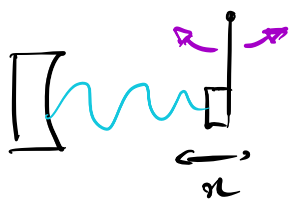
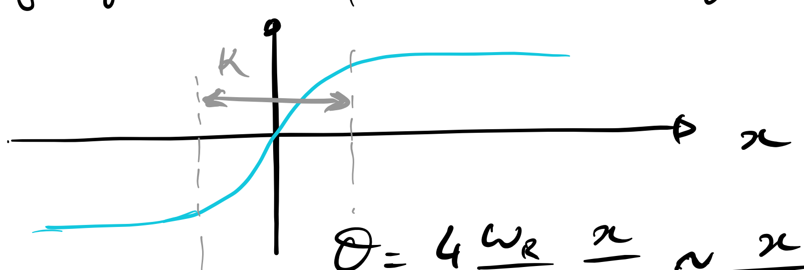
→ General picture, hand-waving:

Typical optomechanical system:



Radiation pressure
⇒ change of cavity length.

Cavity ≡ Harmonic oscillator:
interferometric measurement
relying on phase change



$$\theta = 4\frac{\omega_R}{\kappa}\frac{x}{L} \sim \frac{x}{1\,\mathrm{pm}}$$

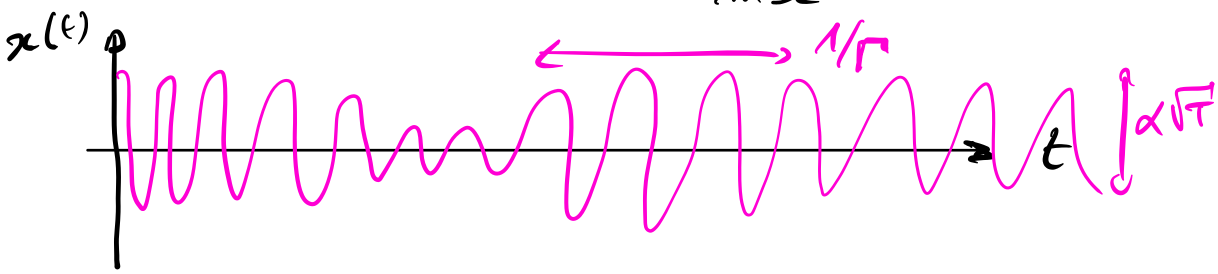
LIGO: best strain noise $8 \cdot 10^{-24}\, m/\sqrt{Hz}$

→ What happens if the mechanical system is thermal?

Simple classical treatment.

Equipartition theorem: $\frac{m\Omega^2}{2}\langle x^2\rangle = \frac{k_B T}{2}$

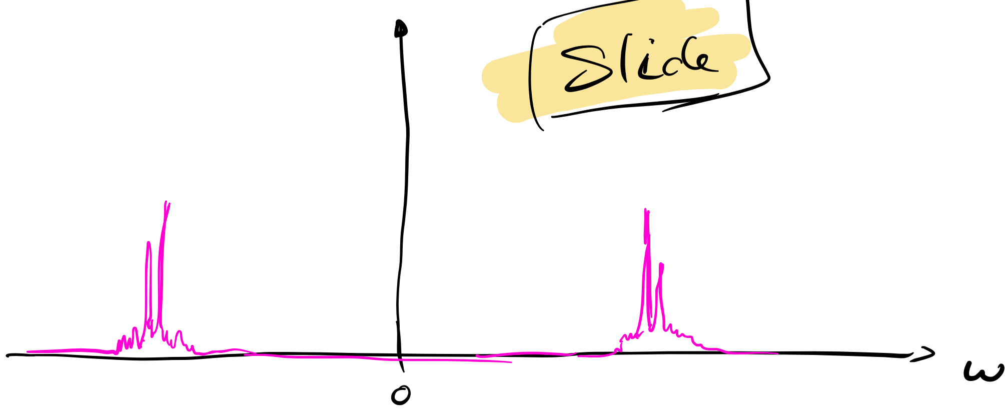
$$\Rightarrow \quad \langle x^2\rangle = \frac{k_B T}{m\Omega^2}$$



↳ Time resolved measurement

↳ Fluctuation spectrum of $x(t)$.

$$\tilde{x}(\omega) = \frac{1}{\sqrt{\tau}}\int_0^\tau dt\, e^{i\omega t}\, x(t).$$



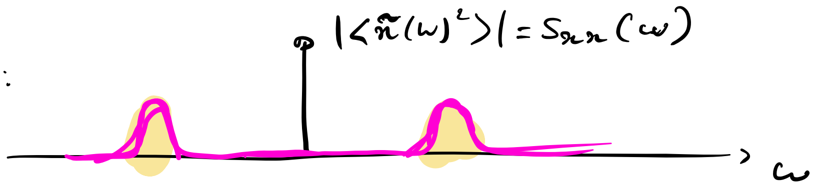
Wigner-Khinchin theorem:

→ in optics : $\frac{2}{\eta}\langle E^-(t)\, E^+(t+\tau)\rangle = \int_0^\infty I(\omega)\, e^{-i\omega\tau} d\omega$

→ here : $S_{xx}(\omega) \equiv \langle|\tilde{x}(\omega)|^2\rangle$

$$\simeq \int_{-\infty}^{\infty} dt\, e^{+i\omega t}\langle x(t)\, x(0)\rangle$$

Remarkably:



$$\rightarrow \quad \int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega = \langle x^2 \rangle = \frac{k_B T}{m \Omega^2}$$

→ What sets the shape of $S_{xx}(\omega)$?
linear response of the mechanics
[damped harmonic oscillator].

Displacement $\quad \langle \delta x \rangle(\omega) = \chi_{xx}(\omega)\, F(\omega)$

mechanical susceptibility

Forcing term

$$F \leftrightarrow \text{thermal} \quad \Rightarrow \quad S_{xx}(\omega) = \frac{k_B T}{\omega}\, \mathrm{Im}\, \chi_{xx}$$

$$m\ddot{x} + 2i\gamma \dot{x} + m\Omega^2 x = F$$

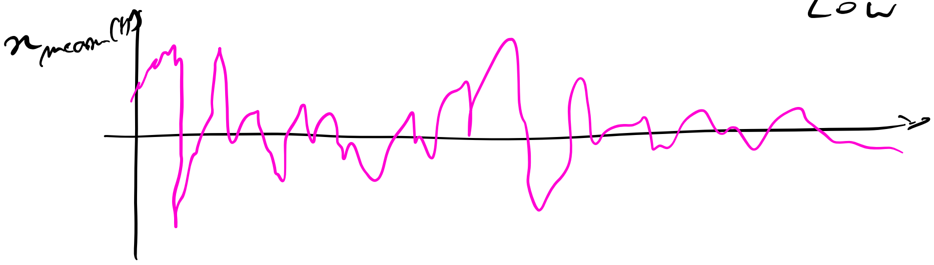
$$x(\omega) = \underbrace{\frac{1/m}{\Omega^2 - \omega^2 + 2i\gamma\omega}}_{\chi_{xx}(\omega).} F(\omega).$$

→ How precisely can we measure?

usually signal contains noise:

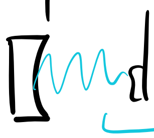
$x_{measured}(t) = x(t) + x_{noise}(t)$.

Low signal.



→ Using few photon, low Signal / noise ratio

→ Using lots of photon, what happens?



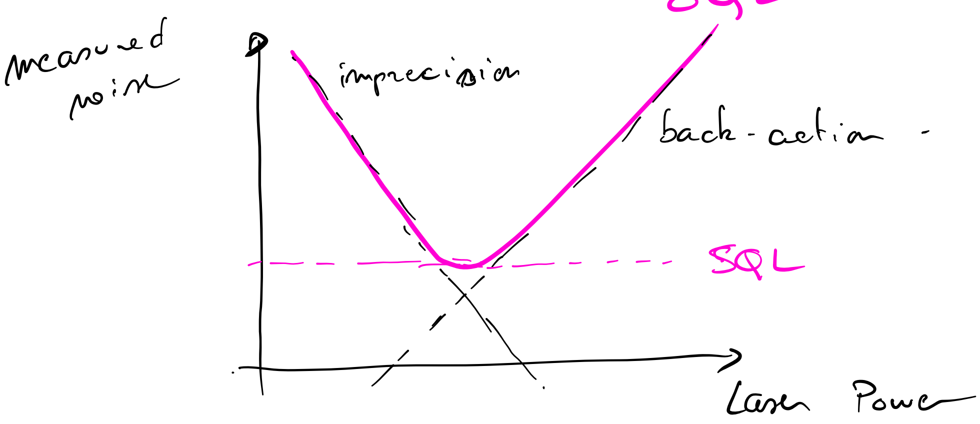
↳ photon shot noise

⇒ Radiation force noise!!

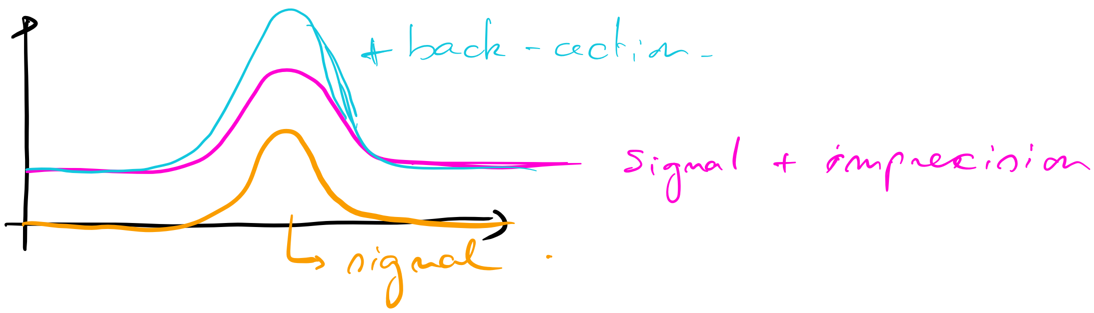
↳ There exist the perfect balance between low and high intensity.

STANDARD QUANTUM LIMIT

SQL.



In terms of signal:



Turns out that the best we can do in measuring at $T=0$ and adding ½ photon of noise

$$S_{xx}^{meas}(\omega) \geq 2 \cdot S_{xx}^{T=0}(\omega)$$

→ Limit due to ?

→ $\boxed{\text{Weak measurement}}$ ⇒ average over time

→ Slowly varying quadratures:

$$\hat{x}(t) = \hat{X}_1 \cos(\Omega t) + \hat{X}_2 \sin(\Omega t).$$

with this notation $[\hat{X}_1, \hat{X}_2] = 2 x_{ZPF}^2$

↳ SQL related to the Heisenberg uncertainty

→ No limit in instantaneous measurement of $x(t)$!

↳ careful if strong measu— with non-QND measurements

SQL $\Rightarrow$ detect $\hat{X}_{12}$ down to x_{ZPF}

$\boxed{\text{time scale } 1/\Gamma}$ $\Rightarrow$ precision $\sim 10^{-15}\,m$

$\rightarrow$ Hand wavy picture for backaction!

Simple reflection :

θ phase shift $\rightarrow$ $\theta = 2kx$

N photon arrive

coherent state $\Rightarrow$ $\delta N = \sqrt{N}$.

Better phase estimation $\Rightarrow$ higher laser power $\Rightarrow$ more shot noise

Uncertainty in phase estimation:

$$\delta N \, \delta\theta \geq 1/2$$

$\Rightarrow$ $\delta\theta \geq \frac{1}{2\sqrt{N}}$ $\qquad$ $\delta x = \frac{\delta\theta}{2k} \sim \frac{1}{2\sqrt{N}\,2k}$

or $\quad \Delta p = 2\hbar k N$

$$\delta p = \sqrt{\mathrm{Var}\,\Delta p} = 2\hbar k \sqrt{N}$$

$\Rightarrow$ $\delta x \, \delta p \geq \hbar/2$ $\quad$ Heisenberg-

→ Fluctuation spectrum, quantum version

Simple system: Harmonic oscillator

$$\hat{H} = \frac{k\hat{q}^2}{2} + \frac{\hat{p}^2}{2m}$$

mass m, freq $k = m\Omega^2$.

Ladder operators: $[\hat{b}, \hat{b}^+] = 1$

$$\hat{b}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{b}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

Thermal case: $p(n) = e^{-\frac{\hbar\Omega n}{k_B T}}\left[1 - e^{-\frac{\hbar\Omega}{k_B T}}\right]$

$$\bar{n} = \langle\hat{n}\rangle = \sum_0^\infty n\,p(n) = \left[e^{\frac{\hbar\Omega}{k_B T}} - 1\right]^{-1}$$

high T_p° : $\bar{n} \simeq \frac{k_B T}{\hbar\Omega}.$

Massive systems $\Rightarrow$ $\Omega/2\pi \simeq 1 - 1000\,\mathrm{MHz}$

$$\frac{k_B T}{\hbar\Omega} = 10^4 - 10^7$$

$$\hat{q} = x_{ZPF}(\hat{b}^+ + \hat{b}), \quad \hat{p} = i\,p_{ZPF}(\hat{b}^+ - \hat{b})$$

$$x_{ZPF} = \sqrt{\frac{\hbar}{2m\Omega}} \qquad p_{ZPF} = \sqrt{\frac{\hbar m\Omega}{2}}$$

$$[\hat{q}, \hat{p}] = i\hbar .$$

$$\sigma(\hat{p})\,\sigma(\hat{q}) \geq \frac{1}{2} |\langle [\hat{q}, \hat{p}] \rangle| = \frac{\hbar}{2} .$$

$$\hat{\varphi} = \frac{1}{\sqrt{2}} \frac{\hat{q}}{x_{ZPF}} \qquad \hat{P} = \frac{1}{\sqrt{2}} \frac{\hat{p}}{p_{ZPF}}$$

$$\hat{H}_0 = \frac{\hbar\Omega}{2}(\hat{P}^2 + \hat{\varphi}^2) = \hbar\Omega \hat{b}^\dagger \hat{b}$$

↳ Forcing ⇒ fluctuations: $\hat{F}$

Acting via radiation pressure $\quad \hat{V} = \hat{q}\,\hat{F}$

$$\hat{H} = \hat{H}_0 + \hat{V} .$$

Transition proba from $|\psi(0)\rangle$ (eigenstate) to $|\psi_f\rangle$

In the interaction picture:

$$|\psi_I(t)\rangle \equiv \hat{U}_0^\dagger(t)\,|\psi(t)\rangle = \hat{U}_I\,|\psi(0)\rangle$$

$$\hat{U}_0(t) = e^{-i\hat{H}_0 t/\hbar} \qquad \hat{U}_I(t) = e^{-i\hat{V}t/\hbar}$$

$$A_{i\to f} = \langle \psi_f | \psi(t) \rangle$$

$$= \langle \psi_f | \hat{U}_0(t)\,\hat{U}_I(t) | \psi(0) \rangle$$

$$= e^{-iE_f t/\hbar} \langle \psi_f | \psi_I(t) \rangle$$

Evolution given by Dyson series:

$$|\psi_I(t)\rangle = |\psi_I(0)\rangle + \frac{1}{i\hbar}\int_0^t d\tau_1 \, \hat{V}_I(\tau_1) |\psi_I(\tau_1)\rangle$$

$$= |\psi_I(0)\rangle + \frac{1}{i\hbar}\int_0^t d\tau_1 \, \hat{V}_I(\tau_1) |\psi_I(0)\rangle$$

$$+ \frac{1}{(i\hbar)^2}\int_0^t \int_0^{\tau_1} d\tau_1 d\tau_2 \, \hat{V}_I(\tau_1) \hat{V}_I(\tau_2) |\psi_I(\tau_2)\rangle$$

$\rightarrow$ $\hat{U}_0(0) = \mathbb{1}$ $\quad |\psi_I(0)\rangle = |\psi(0)\rangle$.

neglect last term (first order).

Furthermore: Born approximat°:

$$|\psi(0)\rangle = |\psi_{sys}(0)\rangle \otimes |\psi_{bath}(0)\rangle \doteq |n, j\rangle.$$

$\rightarrow$ Bath excites the system: $n \rightarrow n+1$

$\langle \psi_f | \psi(0) \rangle = 0$ $\qquad A_{i\to f} = \frac{1}{i\hbar}\int_0^t d\tau_1 \langle n+1, k | \hat{V}_I(\tau_1) | n, j \rangle.$

$$A_{i\to f}(t) = \frac{1}{i\hbar}\int_0^t d\tau_1 \langle n+1 | \hat{q}_I | n \rangle \langle k | \hat{F}_I(\tau_1) | j \rangle$$

$$A_{i\to f}(t) = \frac{x_{ZPF}}{i\hbar}\int_0^t d\tau_1\, e^{i\Omega\tau_1} \langle n+1|\hat{b}^\dagger+\hat{b}|n\rangle \langle k|\hat{F}_I(\tau_1)|j\rangle .$$

$$= \frac{x_{ZPF}\sqrt{n+1}}{i\hbar}\int_0^t d\tau_1\, e^{i\Omega\tau_1} \langle k|\hat{F}_I(\tau_1)|j\rangle$$

We need to sum on all state of the bath:

$$P_{n\to n+1} = \sum_k |A_{i\to f}|^2$$

$$= \frac{x_{ZPF}^2}{\hbar^2}(n+1)\iint_0^t d\tau_1 d\tau_2\, e^{i\Omega(\tau_2-\tau_1)} \sum_k \langle j|\hat{F}_I(\tau_1)|k\rangle\langle k|\hat{F}_I(\tau_2)|j\rangle$$

With completenes of k: $\sum_k |k\rangle\langle k| = \mathbb{1}$
and $\hat{F}^* = \hat{F}$

$$P_{n\to n+1} = \frac{x_{ZPF}^2}{\hbar^2}(n+1)\iint d\tau_1 d\tau_2\, e^{i\Omega(\tau_2-\tau_1)} \langle \hat{F}(\tau_1)\hat{F}(\tau_2)\rangle$$

Return on power spectral density for operators:

$$S_{OO}(\omega) = \lim_{\tau\to\infty} \frac{1}{\tau}\langle O_\tau^\dagger(\omega) O_\tau(\omega)\rangle .$$

$\boxed{O_\tau(\omega)}$: Fourier transform of $O(t)$ on $-\tau/2 < t < \tau/2$.

We have the Wiener Khinchin again:

$$S_{\mathcal{O}\mathcal{O}}(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} \langle \mathcal{O}^{+}(t+\tau)\mathcal{O}(t)\rangle_{t=0}$$

$$= \int_{-\infty}^{\infty} d\omega' \langle \mathcal{O}^{+}(-\omega)\mathcal{O}(\omega')\rangle$$

For quantum operator: $\mathcal{O}_{\tau}(\omega)$ et $\mathcal{O}_{\tau}(\omega')$
do not neccessarly commute

$$\Rightarrow \boxed{S_{\mathcal{O}\mathcal{O}}(\omega) \neq S_{\mathcal{O}\mathcal{O}}(-\omega)}$$

Going back to $P_{n\to n+1}$

$$P_{n\to n+1} = \frac{x_{ZPF}^2}{\hbar^2}(n+1)\int_0^t dt' \int_{-t'}^{t-t'} d\tau\, e^{-i\Omega\tau} \langle \hat{F}_I(t'+\tau)\hat{F}_I(t')\rangle$$

Markov approximation:
the bath is delta correlated.
on the second integral we can change
bounds to $\pm\infty$.
But careful, BMA $\Rightarrow$ Coarse graining.
Bath separable for Δt longer
than τ_c.

$$P_{n\to n+1} = \frac{x_{ZPF}^2 (n+1)}{\hbar^2} \int_0^t dt' \, S_{FF}(-\Omega).$$

$$= \frac{x_{ZPF}^2 (n+1)}{\hbar^2} S_{FF}(-\Omega).$$

idem $\quad P_{n\to n-1} = \frac{x_{ZPF}^2 n}{\hbar^2} S_{FF}(\Omega).$

Thermal equilibrium :

$$p(1)\,\gamma_{1\to 2} = p(2)\,\gamma_{2\to 1}.$$

$$\frac{p(n+1)}{p(n)} = \frac{S_{FF}(-\Omega)}{S_{FF}(\Omega)}$$

At thermal eq $\Rightarrow$ Bose einstein distrib.

$$p(n) = e^{-\frac{\hbar\Omega n}{\hbar k_B T}} \left[1 - e^{-\hbar\Omega/k_B T}\right]$$

$$\frac{p(n+1)}{p(n)} = e^{\hbar\Omega/k_B T} = 1 + \frac{1}{\bar{n}}.$$

$$T = \frac{\hbar\Omega}{k_B} \left[\ln\left(\frac{S_{FF}(\Omega)}{S_{FF}(-\Omega)}\right)\right]^{-1}$$

$$\bar{n} = \frac{S_{FF}(-\Omega)}{S_{FF}(\Omega) - S_{FF}(-\Omega)}.$$

Classical :

$$S_{xx}(\omega) = \frac{k_B T}{\omega} \, Im\left[\chi_{xx}(\omega)\right]$$

$$\chi_{xx}(\omega) = \frac{1}{m} \frac{}{(\omega^2-\Omega^2) + 2i\Gamma\omega}$$

$$S_{xx}(\Omega) = S_{xx}(-\Omega)$$

$\rightarrow$ the integral gives T.

Quantum :

$$\gamma_{n\to n+1} = \frac{x_{ZPF}^2}{\hbar^2}(n+1)\, S_{FF}(-\Omega)$$

$$\gamma_{n\to n-1} = \frac{x_{ZPF}^2}{\hbar^2}\, n \, S_{FF}(\Omega)$$

$\rightarrow$ the ratio of $S_{FF}(\pm\Omega)$ gives n
(Side band asymetry)