M$. Likewise (see Subsection 3.3), M acts on the right on the set of couples (G, π), where π is an embedding of G into $\mathbb{Z}^2$. Note that, by definition of the mutations, we have a compatibility condition between both actions: $(G, \pi).m = (G.m, \pi')$.

Denote by $\mathcal{G}_0$ the finite set of couples (G,u), $G \in \mathcal{G}$, where u is a point of G. There is a natural action of M on the couples $(G,u) \in \mathcal{G}_0$, that we denote by $(G,u).m$ for $m \in M$. This action has been defined on the generators at the end of Subsection 3.3.

To G and π as above, we have associated in Subsection 3.3 a point $u = u(G,\pi)$ of G. The following lemma is a consequence of an observation at the end of Subsection 3.3.

Lemma 3.9. *For any $m \in M$, one has $u((G,\pi).m) = (G,u).m$.*

We now define a function $\delta : \mathcal{G}_0 \times M \to \mathbb{Z}[x,x^{-1}]$, where M is the group of mutations. This mapping is defined as follows: let $(G,u) \in \mathcal{G}_0$, $m \in M$; let π be some embedding of $G \in \mathcal{G}$ into $\mathbb{Z}^2$ such that $u = u(G,\pi)$; and define $(G',\pi') = (G,\pi).m$; let $i = \xi(G',\pi') - \xi(G,\pi)$; then $\delta((G,u),m)$ is the Laurent monomial x^i. This is well-defined, that is, does not depend on the chosen embedding π satisfying $u = u(G,\pi)$.

From this construction follows

Lemma 3.10. *One has for any $m, m' \in M$,*

$$\delta((G,u),mm') = \delta((G,u),m)\delta((G,u).m,m').$$

We can now define a linear representation of the group of mutations.

Lemma 3.11. *For $m \in M$, define a matrix $\mu(m)$, indexed by $\mathcal{G}_0$ as follows: for any $(G,u) \in \mathcal{G}_0$, the $((G,u),(G,u).m)$-entry is equal to $\delta((G,u),m)$. The other entries are 0. Then μ is a homomorphism from M into $GL_N(\mathbb{Z}[x,x^{-1}])$, where N is the cardinality of $\mathcal{G}_0$.*

Proof. The matrix $\mu(m)$ has exactly one nonzero entry in each row and each column, and this entry is a Laurent monomial. Hence it is in $GL_N(\mathbb{Z}[x,x^{-1}])$. The fact that it is a homomorphism follows from the above equation. $\square$

Theorem 3.12. *Let $k \in \{1,\ldots,n\}$. Then the function*

$$M \to R = \mathbb{Z}[x_1^{\pm 1},\ldots,x_n^{\pm 1}], m \mapsto y_k(S_0^m)$$

is a representative function of the group of mutations M with values in R.

Proof. We define the initial embedding π_0 of the initial seed S_0 as at the beginning of the proof of Th. 3.3. Let $u_0 = u(G,\pi)$. Define $\lambda \in R^{1\times\mathcal{G}_0}$ by $\lambda_{(Q_0,u_0)} = x^{i_0}$, where $i_0 = \xi(Q_0,\pi_0)$, while the other components of λ are 0. Define now $\gamma \in R^{\mathcal{G}_0\times 1}$ by: if k is not on the cyclic part of G, then $\gamma_{(G,u)} = 0$;

if k is on the cyclic part of G, choose some vector $(i,j) \in \mathbb{Z}^2$ such that for any embedding π of G, $\pi(k) = u(G,\pi) + (i,j) + \mathbb{Z}(p,-q)$ (see Lemma 3.1); then let $\gamma_{(G,u)} = x^i y^j$.

It is seen that then one has $\lambda\mu(m)\gamma = x^i y^j$ where $(Q_0, \pi_0).m = (G,\pi)$, with $\pi(k) = (i,j)+\mathbb{Z}(p,-q)$ if k is on the cyclic part of G, and $= 0$ otherwise. Call ϕ the representative function $\phi(m) = \lambda\mu(m)\gamma$ of M with value in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$.

Now, by Prop.5.1 and Prop.5.4, the composition $\psi = t \circ \phi$ is a representative function of M with value in R such that $\psi(m)$ is equal, by Th.3.3 to $y_k(S_0^m)$ if k lies in the cyclic part of $Q_0.m$, and $= 0$ otherwise. Moreover the set of $m \in M$ such that k does not lie in the cyclic part of $Q_0.m$ and has as associated variable a fixed variable y is by Th.3.6 a finite union of cosets of a normal subgroup of M; thus the theorem follows from Corollary 5.3. □

3.7 A noncommutative rational series

Consider the free monoid $\mathcal{M}$ generated by the set of mutations $\{\mu_1, \ldots, \mu_n\}$. In this monoid consider the subset L of words m that do not contain two successive occurrences of the same letter.

Theorem 3.13. *Let $k \in \{1, \ldots, n\}$. The series $\sum_{m \in L} y_k(S_0^m)m$ is rational over the ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.*

Proof. This follows from the Kleene-Schützenberger theorem, see [8], Th.7.1: a series is rational if and only it is a representative function of the monoid $\mathcal{M}$. It implies that the series $\sum y_k(S_0^m)m$, where the sum is over all elements m of $\mathcal{M}$, is rational over the ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Now, by Hadamard product with the language L (which is a rational language), the series of the theorem is rational (Cor. III.2.3 in [8]). □

In view of the positivity conjecture of [22] and the rationality over $\mathbb{N}$ of the sequences considered in [5], it is legitimate to ask if the series of the theorem is also rational over the semiring $\mathbb{N}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We show by a counterexample that this is not true in general.

Indeed, consider the $\tilde{A}_2$-quiver with the arrows $1 \to 2 \to 3$ and $1 \to 3$. The corresponding function δ is shown in Figure 11, with the following conventions: ijk denotes the decorated $\tilde{A}_2$-quiver with arrows $i \to j \to k$ and $i \to k$ with no attached pre-continuant tree; ik denotes the decorated $\tilde{A}_1$ quiver with a double arrow $i \to k$ and with a one node pre-continuant tree j attached by the arrows $i \leftarrow j \leftarrow k$. Note that, due to the special form of these quivers, the distinguished point $u(G,\pi)$ is always equal to i. On the figure, an arrow from vertex a to vertex b labelled μ_i/x^l means that $a.\mu_i = b$ (action of the mutations on the set $\mathcal{G}$) and $\delta(a, \mu_i) = x^l$ (we use here the formalism of input/output automata); we have represented only one half of the arrows: to the previous arrow is associated the reverse arrow $b \to a$ with label μ_i/x^{-l}.

All this allows to compute the function δ. For example, let $w(p, q) = (\mu_1\mu_2\mu_3)^p\mu_2\mu_1\mu_2(\mu_3\mu_2\mu_1)^q$. Then we have

$$123.w(p, q) = 321, \delta(123, w(p, q)) = x^{3p}x^2x^{3q} = x^{3p+2+3q}.$$

This implies that starting from the initial seed (123, $\{x_1, x_2, x_3\}$) and initial embedding π_0, and applying the sequence of mutations $w(p, q)$ gives the seed ((321, $\{y_1, y_2, y_3\}$), and the embedding π with $\pi(1) = \pi_0(1) + (3p + 3q, 0)$.

Suppose now that the series $S(x_1, x_2, x_3)$ of the theorem is, for $k = 1$, rational over the semiring $\mathbb{N}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then replacing each variable x_i by 1, we obtain a series $T = S(1, 1, 1)$ which is rational over the series $\mathbb{N}$. Since the set of $w(p, q)$, $p, q \in \mathbb{N}$ is a rational language, the series $W = \sum(T, w(p, q))w(p, q)$ is also rational over $\mathbb{N}$, see [8] Cor. III.2.3. Hence the set of words $w(p, q)$ whose coefficient in W is 1 must be a rational language, by [8] Cor. III.2.7. Now, the SL_2-tiling over $\mathbb{N}$ obtained by replacing the variables by 1 has 1's only on its frontier (compare with the SL_2-tiling give on p.3152 in [5]); moreover the only linearization coefficient which come into play here is 2. Hence this language is the set of words $w(p, p)$: this set is well-known to be not a rational language.

4 Case A_n

4.1 Mutation of a continuant tree at the root

We show that, under suitable hypothesis (which are satisfied in the case A_n), each continuant tree has at least two structures of continuant tree, if one changes the parameters.

Lemma 4.1. *Let $a_1, \dots, a_{n+3}$ be a sequence of elements of R such that*

$$Q(a_1)Q(a_2)\cdots Q(a_{n+3}) = -1.$$

Let G be a continuant tree with root labelled by the word $a_1 \cdots a_n$. Then for some k with $1 \leq k \leq n$, G has a leaf labelled k and there is a continuant tree G' isomorphic to G as labelled quiver, with the root of G' corresponding to k in G.

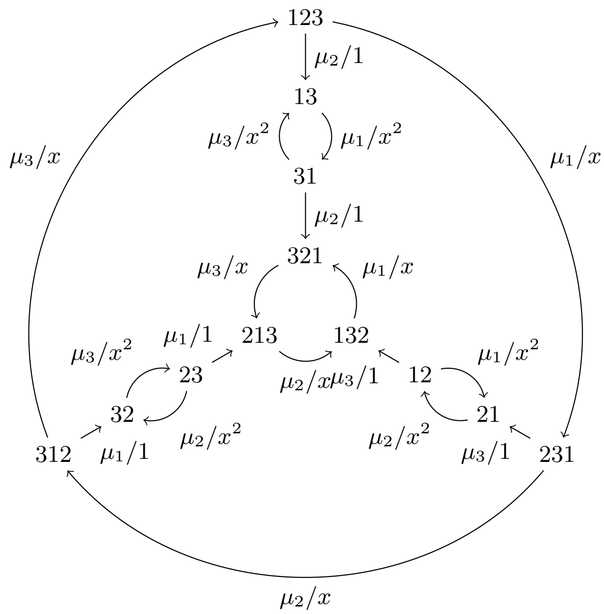

Figure 11: The function δ

The lemma is illustrated by Figure 12, which shows the same quiver as the one in Figure 1: at the left the nodes get new words, which give the same continuant polynomials thanks to lemma 2.3 (under the hypothesis $Q(abcdefghij) = -1$; for example, $q(i) = q(abcdefg)$, and also $q(defg) = q(ijab)$ since $Q(defghijabc) = -1$); at the right, this quiver is shown to be a continuant tree.

Proof. In order to prove the lemma, we associate to the continuant tree a triangulation of an $n+3$-gon whose vertices are labelled by $a_1, \dots, a_{n+3}$ in this order: to each node $a_i \dots a_j$ of the continuant tree, associate the diagonal joining the vertices a_{i-1} and a_{j+1}, with the indices taken modulo $n+3$; in particular, the root will give the diagonal from a_{n+1} to a_{n+3}. The construction is illustrated in Figure 13: this triangulation corresponds to the tree of Figure 1, but also to the tree of Figure 12, right part. Inspection of this example shows that for a given triangulation, one obtains a corresponding tree for each isolated vertex (that is, a vertex without incident diagonal), which corresponds to the root of that tree. Since each triangulation has at least two isolated vertices, the lemma follows if one takes into account the identity on continuant polynomials given by Lemma 2.3: indeed, the hypothesis

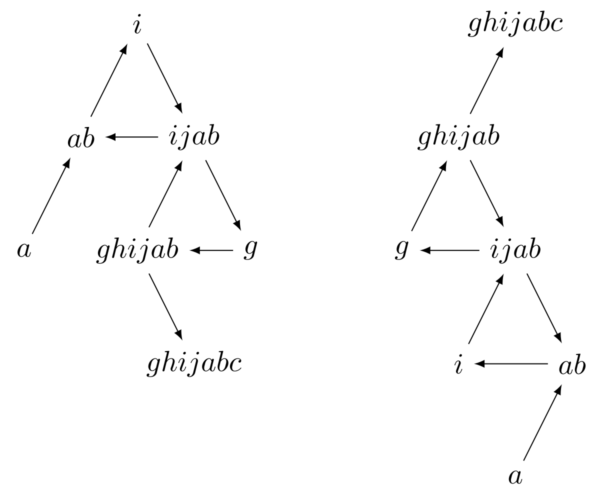

Figure 12: The tree of Figure 1 viewed differently

of this lemma implies that $Q(a_i)Q(a_{i+1})\cdots Q(a_{n+3})Q(a_1)\cdots Q(a_{i-1}) = -1$, so that $q(a_i \cdots a_j) = q(a_{j+2}\cdots a_{n+3}a_1\cdots a_{i-2})$. $\square$

4.2 The mutated seeds in type A

Consider the Dynkin diagram A_n, with vertex set $\{1,\dots n\}$, edge set $\{\{i,i+1\}, i=1,\dots,n-1\}$. We give to this diagram some orientation, obtaining the initial quiver Q_0, which determines the initial seed $S_0 = (Q_0, \{x_1,\dots,x_n\})$.

The following result is essentially due to Conway and Coxeter, see [15] (18) p. 91 and p. 177-178.

Theorem 4.2. *There exists a sequence $a_1,\dots,a_{n+3}$ of Laurent polynomials in the initial variables, with coefficients in $\mathbb{N}$, such that the equality in Lemma 2.3 holds and that each mutated seed is a continuant tree with root $(a_{i+1},\dots,a_{i+n})$, for some $i=1,\dots,n+3$, with indices taken* $\mod n+3$.

Proof. By Lemmas 2.4 and 4.1, a mutated continuant tree is still a continuant tree. Thus it is enough by Lemma 2.3 to show that there exists a sequence as in the statement such that the initial seed S_0 is a continuant tree of the form described in the statement. Consider the frieze with variables associated to the quiver Q_0, see [5] 8.2, [12] Section 5. Its period is $n+3$. The lemma follows from the same method as in [7] 8.1, with $\mathbb{N}$ replaced by

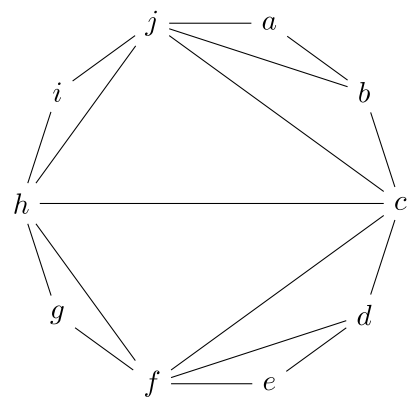

Figure 13: The triangulation corresponding to the continuant trees of Figures 1 and 12, right part

the semiring $\mathbb{N}[x_i^{\pm 1}]$, using the formula of [7] which gives the value of each entry of the tiling using a continuant polynomial. $\square$

Note that mutation of a continuant tree corresponds to the classical flip of a triangulation: in some quadrilateral, replace one diagonal by the opposite one. This is illustrated in Figure 14: the triangulation on the right part corresponds (this correspondance is explained in the proof of 4.1) to the continuant tree on the left part, which, when mutated at vertex *abcde* gives the continuant tree of Figure 1, corresponding to the triangulation of Figure 13: the two triangulations are obtained by exchanging the diagonals hc and jf. This is a particular case of a general construction indicated in [21] 12.2, see also [12] Section 5. Our approach however is different and quite elementary.

5 Appendix: representative functions

Fix a commutative ring R. Let A, B be R-algebras. We say that a function $f : A \to B$ is *representative over R with values in B* if there exists a natural integer n, an R-algebra homomorphism $\mu : A \to B^{n\times n}$, a row matrix $\lambda \in B^{1\times n}$ and a column matrix $\gamma \in B^{n\times 1}$ such that for any $a \in A$

$$f(a) = \lambda\mu(a)\gamma.$$

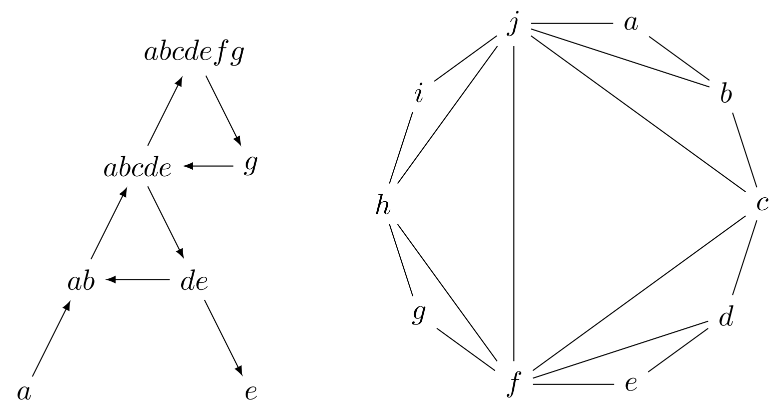

Figure 14: Another continuant tree and its associated triangulation

This definition may seem too general[1], but it is justified by the following result.

Proposition 5.1. *The composition of two representative functions is representative.*

Proof. Let A, B, C be three R-algebras and $f : A \to B$, $g : B \to C$ be representative functions. We have for any $a \in A$ $f(a) = \lambda\mu(a)\gamma$, where the notations are as above; and for any $b \in B$, $g(b) = \kappa\nu(b)\delta$ where for some natural integer p, ν is a R-algebra homomorphism $B \to C^{p\times p}$, $\kappa \in C^{1\times p}$ and $\delta \in C^{p\times 1}$.

1. For a matrix $M \in B^{q\times r}$, denote by $\nu(M)$ the matrix in $(C^{p\times p})^{q\times r}$ by replacing each entry of M by its image under ν. Note that if M, N are matrices over B whose product is defined, then $\nu(MN) = \nu(M)\nu(N)$. Note that, under $p \times p$-block decomposition, the rings $(C^{p\times p})^{q\times r}$ and $C^{pq\times pr}$ are canonically isomorphic, and we identify them.

2. Define the mapping $\pi : A \to (C^{p\times p})^{n\times n}$ by $\pi(a) = \nu(\mu(a))$. Then by 1., π is a ring homomorphism and $\nu(\lambda)\pi(a)\nu(\gamma) = \nu(\lambda)\nu(\mu(a))\nu(\gamma) = \nu(\lambda\mu(a)\gamma) = \nu(f(a))$. Thus $g \circ f(a) = \kappa\nu(f(a))\delta = \kappa\nu(\lambda)\pi(a)\nu(\gamma)\delta$. This shows that $g \circ f$ is a representative function $A \to C$, since $\kappa\nu(\lambda)$ is a row vector of size $1 \times np$, π a ring homomorphism $A \to C^{np\times np}$ and $\nu(\gamma)\delta$ a column vector of size $np \times 1$. $\square$

[1] It could even be more general, by replacing rings by semirings, with applications in Automata Theory and Tropical Geometry

Given a group G and an R-algebra B, we say that a function f from G into B is *representative with values in* B if the natural extension of this function to the group algebra RG is representative. In other words, there exists a homomorphism $\mu : G \to GL_n(B)$, a row vector $\lambda \in B^{1\times n}$ and a column vector $\gamma \in B^{n\times 1}$ such that $f(g) = \lambda\mu(g)\gamma$ for any $g \in G$.

These definitions fit also with the classical case. If A is a K-algebra, K a field, then a linear mapping $A \to K$ is representative in the classical sense if and only if it is representative over K with values in K, in the above meaning. In particular, a function from a group G (or a semigroup) into K is representative (see [1] p.72) if and only if the linear mapping from the group algebra KG into K that it defines is representative over K with values in K (see [17] Ex.1.5.11 p. 41).

By diagonal sum of matrices, it is easily seen that the sum of two representative functions is representative. Moreover, if f is a representative function defined on a group G and a an element of the group, the function $h(g) = f(ag)$ is also representative; we denote it by $h = f.a$.

Lemma 5.2. *Let R be a commutative ring and B an R-algebra, G a group, H a subgroup of finite index and f a representative function of H over R with values in B. Define for any $g \in G$, $\phi(g) = f(g)$ if $g \in H$, $= 0$ otherwise. Then ϕ is a representative function of G with values in B.*

Proof. This is proved by mimicking the matrix construction of an induced character. We know the existence of a group homomorphism μ from H into the group $GL_n(B)$, a row matrix $\lambda \in B^{1\times n}$, a column matrix $\gamma \in B^{n\times 1}$ such that for $h \in H$, one has $f(h) = \lambda\mu(h)\gamma$. Let $x_1, \ldots, x_d$ be representatives of the left classes gH of $G \mod H$. For $j = 1, \ldots, d$ and g in G, let $i = g.j$ and $h_j \in H$ be such that $gx_j = x_i h_j$; they are uniquely defined by this equation. Define the square matrix $R(g)$ of size nd, with $d \times d$ blocks of size n as follows: the (i, j)-block is $\mu(h_j)$; all other blocks are zero. It is then classical that R is a group homomorphism from G into $GL_{nd}(B)$. We may assume that $x_1 = 1$. Then the $(1, 1)$-block of $R(g)$ is nonzero if and only if g is in H, in which case it is equal to $\mu(g)$. Define the $1 \times nd$-row matrix $L = (\lambda, 0, \ldots, 0)$ and the the $nd \times 1$-column matrix $C = (\gamma^T, 0, \ldots, 0)^T$. Then $\phi(g) = LR(g)C$ and ϕ is a representative function of G with values in B. □

Corollary 5.3. *Let R be a commutative ring, B an R-algebra, G a group, and H a subgroup of finite index. For each left coset C of $G \mod H$, with representative a_C, let f_C be a representative function of the group H with* *values in B. Define the function f on G by $f(g) = f_C(h)$ if $g \in C$, $g = a_C h$. Then f is a representative function of G with values in B.*

Proof. Let the function h_C be equal to f_C on H and 0 elsewhere; it is representative by the lemma. Now f is the sum over all cosets C of the functions $h_C.a_c^{-1}$, which proves the corollary. $\square$

Proposition 5.4. *Let t be the SL_2-tiling of the plane over the field K associated to a periodic frontier (see Subsection 2.6). Let R be the subring of K generated by the variables of the frontier and their inverses. Then the function $\mathbb{Z}^2 \to R$, $(x, y) \mapsto t(x, y)$ is a representative function of the group $\mathbb{Z}^2$ with values in R.*

Lemma 5.5. *Consider a function $\mathbb{Z}^2 \to R$, $(x, y) \mapsto s(x, y)$ which is of the following form: $s(x, y)$ is the $(1, 1)$-coefficient of the matrix $A^y BC^x$, where A, B, C are fixed square matrices of the same size over the commutative ring R, with A, C invertible. Then s is a representative function of the group $\mathbb{Z}^2$ with values in R.*

Proof. Consider the free R-module $\mathcal{M}$ of square matrices of the given size over R. Then the group $\mathbf{Z}^2$ acts on it by $(x, y).M = A^y MC^x$. Then $t(x, y) = \phi((x, y).B)$, where ϕ is the linear form on $\mathcal{M}$ which maps M onto its $(1, 1)$-coefficient. Taking a basis of $\mathcal{M}$, we obtain that t is representative. $\square$

Corollary 5.6. *Let n, p be positive integers and $s_{i,j}, 0 \le i \le n-1, 0 \le j \le p-1$ be representative functions of the group $\mathbb{Z}^2$ into R. Define $t : \mathbb{Z}^2 \to R$, $t(x, y) = s_{r_1,r_2}(q_1, q_2)$ if $x = nq_1 + r_1$ and $y = pq_2 + r_2$ (Euclidean division of x by n and of y by p). Then t is representative.*

Proof. Let H be the subgroup of $\mathbb{Z}^2$ generated by the vectors $(n, 0)$ and $(0, p)$. Then we obtain the corollary by applying Corollary 5.3. $\square$

Proof. (Proposition 5.4) We know by Th.2.7 that the tiling is tame; moreover the bi-infinite sequence of column (resp. row) linearization coefficients is periodic. Denote by $(\alpha_j)_{j\in\mathbb{Z}}$ (resp. $(\beta_i)_{i\in\mathbb{Z}}$ this sequence and let n (resp.p) be its period.

Observe that these coefficients belong to R; this follows indeed from Th. 2.7 and from the fact that for each three adjacent columns, one may find three points A, B, C on them, on the same horizontal line, and such that A, B are on the frontier; then the linearization coefficient of the middle column is $\frac{t(A)+t(C)}{t(B)}$, which is in R since $t(A)$ and $t(B)$ are variables of the frontier and $t(C$ is given by Eq.(7).

Denote by C_j the j-th column of the tiling. Then $C_{j+2} = -C_j + \alpha_{j+1}C_{j+1}$. This implies, with the matrix notation $Q(\alpha)$ of Subsection 2.1, that $(C_j, C_{j+1})Q(\alpha_{j+1}) = (C_{j+1}, C_{j+2})$. It follows that for any natural integer j, one has

$$(*) \quad (C_0, C_1)Q(\alpha_1 \dots \alpha_j) = (C_j, C_{j+1}),$$

if we denote $Q(\alpha_1 \dots \alpha_j) = Q(\alpha_1) \cdots Q(\alpha_j)$.

Now, we have $\alpha_{j+n} = \alpha_j$. It follows that for any natural integers q, r, we have $(C_0, C_1)[Q(\alpha_1) \cdots Q(\alpha_n)]^q Q(\alpha_1) \cdots Q(\alpha_r) = (C_{nq+r}, C_{nq+r+1})$.

We claim that this is even true for any integer q. Indeed, an equality similar to $(*)$ holds for negative indices: for $j > 0$,

$$(C_0, C_1)Q(\alpha_0)^{-1} \cdots Q(\alpha_{-i+1})^{-1} = (C_{-j}, C_{-j+1}).$$

From this the claim follows by induction on negative q.

Now, similar calculations apply to rows. Putting this together, we obtain that for any integers x, y with $x = nq_1 + r_1$, $y = pq_2 + r_2$, one obtains that the matrix

$$\begin{pmatrix} t(x,y) & t(x+1,y) \\ t(x,y+1) & t(x+1,y+1) \end{pmatrix}$$

is equal to

$$Q(\beta_{r_1} \dots \beta_1)[Q(\beta_p \dots \beta_1)]^{q_2} \begin{pmatrix} t(0,0) & t(0,1) \\ t(1,0) & t(1,1) \end{pmatrix} [Q(\alpha_1 \dots \alpha_n)]^{q_1} Q(\alpha_1 \dots \alpha_{r_2}).$$

Note that

$$[Q(\alpha_1 \dots \alpha_n)]^{q_1} Q(\alpha_1 \dots \alpha_{r_2}) = Q(\alpha_1 \dots \alpha_{r_2})[Q(\alpha_{r_2+1}\alpha_{r_2+2} \dots \alpha_{r_2})]^{q_1},$$

and similarly for the β's. This implies the proposition, by Lemma 5.5 and Corollary 5.6.

$\square$

6 Conjectures

We conjecture that Th. 3.6, Th. 3.12 and Th. 3.13 extend to all Euclidean diagrams. Note that for Dynkin diagrams, these extensions are immediate since the mutation classes of seeds are finite, by Fomin and Zelevinsky's finite type classification [23].

References

[1] E. Abe, *Hopf algebras*, Cambridge University Press, 1980.

[2] I. Assem, T. Brüstle, G. Charbonneau-Jodoin and P.-G. Plamondon, *Gentle algebras arising from surface triangulations*, Algebra and Number Theory **4**, 2010, 201-229.

[3] I. Assem, T. Brüstle, and R. Schiffler, *Cluster-tilted algebras as trivial extensions*, Bulletin of the London Mathematical Society **40**, 2008, 151-162.

[4] I. Assem and G. Dupont, *Friezes and a construction of the Euclidean cluster variables*, Journal of Pure and Applied Algebra **215**, 2011, 2322-2340.

[5] I. Assem, C. Reutenauer and D. Smith, *Friezes*, Advances in Mathematics **225** (2010) 3134-3165.

[6] J. Bastian, *Mutation classes of $\tilde{A}_n$-quivers and derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$*, to appear, arXiv:0901.1515.

[7] F. Bergeron and C. Reutenauer, *SL_k-tilings*, Illinois Journal of Mathematics **54** (2010) 263-300.

[8] J. Berstel and C. Reutenauer, *Noncommutative rational series with applications*, Cambridge University Press, 2010.

[9] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics, **204** (2006) 572–618.

[10] A. B. Buan, R. Marsh and I. Reiten, *Cluster mutation via quiver representations*, Commentarii Mathematici Helvetici **83** (2008) 143-177.

[11] A. B. Buan and D. F. Vatne, *Derived equivalence classication for cluster-tilted algebras of type A_n*, Journal of Algebra **319** (2008) 2723-2738.

[12] P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Commentarii Mathematici Helvetici 81 (2006), 595–616.

[13] P. Caldero and B. Keller, *From triangulated categories to cluster algebras*, Inventiones Mathematicae 172 (2008) 169–211.

[14] P.M. Cohn, *Free rings and their relations*, 2nd ed., Oxford University Press, 1985.

[15] J. Conway and H.S.M. Coxeter, *Triangulated polygons and frieze patterns*, The Mathematical Gazette **57** (1973), 87–94 and 175-183.

[16] H.S.M. Coxeter, *Frieze Patterns*, Acta Arithmetica **XVIII** (1971), 297–310.

[17] S. Dascalescu,C. Nastasescu and S. Raianu, *Hopf algebras, an introduction*, Dekker, 2001.

[18] G. Dupont *Cluster multiplication in regular components via generalized Chebyshev polynomials*, Algebras and Representation Theory, in press.

[19] G. Dupont *Quantized Chebyshev Polynomials and Cluster Characters with Coefficients*, Journal of Algebraic Combinatorics **31**, 501–532, 2010.

[20] G. Dupont *Generalized Chebyshev Polynomials and Positivity for Regular Cluster Characters*, arXiv :0911.0714.

[21] S. Fomin, M. Shapiro and D. Thurston, *Cluster algebras and triangulated surfaces. Part I. Cluster Complexes*, Acta Mathematica **201** (2008), 83–146.

[22] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, Journal of the American Mathematical Society **15** (2002), 497–529.

[23] S. Fomin and A. Zelevinsky, *Cluster algebras. II. Finite type classification*, Inventiones Mathematicae **154**, 2003, 63-121.

[24] B. Keller, *Cluster algebras, quiver representations and triangulated categories*, London Mathematical Society Lecture Note Series, 375, Cambridge Univ. Press, Cambridge, 2010, 76-160.

[25] R.L. Graham and D.E. Knuth and O. Patashnik, *Concrete mathematics*, Addison Wesley, 1994.

[26] D. Happel and C. M. Ringel, *Construction of tilted Algebras*, Lecture Notes in mathematics **903**, 1982, 125-144.

[27] G. P. Hochschild, *Basic Theory of Algebraic Groups and Lie Algebras*, Springer, 1981.

[28] M. E. Sweedler, *Hopf algebras*, Benjamin, 1969.

[29] E. B. Vinberg, *Linear representations of groups*, Birkhäuser/Springer, 2010.

[30] H.M. Wedderburn, *Non-commutative domains of integrity*, Journal für Reine und Angewandte Mathematik **167** (1932), 129–141.