## Linear forms, quadratic forms and the normal distribution [1]

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**Abstract**

Among underlying possible distributions for random vectors $X = (X_1, \ldots, X_n)$ with independently distributed components $X_i$ and identically distributed $X_i - E[X_i]$'s, we motivate and study the problem of findings solutions to the equation $\mathbb{C}ov(X^\top QX, \exp(itc^\top X)) = 0$, for all $t \in \mathbb{R}$, and for pairs $(Q, c)$ meeting certain conditions. The problem is an extension of the classic and historically resonating case where the $X_i$'s are identically distributed, the first two moments are assumed to exist, $c^\top X$ and $X^\top QX$ are respectively the sample mean $\bar{X}$ and sample variance $S^2$, and where the only solutions are normal distributions. We give various conditions, namely on $(Q, c)$, that duplicate and generalize the normal characterization result for $(\bar{X}, S^2)$, but we also provide a counterexample where solutions can be found outside the normal class of distributions. Our findings exploit nad exhibit elegant representations in terms of the cumulant generating function, and the cumulants when all the moments of the $X_i$'s are assumed to exist. Finally, we show how a result of Seneta and Szekely (*Journal of the Australian Mathematical Society*, 2006) follows and can be generalized.

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# 1 Introduction

Characterizations of probability distributions have a long and rich history in probability and statistics. For the normal distribution, one of its best known properties, the independence between sample mean $\bar{X}$ and sample variance $S^2$ actually characterizes the normal as the common distribution (among those with finite variance) based on a sample $X_1, \ldots, X_n$ with $\bar{X} = \sum_{i=1}^n X_i/n$ and $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$. This result dates back to

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Geary (1936). Geary's proof is based on the cumulants and it is assumed that all the cumulants exist. The same result was established by Lukacs (1942, 1956), Tweedie (1946) and Geisser (1956) under the weaker assumption of finite variance and with proofs based on the cumulant generating function. In fact, if we suppose that $\bar{X}$ and $S^2$ are independent, then

$$\mathbb{C}ov(S^2, \exp(it\bar{X})) = 0\,, \text{ for all } t \in \mathbb{R}. \tag{1.1}$$

It seems that Equation (1.1) requires less than independence, but it implies that

$$\Psi''(t) - \Psi''(0) = 0\,, \text{ for all } t \in \mathbb{R}, \tag{1.2}$$

where $\Psi$ is the cumulant generating function of $X_1$ (see Example 1 below as well). And only the cumulant generating function of a normal distribution can solve equation (1.2) so that (1.1) is indeed equivalent to the independence of $\bar{X}$ and $S^2$. Laha (1953) considers an extension with $S^2$ replaced by a quadratic form $X^\top QX$ and assumes that $\bar{X}$ and $X^\top QX$ are independent. When the sum of all of the elements in $Q$ is equal to zero, the equation $\mathbb{C}ov(X^\top QX, \exp(it\bar{X})) = 0$, for all $t \in \mathbb{R}$, leads to the equation (1.2) again, yielding a more general characterization result. Other interesting normal distribution characterization results can be found in the book of Bryc (1995).

The main objective of this paper is to explore extensions of Laha's (1953) result by replacing the identically distributed assumption on the $X_i$'s by an identically distributed assumption on the variables $X_i - E[X_i]$, $i = 1, \ldots, n$, and by replacing $\bar{X}$ by a more general linear form $c^\top X$. The problem regarding the independence between a quadratic form and a linear form finds statistical applications in linear models. We will study the equation

$$\mathbb{C}ov(X^\top QX, \exp(itc^\top X)) = 0\,, \text{ for all } t \in \mathbb{R}. \tag{1.3}$$

In Section 2, starting with equation (1.3), we develop a second order differential equation involving $\Psi$ (Lemma 2.1), the cumulant generating function of $X_1 - \mathbb{E}[X_1]$. We identify conditions on $Q$ and $c$, such that equation (1.3) characterizes the normal distribution. These conditions include Laha's case with $c^\top X = \bar{X}$ (Theorem 2). On the other hand, we provide counterexamples where the characterization does not hold. In other words, equation (1.3) is not necessarily informative enough to characterize the normal distribution.

In Section 3, we proceed as in Geary (1936) assuming that all the moments of $X_1$ exist. We are thus led to the equivalent to (1.3) condition

$$\mathbb{C}ov(X^\top QX, (c^\top X)^m) = 0\,, \text{ for all } m \in \mathbb{N}. \tag{1.4}$$

With this latter condition being also of interest on its own, we obtain sharper sufficient conditions on $Q$ and $c$ such that condition (1.3) characterizes the normal distribution. As a by-product, elegant representations of the covariance in (1.4) are obtained (e.g., Remark 1).

Finally, Seneta and Szekely (2006) have a result saying that if,

$$\mathbb{C}ov(S^2, (\bar{X})^m) = 0\,, \text{ for all } m \in \{0, 1, \ldots, M\}, M > 0,$$

then there exists a normally distributed random variable $Y$ such that $\mathbb{E}(X_1^{m+2}) = \mathbb{E}(Y^{m+2})$ for all $m \in \{0, 1, \ldots, M\}$. In Section 4, we consider extensions consisting in finding conditions on $Q$ and $c$ such that if

$$\mathbb{C}ov(X^\top QX, (c^\top X)^m) = 0\,, \text{ for all } m \in \{0, 1, \ldots, M\}, M > 0, \tag{1.5}$$

then there exists a normally distributed random variable $Y$ which satisfies $\mathbb{E}(X_1^{m+2}) = \mathbb{E}(Y^{m+2})$ for all $m \in \{0, 1, \ldots, M\}$. Our development relies on an expansion of $\mathbb{C}ov(X^\top QX, \exp(itc^\top X)$ to derive the terms $\mathbb{C}ov(X^\top QX, (c^\top X)^m); m = 0, 1, \ldots;$ and an analysis of the recursive structure present in (1.5). Our extension is unified with respect to $Q$ and $c$ and provides an arguably much simplified alternative to Seneta and Szekely's proof in the $\bar{X}$ and $S^2$ case.

## 2 Analysis with the cumulant generating function

We begin with a description of the setting along with some notations and definitions. We consider $X = (X_1, X_2, \ldots, X_n)^\top$ a random vector. We denote by $(q_{jk})$, $j, k = 1, \ldots, n$, the matrix $Q$ generating the quadratic form $X^\top QX$, and by $(c_1, c_2, \ldots, c_n)^\top$ the vector $c$ generating the linear form $c^\top X$. We assume that $Q$ is symmetric and positive semi-definite. We write $X = \epsilon + \mu$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^\top$ is the mean of $X$. We assume that the components of $\epsilon$ are independent, identically distributed, and have a finite variance $\sigma^2$. We carry along standard assumptions in linear models, that is $Qc = 0$ and $Q\mu = 0$. Notice that when $\sum_j q_{jj} c_j^2 = 0$ it implies that $X^\top QX$ depends on some of the $X_j$'s, $j = 1, \ldots, n$, while $c^\top X$ depends solely on different $X_j$'s, $j = 1, \ldots, n$. This means that $X^\top QX$ and $c^\top X$ are independent for all possible distributions on $X_1$. Let us say that it is the trivial situation. In order to avoid this situation, we shall assume throughout that $\sum_j q_{jj} c_j^2 > 0$ (in other words that $\{j : q_{jj} \neq 0 \text{ and } c_j \neq 0\} \neq \emptyset$).

We denote by $\Phi$ and $\Psi$ the characteristic and cumulant generating functions of $\epsilon_1$, that is $\Phi(t) = \mathbb{E}(exp(it\epsilon_1))$ for all $t \in \mathbb{R}$, and $\Psi(t) = \log \Phi(t)$ for all $t \in \mathbb{R}$ where $\Phi(t) \neq 0$. We now proceed with a useful covariance expression.

**Lemma 1.** *Consider a random vector $X$ with mean $\mu$ and covariance matrix $\sigma^2 I_n$, such that the components $\epsilon_1, \ldots, \epsilon_n$ of $\epsilon = X - \mu$ are independent and identically distributed with characteristic function $\Phi$ and cumulant generating function $\Psi$. Consider further the quadratic and linear forms $X^\top QX$ and $c^\top X$, where $Q$ is symmetric, semi-positive definite, and assume that $Q\mu = 0$.*

*Then, we have for $t \in \mathbb{V}$; $\mathbb{V}$ being some neighbourhood of 0;*

$$Cov(X^\top QX, e^{itc^\top X}) = -e^{itc^\top \mu} \prod_j \Phi(c_j t) \left\{ \sum_j q_{jj}(\Psi''(c_j t) + \sigma^2) + \sum_{j,k} q_{j,k} \Psi'(c_j t)\Psi'(c_k t) \right\}. \tag{2.6}$$

**Proof.** First, since $Q\mu = 0$, we have

$$\begin{aligned} \mathrm{Cov}(X^\top QX, e^{itc^\top X}) &= e^{itc^\top \mu}\, \mathrm{Cov}(\epsilon^\top Q\epsilon, e^{itc^\top \epsilon}) \\ &= e^{itc^\top \mu} \left\{ E((\epsilon^\top Q\epsilon)\, e^{itc^\top \epsilon}) - \sigma^2 tr(Q) \prod_j \Phi(c_j t) \right\}. \end{aligned} \tag{2.7}$$

Expanding the left-side bracketed term, we obtain

$$\begin{aligned} E((\epsilon^\top Q\epsilon)\, e^{itc^\top \epsilon}) &= E(\sum_j q_{jj}\, \epsilon_j^2 exp\{it(c_j\epsilon_j + \sum_{k:k\neq j} c_k\epsilon_k)\} \\ &+ \sum_{j,k;j\neq k} q_{jk}\epsilon_j\epsilon_k exp\{(it(c_j\epsilon_j + c_k\epsilon_k + \sum_{l:l\neq j,k} c_l\epsilon_l)\} \\ &= -\sum_j q_{jj}\Phi''(c_j t)\frac{\prod_i \Phi(c_i t)}{\Phi(c_j t)} - \sum_{j,k;j\neq k} q_{jk}\Phi'(c_j t)\Phi'(c_k t)\frac{\prod_i \Phi(c_i t)}{\Phi(c_j t)\Phi(c_k t)} \\ &= -\prod_i \Phi(c_i t) \left\{ \sum_j q_{jj}[\Psi''(c_j t) + (\Psi'(c_j t))^2] + \sum_{j,k;j\neq k} q_{jk}\Psi'(c_j t)\Psi'(c_k t) \right\}, \end{aligned} \tag{2.8}$$

by making use of the independence of the $\epsilon_j$'s, the identities $\frac{\partial^k}{\partial t^k}\Phi(t) = i^k E(\epsilon_j^k e^{it\epsilon_j})$, $\Psi'(t) = \frac{\Phi'(t)}{\Phi(t)}$, and $\Psi''(t) + (\Psi'(t))^2 = \frac{\Phi''(t)}{\Phi(t)}$; $t \in \mathbb{V}$. Finally, (2.7) and (2.8) lead directly to the stated result. $\square$

**Example 1.** *As a first illustration, Lemma 1 applies for a sample mean $\bar{X}$ and sample variance $S^2$, with $c = \frac{1}{n}1$, $Q = \frac{1}{n-1}(I_n - \frac{1}{n}11^\top)$, $1 = (1,\ldots,1)^\top$, and with $\mu = \alpha 1$ for some $\alpha \in \mathbb{R}$. Expression (2.6) yields (notice the simplification here arising from the fact that $Qc = 0$ with the given choice of $Q$ and $c$)*

$$Cov(S^2, e^{it\bar{X}}) = -e^{it\alpha}\, (\Phi(\frac{t}{n}))^n\, (\Psi''(\frac{t}{n}) + \sigma^2)\,.$$

*Now, interestingly, observe that the assumption $Cov(S^2, e^{it\bar{X}}) = 0$, for all $t \in \mathbb{V}$, implies that $\Psi''(t) = -\sigma^2$ for all $t$ in a neighbourhood of zero, and hence characterizes $X_1$ as having a normal distribution. Alternatively, with $\Psi''(0) = -\sigma^2$ in general, we recover the equivalence between equations (1.1) and (1.2).*

In the same spirit as in the previous example, we obtain for other pairs $(Q, c)$ the following immediate consequence.

**Corollary 1.** *Under the assumptions of Lemma 1 and with $Qc = 0$, we have $Cov[(X^\top QX), \exp(ic^\top X\, t)] = 0$ for all $t$ in a neighbourhood of zero if and only if*

$$\sum_{j=1}^{n} q_{jj}\{\Psi''(c_j t) - \Psi''(0)\} + \sum_{j=1}^{n}\sum_{k=1}^{n} q_{jk}\Psi'(c_j t)\Psi'(c_k t) = 0\,, \tag{2.9}$$

*for all $t$ in a neighbourhood of zero.*

The general problem is now stated as follows.

*General problem: Among all possible cumulant generating functions satisfying $\Psi'(0) = 0$, what are the ones that solve equation (2.9) under the assumptions of Corollary 1 for given $Q, c, \mu$ ?*

Here is a first important observation. If $\Psi$ is the cumulant generating function of a $\mathcal{N}(0, \sigma^2)$ distribution, i.e., $\Psi(t) = -t^2\sigma^2/2$ for all $t \in \mathbb{R}$, then $\Psi$ is always a solution as seen by the fact that $\Psi''$ is constant for such $\Psi$'s, and $c^\top Qc = 0$ given the assumption $Qc = 0$. We continue with situations (Theorems 1 and 2) where a solution to the general problem is necessarily the cumulant generating function of a normal distribution.

**Theorem 1.** *If $c_j \in \{-1, 0, 1\}$ for all $j = 1, \ldots, n$, $\sum_{j=1}^{n} c_j = 0$, $\Psi$ is symmetric, and if $\Psi$ solves (2.9) with the given assumptions on $Q, c, \mu$, then $\Psi$ is necessarily the cumulant generating function of a $\mathcal{N}(0, \sigma^2)$ distribution, for some $\sigma^2 \geq 0$.*

**Proof.** With the symmetry of $\Psi$, we have for $c_j \in \{-1, 0, 1\}$: $\Psi'(c_j t) = c_j\Psi'(t)$ and $\Psi''(c_j t) - \Psi''(0) = c_j^2\{\Psi''(t) - \Psi''(0)\}$, for all $t$ in a neighbourhood of zero. Hence, an expansion of the lhs of (2.9) yields

$$\begin{aligned} \sum_{j=1}^{n} q_{jj}\{\Psi''(c_j t) - \Psi''(0)\} &= \sum_{j=1}^{n} q_{jj} c_j^2\{\Psi''(t) - \Psi''(0)\}\,; \\ \sum_{j=1}^{n}\sum_{k=1}^{n} q_{jk}\Psi'(c_j t)\Psi'(c_k t) &= c^\top Qc\{\Psi'(t)\}^2 = 0\,. \end{aligned}$$

Therefore, with the given assumptions, (2.9) becomes equivalent to (1.2), and the result follows. □

**Theorem 2.** *If $c_j \in \{0, 1\}$ for all $j = 1, \ldots, n$, and if $\Psi$ solves (2.9) with the given assumptions on $Q, c, \mu$, then $\Psi$ is necessarily the cumulant generating function of a $\mathcal{N}(0, \sigma^2)$ distribution, for some $\sigma^2 \geq 0$.*

**Proof.** The proof of Theorem 1 applies without the symmetry assumption of $\Psi$, but with $c_j \in \{0, 1\}$ for all $j = 1, \ldots, n$. □

We pursue with a counterexample, where a solution to the general problem is not necessarily generated by a normal distribution.

**Example 2.** *Consider the general problem with* $n = 2$*,* $c = (1, -1)^\top$*,* $Q = ee^\top$*, where* $e = (1, 1)^\top$*. We take* $\mu = (0, 0)^\top$ *and* $X_1 + p$ *as having a Bernoulli(p) distribution. A direct evaluation yields*

$$\begin{aligned} \Psi(t) &= \log\{q\exp(-ipt) + p\exp(iqt)\}, \quad q = 1 - p\,, \\ Cov[(X^\top QX), \exp(ic^\top X\,t)] &= 2pq(1 - 6pq)(cos(t) - 1), \qquad \text{for all } t \in \mathbb{R}. \end{aligned}$$

*Therefore, when* $pq = 1/6$*,* $\Psi$ *is a solution to equation (2.9).*

Example 2 provides a particularly revealing counterexample to the conjecture that only the cumulant generating function of a normal random variable solves the general problem. Consequently, the normal distribution characterizations given in Example 1, Theorem 1 and Theorem 2 cannot be inferred from a much more general context. Notice as well that in Example 2, $X^\top QX$ and $c^\top X$ are not independent showing that, for this particular choice of $Q$ and $c$, the condition given by Equation (1.3) is indeed weaker than requiring the independence between $X^\top QX$ and $c^\top X$. Finally, the next observation implies that, and shows how, many other solutions to (1.3) can be generated in the context of Example 2.

**Lemma 2.** *If* $\Psi_{\epsilon_1}$ *is a solution to (2.9) and* $Z$ *is a* $\mathcal{N}(0, 1)$ *random variable independent of* $\epsilon_1$ *then* $\Psi_{a\epsilon_1+bZ} = \Psi_{a\epsilon_1} + \Psi_{bZ}$*, and* $\Psi_{a\epsilon_1+bZ}$ *is hence a solution to (2.9) for all* $a, b \in \mathbb{R}$*.*

To summarize findings up to now, we have been able to solve the general problem and infer a normal characterization in some cases, but not in general. In other words, as seen in Example 2, differential equation (1.3) can possess many feasible solutions outside the class of normal cumulant generating function solutions. In the Appendix, we push the analysis further and provide an alternative approach to obtain many solutions to (1.3) for the settings for $Q, c$ of Example 2. In the next section, sharper characterizations are obtained by assuming that all the moments of $\epsilon$ exist.

# 3 Analysis with all of the cumulants

The cumulant of order $m$ for $\epsilon_1$ will be denoted by $\kappa_m$. When the cumulant of order $m$ exists, it is given by

$$\kappa_m = i^{-m}\Psi^{(m)}(0)\,,$$

where $\Psi^{(m)}$ is the derivative of order $m$ of $\Psi$. Since $\epsilon_1$ has mean zero and second moment $\sigma^2$, we have $\kappa_1 = 0$ and $\kappa_2 = -\sigma^2$. A normal distribution is characterized by having all cumulants of order three or more equal to zero.

We now proceed by showing that if one assumes existence of all of the moments, we can move from the problem of solving equation (2.9) to the one of solving equation (1.4). This brings into play a representation for $\mathbb{C}ov[(X^\top QX),(c^\top X)^m]$; $m=1,2,\ldots$; which we will also make use of in Section 4.

**Lemma 3.** *Let $\gamma_m = \mathbb{C}ov[(X^\top QX),(c^\top X)^m]$ for $m=1,2,\ldots$. If the moment of order $m+2$ of $\epsilon_1$ (or $X_1$) exists, then we have*

$$\begin{aligned}\gamma_m &= \sum_{\ell_1=1}^{m}\binom{m}{l_1}\{\mathbb{E}[(c^\top X)^{m-\ell_1}]\}[\{\sum_{j=1}^{n} q_{jj}c_j^{\ell_1}\}\kappa_{2+\ell_1} \\ &\quad+\sum_{\ell_2=0}^{\ell_1}\binom{\ell_1}{\ell_2}\{\sum_{j=1}^{n}\sum_{k=1}^{n} q_{jk}c_j^{\ell_1-\ell_2}c_k^{\ell_2}\}\kappa_{1+\ell_1-\ell_2}\kappa_{1+\ell_2}]\,. \end{aligned} \tag{3.10}$$

*Furthermore, if all the moments of $\epsilon_1$ exist, then equation (2.9) is equivalent to the system of equations : $\gamma_m = 0$ for $m=1,2,\ldots$.*

**Proof.** Expression (3.10) follows by making use of Lemma 1, the fact that

$$\mathbb{C}ov[(X^\top QX),(c^\top X)^m] = \frac{1}{i^m}\frac{d^m}{dt^m}\mathbb{C}ov[(X^\top QX),\exp(ic^\top X\,t)]\Big|_{t=0},$$

for all $m=1,2,\ldots$ , and a careful expansion. The equivalence of (2.9) with $\gamma_m=0$ for $m=1,2,\ldots$ follows from (3.10), and since (2.9) is equivalent to $\mathbb{C}ov[(X^\top QX),\exp(ic^\top X\,t)]\Big|_{t=0}=0$ when all the moments exist. □

In Lemma 3's representation of $\gamma_m$, the dominating term is given by $\sum_{j=1}^n\{q_{jj}c_j^m\}\kappa_{2+m}$ and it is affected by the cumulant of order $2+m$, while the remaining terms are affected by the cumulants of lower orders. This key observation leads to the following characterizations.

**Corollary 2.** *If $\sum_{j=1}^n q_{jj}c_j^m \neq 0$ for all $m=1,2\ldots$ [3], if all of the moments of $\epsilon_1$ exist, and if $\Psi$ solves (2.9) with the given assumptions on $Q,c,\mu$, then $\Psi$ is necessarily the cumulant generating function of a $\mathcal{N}(0,\sigma^2)$ distribution, for some $\sigma^2\geq 0$.*

**Proof.** The result follows by Lemma 3's equivalence under the assumption that all the moments exist, and a recursive analysis where, for all $M=1,2,\ldots$, the equations $\gamma_m=0$ for $m=1,\ldots,M$ implies that that the first $M+2$ cumulants of $\epsilon_1$ are equal to zero. Since this is true for all $M$, we infer that the cumulants of $\epsilon_1$ of order greater than 2 are all equal to zero, and hence that $\epsilon_1$ is normally distributed. □

[3] In fact, we only require this for odd $m$ because we have $\sum_{j=1}^n q_{jj}c_j^m > 0$ whenever $m$ is even given the assumptions on $Q,c$.

**Corollary 3.** *If the distribution of $\epsilon_1$ is symmetric, if all of the moments of $\epsilon_1$ exist and if $\Psi$ solves (2.9) with the given assumptions on $Q, c, \mu$, then $\Psi$ is necessarily the cumulant generating function of a $\mathcal{N}(0, \sigma^2)$ distribution, for some $\sigma^2 \geq 0$.*

**Proof.** Since the distribution of $\epsilon_1$ is symmetric about 0, the cumulants of odd orders are zero. Using this and the positivity of $\sum_{j=1}^n q_{jj} c_j^m$ for even $m$ (assumption), we infer, as above in Corollary 2, from Lemma 3 that all the cumulants of even orders greater than 2 are equal to zero as well, which establishes the result. □

Corollaries 2 and 3 contain a vast number of interesting cases. For instance, they cover all cases where the coefficients of $c$ are nonnegative, as well as Laha's result with $c^\top X = \bar{X}$. Revisiting Example 2, where all the moments exist, where $\gamma_m = 0$ for all $m = 1, 2, \ldots$, observe that its choice of $Q$ and $c$ leads to $\sum_{j=1}^n q_{jj} c_j^m = 0$ for all odd $m$, and pinpoints to where Corollary 2 (of course) does not apply. A similar observation relates to Corollary 3 with an asymmetric counterexample and the impossibility of a symmetrically distributed counterexample. We conclude this section by pointing out a lovely particular case of (3.10).

**Remark 1.** *For a sample mean $\bar{X}$ and sample variance $S^2$, we obtain as an application of (3.10); with $c = 1$, $Q = \frac{1}{n-1}(I_n - \frac{1}{n}11^\top)$, $1 = (1, \ldots, 1)^\top$; the formula*

$$\mathbb{C}ov[(S^2, (n\bar{X})^m] = \sum_{\ell_1=1}^{m} \binom{m}{l_1} \{\mathbb{E}[(n\bar{X})^{m-\ell_1}]\} \kappa_{2+\ell_1} \, . \tag{3.11}$$

## 4 Analysis with the first cumulants

We revisit here a result due to Seneta and Szekely (2006) and show how our results above provide near immediate extensions. To describe this result, consider first the covariance between $S^2$ and $\bar{X}$. Expression (3.11) yields the known result $Cov(S^2, \bar{X}) = \kappa_3/n$ (e.g., Casella and Berger, 2002, problem 58c on page 257). So a null covariance $Cov(S^2, \bar{X})$ implies that the third cumulant of $X_1$ is zero, or in other words that the first three moments of $\epsilon_1$ (or central moments of $X_1$) coïncide with those of a $N(0, \sigma^2)$ distribution. Now, Seneta and Szekely's result says that if $Cov(S^2, (\bar{X})^m) = 0$ for $m = 1, 2, \ldots, M$, then the first $M+2$ central moments of $X_1$ match with those of a of a $N(0, \sigma^2)$ distribution, this holding for all $M = 1, 2, \ldots$ . A reformulation of Corollary 2 and Corollary 3 produces this result and extensions to other pairs of linear and quadratic forms as follows.

**Corollary 4.** *Assume that $\sum_{j=1}^n q_{jj} c_j^m \neq 0$ for all $m = 1, \ldots, M$, and that the moment of order $M + 2$ of $\epsilon_1$ exists. Then the condition $\gamma_m =$*

$\mathbb{C}ov[(X^\top QX), (c^\top X)^m] = 0$ *for* $m = 1, \ldots, M$ *implies, under the given assumptions (Corollary 1) on* $Q, c, \mu$*, that the cumulants of* $\epsilon_1$ *of order* $3, \ldots, M+2$ *are all equal to zero, so the first* $M+2$ *moments of* $\epsilon_1$ *match the moments of a normally distributed random variable with mean 0 and variance* $\sigma^2$ *for some* $\sigma^2 \geq 0$*.*

**Corollary 5.** *If the distribution of* $\epsilon_1$ *is symmetric and the moment of order* $M+2$ *of* $\epsilon_1$ *exists, then the condition* $\gamma_m = \mathbb{C}ov[(X^\top QX), (c^\top X)^m] = 0$ *for* $m = 1, \ldots, M$ *implies, under the given assumptions (Corollary 1) on* $Q, c, \mu$*, that the cumulants of* $\epsilon_1$ *of order* $3, \ldots, M+2$ *are all equal to zero, so the first* $M+2$ *moments of* $\epsilon_1$ *match the moments of a normally distributed random variable with mean 0 and variance* $\sigma^2$ *for some* $\sigma^2 \geq 0$*.*

# Concluding remarks

# Appendix

As in Example 2, we consider (2.9) with $n = 2$, $c = (1, -1)^\top$, $Q = ee^\top$, where $e = (1, 1)^\top$ We hence want to find and characterize all solutions in $\Psi$ to the following equation

$$\{\Psi''(t) + \Psi"(-t) - 2\Psi''(0)\} + \{\Psi'(t) + \Psi'(-t)\}^2 = 0\,. \tag{4.12}$$

To do so, consider the functions $f$ and $g$ given by

$$f(t) = \frac{\Psi(t) + \Psi(-t)}{2} \quad \text{and} \quad g(t) = \frac{\Psi(t) - \Psi(-t)}{2}\,.$$

The boundary conditions tell us that $\Psi(0) = 0$, $\Psi'(0) = 0$ and $\Psi''(0) = -\sigma^2$. Therefore, we require that $f(0) = 0 = g(0)$, $f'(0) = 0 = g'(0)$, $f''(0) = -\sigma^2$ and $g''(0) = 0$. Moreover, $f$ is even and $g$ is odd. The new equation becomes

$$2\{f''(t) - f''(0)\} + 4\{g'(t)\}^2 = 0\,, \tag{4.13}$$

and the general solution to equation (4.12) is given by

$$\begin{aligned}
\Psi''(t) &= f''(t) + g''(t) \\
&= f''(0) - 2\{g'(t)\}^2 + g''(t) \\
&= -\sigma^2 - 2\{g'(t)\}^2 + g''(t)\,,
\end{aligned}$$

given (4.13). We conclude that cumulant generating function solutions $\psi$ to (4.13), such as Example 2's cumulant generating function of a Bernoulli($p$) distribution with $6p(1-p) = 1$ and those that can be further generated via Lemma 2, possess necessarily a second derivative such that $\Psi''(t) =$

$-\sigma^2 - 2\{g'(t)\}^2 + g''(t)$ for some odd $g$ with $g'(0) = 0$. The above development here illustrates differently than in Section 2 why there may well exist for some selections of $(Q, c, n)$ other solutions than the normal cumulant generating function to the *General problem.*

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