Articulations of algebras and their homological properties

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Abstract. We introduce a new construction of algebras, called articulation, which is a specific gluing of two non-simple algebras. Once this is done, we describe the Auslander-Reiten quiver of an algebra obtained in this way as well as some of its elementary properties. Finally, we characterize the articulated algebras which are laura, left (or right) glued, weakly shod, shod, quasi-tilted or tilted.

Introduction

In the representation theory of algebras, a frequently used technique to study a given algebra is to use the well-understood representation theory of another algebra which is somehow related to the original one. In this sense, many distinct approaches have been used. One of them consists in studying an algebra by using its quotient algebras or some appropriate manipulations of its ordinary quiver (see [**19**], for instance).

In this paper, we use this approach in order to define a new construction of algebras inspired by the works of Igusa, Platzeck, Todorov and Zacharia [**17**], Ringel [**19**] and Lévesque [**18**]. We call *articulated algebras* the algebras obtained in this way (see Section 2 for details). Briefly, an algebra A is said to be articulated if it is obtained by gluing (in a specific way) two non-simple algebras B and C. We are motivated by the fact that, in this case, we have a good description of the representation theory of A in terms of B and C and, moreover, the algebras A and $B \times C$ are stably equivalent (see Section 3). This allows us to present in Section 4 a characterization of the articulated algebras which are laura [**2, 22**], left (or right) glued [**1**], weakly shod [**13**], shod [**11**], quasi-tilted [**15**] or tilted [**16**]. In particular, we show that an articulated algebra is quasi-tilted if and only if it is tilted. Moreover, in this case, the algebra is left and right supported [**3**]. Finally, we end our discussion with an observation on the global dimension of the articulated algebras, see (4.16).

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1. Notations

In this paper, all algebras are basic and connected finite dimensional over an algebraically closed field k. For a quiver Q, we denote by Q_0 the set of points of Q, by Q_1 the set of arrows in Q and by $Q(x,y)$ the set of arrows from x to y. For an algebra A, we denote by mod$A$ the category of finitely generated right A-modules, and by ind$A$ a full subcategory of mod$A$ generated by exactly one representative of each isomorphism classes of indecomposable modules. Moreover, we denote by proj$A$ and inj$A$ respectively the subcategories of projectives and injectives in ind$A$. We denote by pd$X$ and id$X$ the projective dimension and the injective dimension of an A-module X, respectively. Finally, we denote by $\Gamma(\text{mod}A)$ the Auslander-Reiten quiver of the algebra A and by $\tau_A = \text{DTr}$ and $\tau_A^{-1} = \text{TrD}$ the Auslander-Reiten translations. For further definitions or facts needed on mod$A$ and its Auslander-Reiten quiver $\Gamma(\text{mod}A)$, we refer the reader to [**6, 21**].

Given two modules X, Y in ind$A$, a **path from X to Y of length t** in ind$A$ is a sequence of non-zero morphisms

$$(*):\ X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} X_t = Y,$$

where $t \geq 0$ and $X_i \in \text{ind}A$ for each i. In this case, we write $(*) : X \rightsquigarrow Y$ or $X \overset{(*)}{\rightsquigarrow} Y$ and we say that Y is a **successor** of X, while X is a **predecessor** of Y. When each f_i in $(*)$ is irreducible, $(*)$ is a **path of irreducible morphisms**. A path $(*)$ is called a **cycle** in ind$A$ if $X = Y$, $t > 0$ and at least one f_i is a not an isomorphism. A module X in ind$A$ is **directing** if it does not belong to any cycle in ind$A$ and a component Γ of $\Gamma(\text{mod}A)$ is **directed** if it contains only directing modules. Moreover, Γ is called **non-semiregular** if it simultaneously contains a projective module and an injective module. A **refinement** of the path $(*)$ is a path

$$X = X'_0 \xrightarrow{f'_1} X'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_s} X'_s = Y$$

in ind$A$, where $s \geq t$ together with an order-preserving function $\sigma : \{1,\ldots,t-1\} \to \{1,\ldots,s-1\}$ such that $X_i = X'_{\sigma(i)}$ for each i with $1 \leq i \leq t-1$. Finally, if $(*)$ is a path of irreducible morphisms, a module X_i is a **hook** in $(*)$ whenever $\tau_A X_{i+1} = X_{i-1}$. A path of irreducible morphisms is **sectional** if it contains no hook. It follows from [**7**, (3.2)] that if a path $X \rightsquigarrow Y$ in ind$A$ is sectional, then the composition of the morphisms in this path is non-zero, and hence $\text{Hom}_A(X,Y) \neq 0$.

2. Definitions and examples

We recall that any basic algebra over an algebraically closed field k can be seen as a locally bounded k-category, as well as a bound quiver algebra [**8**].

Definition 2.1. *Let (Q_B, I_B) and (Q_C, I_C) be two connected bound quivers containing at least one arrow. A bound quiver (Q_A, I_A) is called* ***articulated along*** *(Q_B, I_B)* ***and*** *(Q_C, I_C), or simply* ***articulated****, if*

1. *Q_B and Q_C are subquivers of Q_A;*
2. *$(Q_A)_0 = (Q_B)_0 \cup (Q_C)_0$ and $(Q_A)_1 = (Q_B)_1 \cup (Q_C)_1$;*
3. *$(Q_B)_0 \cap (Q_C)_0 \neq \emptyset$ and for each $x \in (Q_B)_0 \cap (Q_C)_0$, x is either:*
   (a) *a source of Q_B and a sink of Q_C or;*

(b) *a source of Q_C and a sink Q_B;*

4. $I_A = I_B + I_C + < \overline{\rho} >$, *with $\overline{\rho}$ the set of paths linking $(Q_B)_0 \setminus (Q_C)_0$ and $(Q_C)_0 \setminus (Q_B)_0$ in Q_A.*

If there exist some presentations $A = kQ_A/I_A$ $B = kQ_B/I_B$ and $C = kQ_C/I_C$ such that (Q_A, I_A) is articulated along (Q_B, I_B) and (Q_C, I_C) we say that the algebra A is ***articulated along B and C****.*

If A, B and C are as above, we write $A = (B, C)$ and we say that the pair (B, C) is an **articulation** of A. Moreover, we respectively denote by X_{BC} and X_{CB} the sets of elements in $(Q_B)_0 \cap (Q_C)_0$ satisfying the conditions $3(a)$ and $3(b)$. Finally, an articulation $A = (B, C)$ is said to be **unidirectional** if $X_{BC} = \emptyset$ or $X_{CB} = \emptyset$.

It is an easy checking that, if $A = (B, C)$, then

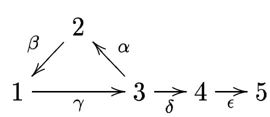
$$A(x,y) = \begin{cases} B(x,y), & \text{if } x, y \in B_0 \text{ and } B(x,y) \neq 0 \\ C(x,y), & \text{if } x, y \in C_0 \text{ and } C(x,y) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

for all $x, y \in A_0$.

This remark shows us that the concept of articulated algebras can be expressed elegantly in terms of categories. However, making use of presentations does not carry any difficulties since it is shown in (3.10) that the study of an articulated algebra is independent of the choice of its presentation. Moreover, this approach will be useful in a forthcoming paper.

It is an easy consequence of the definition that if $A = (B,\ C)$, then $B \cap C$ is semisimple. Moreover, the above definition does not require the algebras B and C to be full or convex subcategories of A. The example 2.2 below illustrates this point.

Example 2.2. Let A be the algebra given by the quiver



bound by the relations $\alpha\beta = 0$, $\gamma\alpha = 0$ $\gamma\delta = 0$ and $\delta\epsilon = 0$. Then A admits the pairs (B_1, C_1) and (B_2, C_2) as articulations, where

- B_1 and C_1 are the full subcategories respectively generated by the sets of points $\{1, 2, 3, 4\}$ and $\{4, 5\}$. In particular, $X_{C_1B_1} = \{4\}$ and $X_{B_1C_1} = \emptyset$, and hence the articulation (B_1, C_1) is unidirectional;
- B_2 and C_2 are the subcategories respectively generated by the sets of arrows $\{\beta, \gamma\}$ and $\{\alpha, \delta, \epsilon\}$. In particular, $X_{C_2B_2} = \{3\}$ and $X_{B_2C_2} = \{2\}$, and hence the articulation (B_2, C_2) is not unidirectional. Moreover, the algebra C_2 is a subcategory of A which is neither full nor convex.

It would be easy to define a concept of *multi-articulated algebras*, which are obtained by a finite number of successive articulations. However, since any multi-articulated algebra is, in particular, articulated, it suffices to study articulated algebras.

3. Auslander-Reiten theory of articulated algebras

In this section, we study the representation theory of an articulated algebra A in function of the algebras involved in an articulation of A. In particular, we describe the shapes of the almost split sequences in mod$A$ and its Auslander-Reiten quiver. Finally, we end this section with some results which will be useful in Section 4.

Our first step is to show that any algebras B and C involved in an articulation are quotient algebras of the articulated algebra they express.

Lemma 3.1. *Let A be an articulated algebra along two algebras B and C. Then, B and C are quotient algebras of A. In particular,* mod$B$ *and* mod$C$ *are full subcategories of* mod$A$.

Proof : We only prove that B is a quotient algebra of A since the proof for C is similar. Let (Q_A, I_A), (Q_B, I_B) and (Q_C, I_C) be presentations of A, B and C, respectively, such that (Q_A, I_A) is articulated along (Q_B, I_B) and (Q_C, I_C). Consider the function ϕ which maps any path ω of Q_A onto the element $\phi(\omega)$ of kQ_B/I_B, where

$$\phi(\omega) = \begin{cases} \omega + I_B, & \text{if } \omega \in kQ_B, \\ 0, & \text{otherwise.} \end{cases}$$

Since the paths of Q_A form a basis of kQ_A, we obtain, by extending ϕ by linearity, a surjective morphism of k-algebras $\phi' : kQ_A \to kQ_B/I_B$: the morphism ϕ' preserves multiplication since any path passing through an articulation point vanishes. We show that $I_A \subseteq \text{Ker}\phi'$. Indeed, assume to the contrary that ρ is a relation in I_A which does not belong to $\text{Ker}\phi'$. Then $\rho \in kQ_B \setminus I_B$, and hence, if x and y are respectively the source and the sink of ρ, then $A(x,y) \neq B(x,y)$ which is a contradiction. Therefore, $I_A \subseteq \text{Ker}\phi'$ and B is a quotient algebra of A. □

We now describe the modules over an articulated algebra, and especially the projective and the injective modules. We begin with the following notations.

Notations 3.2. Let A be an articulated algebra along two algebras B and C. We denote by S_{BC} and S_{CB} the set of simple A-modules corresponding to the points of X_{BC} and X_{CB} respectively. In particular, any element of S_{BC} (or S_{CB}) is a simple injective B-module (or C-module) and a simple projective C-module (or B-module, respectively).

Proposition 3.3. *Let A be an articulated algebra along two algebras B and C. Then,*

(i) $\text{ind}A = \text{ind}B \cup \text{ind}C$.
(ii) $\text{ind}B \cap \text{ind}C = S_{BC} \cup S_{CB}$.
(iii) $\text{proj}A = (\text{proj}B \setminus S_{CB}) \cup (\text{proj}C \setminus S_{BC})$
(iv) $\text{inj}A = (\text{inj}B \setminus S_{BC}) \cup (\text{inj}C \setminus S_{CB})$

Proof : (i). Since the algebras B and C can be obtained from A by deleting some points and arrows, it follows from [**19**, (p.148)] that $\text{ind}B \cup \text{ind}C \subseteq \text{ind}A$.
Conversely, let (Q_A, I_A) be a presentation of A and M be a representation of (Q_A, I_A). Let $M(x)$ be the vector space at the point x and, for each $x \in X_{BC} \cup X_{CB}$, let $L(x) \subseteq M(x)$ be the sum of all the images of the maps $M(\alpha) : M(y) \to M(x)$ induced by the arrows $\alpha : y \to x$ in Q_A.
Consider now the representation M'_B and M''_C of (Q_A, I_A), where M'_B is given by:

$$M'_B\text{:}\quad M'_B(x) = \begin{cases} M(x)/L(x), & \text{if } x \in X_{BC} \\ M(x), & \text{if } x \in B_0 \backslash C_0 \\ L(x), & \text{if } x \in X_{CB} \\ 0, & \text{if } x \in C_0 \backslash B_0 \end{cases}$$

where $M'_B(\alpha) = M(\alpha)|_{M'_B(x)} : M'_B(x) \to M'_B(y)$ for each arrow $\alpha : x \to y$ in Q_A, and M''_C is defined dually.

Then it is easily seen that $M(x) = M'_B(x) \oplus M''_C(x)$ as k-vector spaces for each $x \in A_0$. Moreover, since any composition $M(y) \stackrel{M(\alpha)}{\longrightarrow} M(x) \stackrel{M(\beta)}{\longrightarrow} M(z)$ vanishes when $x \in B_0 \cap C_0$, this yields $M = M'_B \oplus M''_C$ as a representation of (Q_A, I_A). Consequently, if M is indecomposable, then M is either a B-module or a C-module.

(ii). This clearly follows from the fact that $B \cap C$ is a semisimple algebra.

(iii) and (iv). We only prove (iii) since the proof of (iv) is dual. Let P be an indecomposable projective A-module. By (i), $P \in \text{ind}B \cup \text{ind}C$. Assume $P \in \text{ind}B$. Then P is an indecomposable projective B-module because mod$B$ is a full subcategory of mod$A$ by (3.1). On the other hand, if $P \in S_{CB}$, then P is a simple injective C-module. Moreover, $\tau_C P \subseteq \tau_A P = 0$ since C is a quotient algebra of A (see [**21**, (2.4)] for instance). This means that P is a simple projective-injective C-module, a contradiction to the fact that C is a connected non-simple algebra. Therefore, $P \notin S_{CB}$. This shows that $\text{proj}A \subseteq (\text{proj}B \setminus S_{CB}) \cup (\text{proj}C \setminus S_{BC})$

We conclude the proof by showing that the cardinality of the sets involved in the above inclusion coincide. Indeed, $|\text{proj}A| = |A_0| = |B_0 \cup C_0| = |B_0| + |C_0| - |B_0 \cap C_0| = |(\text{proj}B \cup \text{proj}C) \backslash (S_{BC} \cup S_{CB})| = |(\text{proj}B \backslash S_{CB}) \cup (\text{proj}C \backslash S_{BC})|$, where the last equality comes from the fact that B and C are both connected and non-simple algebras. □

Our next step is to describe the almost split sequences in mod$A$ when A is an articulated algebra.

Lemma 3.4. *Let A be an articulated algebra along two algebras B and C and let X be an indecomposable B-module. Then,* $\tau_A X = \begin{cases} \tau_B X, & \textit{if } X \notin S_{CB}, \\ \tau_C X, & \textit{otherwise.} \end{cases}$

Proof : Assume $X \notin S_{CB}$. First, if $X \in \text{proj}B$, then $X \in \text{proj}A$ by (3.3) and hence $\tau_A X = 0 = \tau_B X$. Otherwise, since B is a quotient algebra, we have $0 \neq \tau_B X \subseteq \tau_A X$ (see [**21**, (2.4)], for instance), and hence $\tau_A X \in \text{ind}B$. Indeed, if $\tau_A X \in \text{ind}C$, then $\tau_B X \in S_{BC} \cup S_{CB}$, that is $\tau_B X \in S_{CB}$ since $\tau_B X$ is not injective in mod$B$. This yields $\tau_A X = \tau_B X \in \text{ind}B$.
On the other hand, if $X \in S_{BC}$, then X is an indecomposable C-module which does not belong to S_{CB}, and the result follows from the dual of the first part. □

We recall the following fact from [**6**, (p.186)]: let $\overline{A} = A/I$ be a quotient algebra of A and $0 \to \tau_A X \to Y \to X \to 0$ be an almost split sequence in mod$A$, with both X and $\tau_A X$ indecomposable $\overline{A}$-modules. Then this sequence is almost split in mod$\overline{A}$. In particular, $\tau_{\overline{A}} X = \tau_A X$. We deduce the following proposition.

Proposition 3.5. *Let A be an articulated algebra along two algebras B and C. Then, any almost split sequence in* mod$A$ *is an almost split sequence in* mod$B$ *or in* mod$C$*, and conversely.*

Proof : Let $(*) :\ 0 \to \tau_A X \to Y \to X \to 0$ be an almost split sequence in mod$A$. In particular, X is not projective in mod$A$, and hence it is an easy consequence of (3.3)

that $X \in \text{ind}B \setminus \text{proj}B$ or $X \in \text{ind}C \setminus \text{proj}C$. Assume without loss of generality that $X \in \text{ind}B \setminus \text{proj}B$. Then, it follows from (3.4) that $\tau_A X = \tau_B X$ and $(*)$ is an almost split sequence in $\text{mod}B$.
Conversely, assume $(*)$: $0 \to \tau_B X \to Y \to X \to 0$ is an almost split sequence in $\text{mod}B$. In particular, $X \in \text{ind}B \setminus \text{proj}B$, and hence $(*)$ is an almost split sequence in $\text{mod}A$ by (3.4). □

Corollary 3.6. *Let A be an articulated algebra along two algebras B and C. Then a morphism $f : M \to N$ is irreducible in* $\text{mod}A$ *if and only if it is irreducible in* $\text{mod}B$ *or in* $\text{mod}C$. □

Lemma 3.7. *Let A be an articulated algebra along two algebras B and C and let S be a simple module in $S_{BC} \cup S_{CB}$. Then*

(i) *there exists an indecomposable A-module P together with an irreducible map $f : S \to P$, and any such module is projective.*
(ii) *there exists an indecomposable A-module I together with an irreducible map $f : I \to S$, and any such module is injective.*

Proof : We only prove (i) since the proof of (ii) is dual. Assume $S \in S_{BC}$. In particular, S is a simple projective non-injective C-module. Moreover, S is not an injective A-module since $0 \neq \tau_C^{-1} X = \tau_A^{-1} X$ by the dual of (3.4). Therefore, let P be an indecomposable A-module together with an irreducible map $f : S \to P$. Since f is not irreducible in $\text{mod}B$ (because S is a simple injective B-module), we infer from (3.6) that f is irreducible in $\text{mod}C$. Therefore, P is an indecomposable projective C-module which is not in S_{BC} since f is not an isomorphism. By (3.3), this implies that P is a projective A-module. The proof is similar when $S \in S_{CB}$. □

Remark 3.8. *Let I_B be an indecomposable injective B-module. Then there exists an injective A-module I'_A and a path in* $\text{ind}A$ *from I'_A to I_B.*

Proof : This follows from (3.3) and (3.7). □

As consequence, we deduce the shape of the Auslander-Reiten quiver of an articulated algebra $A = (B, C)$ where we identify the modules in S_{CB} (see Fig 1).

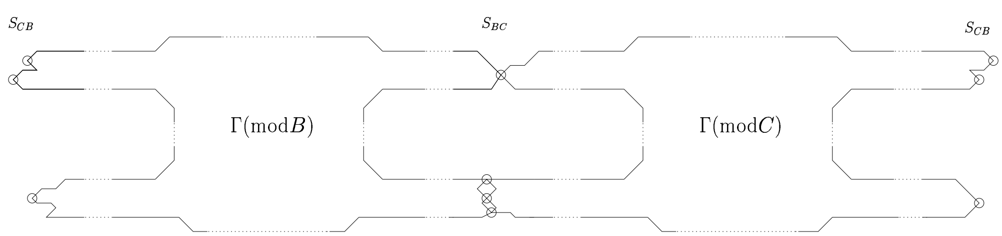


Figure 1. $\Gamma(\text{mod}A)$

In fact, we have a stronger result. We denote by $\Gamma(M, N)$ the set of arrows from M to N in the quiver Γ.

Theorem 3.9. *An algebra A is an articulated algebra along two algebras B and C if and only if:*

(i) $\Gamma(\text{mod}B)_0 \cup \Gamma(\text{mod}C)_0 = \Gamma(\text{mod}A)_0$;

(ii) $$\Gamma(\text{mod}A)(M,N) = \begin{cases} \Gamma(\text{mod}B)(M,N) & \text{if } M,\ N \in (\Gamma(\text{mod}B))_0 \text{ and } \\ & \Gamma(\text{mod}B)(M,N) \neq \emptyset, \\ \Gamma(\text{mod}C)(M,N) & \text{if } M,\ N \in (\Gamma(\text{mod}C))_0 \text{ and } \\ & \Gamma(\text{mod}C)(M,N) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

(iii) *For each $M \in \Gamma(\text{mod}B)_0 \cap \Gamma(\text{mod}C)_0$, M is a sink of $\Gamma(\text{mod}B)$ and a source of $\Gamma(\text{mod}C)$ or M is a sink of $\Gamma(\text{mod}C)$ and a source of $\Gamma(\text{mod}B)$.*

(iv) *For each $M \in \Gamma(\text{mod}A)_0$,*

$$\tau_A(M) = \begin{cases} \tau_B(M) & \text{if } M \in (\Gamma(\text{mod}B))_0 \text{ and } \tau_B(M) \neq 0, \\ \tau_C(M) & \text{if } M \in (\Gamma(\text{mod}C))_0 \text{ and } \tau_C(M) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof : It is a tedious and straightforward proof. □

Corollary 3.10. *An algebra A is an articulated algebra along two algebras B and C if and only if all its presentations are articulated along some presentations of B and C.* □

We end this section with the following useful result.

Proposition 3.11. *Let A be an articulated algebra along two algebras B and C. For each non-zero morphism $f : M \to N$ in $\text{ind}A$, where $M \in \text{ind}B$ and $N \in \text{ind}C$, either M belongs to S_{CB} and $M = N$ or f factors through a module in $\text{add}(S_{BC})$. Moreover, if f is irreducible, then either $M \in S_{BC}$ or $N \in S_{BC}$.*

Proof : Let $f : M \to N$ be a non-zero morphism as above. Clearly, $\text{Im}f \in \text{mod}B \cap \text{mod}C$, and so $\text{Im}f \in \text{add}(S_{BC} \cup S_{CB})$. Of course, if $\text{Im}f \in \text{add}(S_{BC})$, then f factors through a module in $\text{add}(S_{BC})$ and there is nothing to prove. Otherwise, there exists a simple A-module S in S_{CB} such that S is a direct summand of $\text{Im}f$. This implies that there is a direct summand L of $\text{top}M$ such that L is a projective B-module, and hence $L = S = M$ (because a strict quotient of an indecomposable module cannot be projective). In the same way, $L = S = N$ and then $M = N \in S_{CB}$. Finally, the last statement follows from the fact that if f is irreducible and

$$f:\ M \xrightarrow{p} \text{Im}f \xrightarrow{i} N$$

is the canonical factorisation of f, then either p is a section or i is a retraction. □

We also have the following result, suggested by the shape of the Auslander-Reiten quiver of $A = (B, C)$. We first recall that two algebras A and B are called **stably equivalent** if there exists an equivalence of categories between $\text{mod}A/[PA]$ and $\text{mod}B/[PB]$, where $[PA]$ (or $[PB]$) is the ideal of $\text{mod}A$ (or $\text{mod}B$) generated by the morphisms of $\text{mod}A$ (or $\text{mod}B$) which factor through a projective A-module (or B-module, respectively).

Proposition 3.12. *Let A be an articulated algebra along two algebras B and C. Then A is stably equivalent to $B \times C$.*

Proof : We define the map

$$\begin{array}{llll} \Phi: & \underline{\text{mod}}B \times \underline{\text{mod}}C & \longrightarrow & \underline{\text{mod}}A \\ & (M_B, M'_C) & \longmapsto & M_B \oplus M'_C \\ & (\underline{f} : M_B \mapsto N_B, \underline{g} : M'_C \mapsto N'_C) & \longmapsto & \underline{f}|_{M_B \setminus S_{CB}} \oplus \underline{g}|_{M'_C \setminus S_{BC}} \end{array}$$

where $M_B \setminus S_{CB} = \oplus\{L_i | L_i$ is an indecomposable direct summand of M_B such that

$L_i \notin S_{CB}\}$ and $M_C \setminus S_{BC}$ is defined similarly.
A trivial checking shows that this map yields a well defined functor which is full, faithful and dense. □

4. Homological properties

In this section, we investigate the homological properties of the articulated algebras. More precisely, since the classes of laura algebras, left (or right) glued algebras, weakly shod algebras, shod algebras, quasi-tilted algebras and tilted algebras nowadays play a crucial role in the representation theory of algebras, it is reasonable to ask when an articulated algebra belongs to one of those families. In particular, we show that an articulated algebra A is quasi-tilted if and only if it is A tilted. Moreover, in this case, A is left and right supported. We observe that no articulated algebra is hereditary since every non trivial path passing through an articulation point vanishes.

4.1. $\mathcal{L}_A$ and $\mathcal{R}_A$. We start with some definitions and notations. Given an articulated algebra $A = (B, C)$ and $S \in S_{BC} \cup S_{CB}$, we denote by Γ_A^S (or Γ_B^S, or Γ_C^S) the connected component of $\Gamma(\text{mod}A)$ (or $\Gamma(\text{mod}B)$, or $\Gamma(\text{mod}C)$, respectively) containing S. In addition, given a path of morphisms $X \rightsquigarrow Y$, we sometimes denote it by $X \underset{A}{\rightsquigarrow} Y$ (or $X \underset{B}{\rightsquigarrow} Y$, or $X \underset{C}{\rightsquigarrow} Y$) to underline the fact that it is a path in ind$A$ (or ind$B$, or ind$C$, respectively). Finally, we say that a path is **sectionally refinable** in ind$A$ (or ind$B$, or ind$C$) if this path can be refined to a path of irreducible morphisms in ind$A$ (or ind$B$, or ind$C$, respectively), and any such refinement is sectional. The following lemma gives more precision on sectionally refinable paths.

Lemma 4.1. *Let A be an articulated algebra along two algebras B and C and X, Y be two modules in* ind$B$ *(or* ind$C$*). Let $(*) : X \rightsquigarrow Y$ be a path in* ind$B$ *(or* ind$C$*). If $(*)$ is sectionally refinable in* ind$A$*, then it is sectionally refinable in* ind$B$ *(or* ind$C$*, respectively).*

Proof : We only prove it in the case $(*)$ is a path in ind$B$ since the proof for ind$C$ is similar.
Let $(*) : X \rightsquigarrow Y$ be a path in ind$B$ which is sectionally refinable in ind$A$. In order to prove that $(*)$ is sectionally refinable in ind$B$, it suffices to show that any irreducible refinement of $(*)$ in ind$A$ lies in ind$B$. But this follows from (3.11) and the fact that there is no non-trivial sectional path between to A-simple modules. □

Following [**15**], for an artin algebra A, we denote by $\mathcal{L}_A$ and $\mathcal{R}_A$ the following full subcategories of ind$A$:

$$\mathcal{L}_A = \{X \in \text{ind}A \mid \text{pd}_A Y \leq 1 \text{ for each predecessor } Y \text{ of } X\}$$

$$\mathcal{R}_A = \{X \in \text{ind}A \mid \text{id}_A Y \leq 1 \text{ for each successor } Y \text{ of } X\}$$

The classes $\mathcal{L}_A$ and $\mathcal{R}_A$ are respectively called the **left part** and the **right part** of mod$A$. Clearly, $\mathcal{L}_A$ is closed under predecessors and $\mathcal{R}_A$ is closed under successors.

For the sake of brevity, we refrain from now on from stating the dual of each statement and leave the primal-dual translation to the reader. We recall the following key lemma:

Lemma 4.2. [**2**, (1.6)] *Let A be an artin algebra. Then $\mathcal{L}_A$ consists of the modules $N \in \text{ind}A$, such that, if there exists a path in* ind$A$ *from an injective module to N, then this path is sectionally refinable in* ind$A$. □

Proposition 4.3. *Let A be an articulated algebra along two algebras B and C. Then, $\mathcal{L}_A \subseteq \mathcal{L}_B \cup \mathcal{L}_C$.*

Proof : First, let $M \in \mathcal{L}_A$ and suppose, without loss of generality, that $M \in \text{ind}B$. We show that $M \in \mathcal{L}_B$. So, in the spirit of (4.2), assume $(*) : I_B \underset{B}{\rightsquigarrow} M$ is a path in ind$B$ from an injective B-module I_B to M. We must show that $(*)$ is sectionally refinable in ind$B$. If I_B is an injective A-module, then $(*)$ is sectionally refinable in ind$A$ (and so in ind$B$ by (4.1)) since $M \in \mathcal{L}_A$. Otherwise, it follows from (3.3) that $I_B \in S_{BC}$, so that I_B is a simple injective B-module and that the path $(*)$ is trivial since it lies in ind$B$. Hence, $M \in \mathcal{L}_B$. Since a similar argument holds when $M \in \text{ind}C$, this yields $\mathcal{L}_A \subseteq \mathcal{L}_B \cup \mathcal{L}_C$. □

4.2. Articulated laura algebras. In this section, we present a characterization of articulated algebras which are laura. Recall from [**2**] that an algebra A is called a **laura** algebra if $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in ind$A$. We need the following two lemmata.

Lemma 4.4. *Let A be a laura algebra and assume that Γ is a non-semiregular component of* $\Gamma(\text{mod}A)$. *Then, for all $X, Y \in \Gamma$, the set*

$$\mathcal{M}_A^{X,Y} = \{Z \in \text{ind}A \mid \text{there exists a path } X \rightsquigarrow Z \rightsquigarrow Y \text{ in ind}A\}$$

is finite.

Proof : Clearly, $\mathcal{M}_A^{X,Y} = \mathcal{L}_A^{X,Y} \cup \mathcal{N}_A^{X,Y} \cup \mathcal{R}_A^{X,Y}$, where $\mathcal{L}_A^{X,Y} = \mathcal{L}_A \cap \mathcal{M}_A^{X,Y}$, $\mathcal{R}_A^{X,Y} = \mathcal{R}_A \cap \mathcal{M}_A^{X,Y}$ and $\mathcal{N}_A^{X,Y} = \mathcal{M}_A^{X,Y} \setminus (\mathcal{L}_A^{X,Y} \cup \mathcal{R}_A^{X,Y})$. Since the sets $\mathcal{L}_A^{X,Y}$ and $\mathcal{R}_A^{X,Y}$ are finite by [**23**, (3.9)] and its dual, and $\mathcal{N}_A^{X,Y}$ is finite since A is laura, then so is $\mathcal{M}_A^{X,Y}$. □

The previous result is well known when A is a quasi-tilted algebra and Γ is a connecting component of $\Gamma(\text{mod}A)$ since Γ is convex, generalized standard and can be embedded in a translation quiver of the form $\mathbb{Z}\Delta$, where Δ is a finite and acyclic quiver. This observation plays a crucial role in the proof of (4.6).

Lemma 4.5. *Assume A is an articulated laura algebra along two algebras B and C. Then, for each $S \in S_{BC} \cup S_{CB}$, the component Γ_B^S (or Γ_C^S) is a non-semiregular component of* $\Gamma(\text{mod}B)$ *(or* $\Gamma(\text{mod}C)$*), or is such that $\Gamma_B^S \cap \mathcal{L}_B \neq \emptyset$ and $\Gamma_B^S \cap \mathcal{R}_B \neq \emptyset$ (or $\Gamma_C^S \cap \mathcal{L}_C \neq \emptyset$ and $\Gamma_C^S \cap \mathcal{R}_C \neq \emptyset$, respectively).*

Proof : We only prove the statement for Γ_B^S since the proof for Γ_C^S is similar.
Let $S \in S_{BC} \cup S_{CB}$. If $S \in S_{BC}$, then S is a simple injective B-module, and hence $\Gamma_B^S \cap \mathcal{R}_B \neq \emptyset$. To prove our claim, it then suffices to show that Γ_B^S is non-semiregular when $\Gamma_B^S \cap \mathcal{L}_B = \emptyset$. Suppose to the contrary that $\Gamma_B^S \cap \mathcal{L}_B = \emptyset$ and Γ_B^S contains no projective B-module. Then, for all $t \geq 1$, there exists a path of irreducible morphisms

$$\tau_B^t S \to * \to \tau_B^{t-1} S \to \ldots \to \tau_B S \to * \to S$$

where $\tau_B^i S \neq \tau_B^j S$ when $i \neq j$ since S is not periodic. In particular, $\tau_B^t S \notin \mathcal{L}_B$, and so it follows from (4.2) that there exists a path

$$I \rightsquigarrow \tau_B^t S \to * \to \tau_B^{t-1} S \to \ldots \to \tau_B S \to * \to S$$

in ind$A$, where I is an injective B-module. Applying (3.7) and (3.8), this gives a path

$$I' \rightsquigarrow I \rightsquigarrow \tau_B^t S \to * \to \tau_B^{t-1} S \to \ldots \to \tau_B S \to * \to S \to P'$$

in ind$A$, where I' is an injective A-module and P' a projective A-module. In particular, $\tau_B^t S, \tau_B^{t-1} S, \ldots, \tau_B S$ do not belong to $\mathcal{L}_A \cup \mathcal{R}_A$. Since t can be arbitrarily large, this contradicts the fact that A is laura. Since similar arguments hold when $S \in S_{CB}$, this concludes the proof. $\square$

Theorem 4.6. *Let A be an articulated algebra along two algebras B and C. Then A is laura if and only if:*

(i) *B and C are laura algebras;*
(ii) *$S_{BC} \cup S_{CB} \subseteq \Gamma_B \cap \Gamma_C$, where Γ_B and Γ_C are non-semiregular or connecting components of $\Gamma(\text{mod}B)$ and $\Gamma(\text{mod}C)$ respectively.*

Proof : *Necessity.* We only prove the statement for B since the proof for C is similar. Assume A is laura and let M be such that $M \in \text{ind}B \setminus (\mathcal{L}_B \cup \mathcal{R}_B)$. Since $M \notin \mathcal{L}_B$, there exists by (4.2) a non sectionally refinable path $I \rightsquigarrow M$ in ind$B$, where I is an indecomposable injective B-module. By (3.8), this path can be extended (maybe trivially) to a path $I' \rightsquigarrow I \rightsquigarrow M$ in ind$A$, where I' is an indecomposable injective A-module. Since this path is non sectionally refinable in ind$A$ by (4.1), we infer from (4.2) that $M \notin \mathcal{L}_A$. Dually, $M \notin \mathcal{R}_B$ implies $M \notin \mathcal{R}_A$. Consequently, $\text{ind}B \setminus (\mathcal{L}_B \cup \mathcal{R}_B) \subseteq \text{ind}A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$, and then B is laura since so is A.
In order to show (ii), we consider two cases:
Firstly, if B is laura not quasi-tilted, then Γ_B^S is the unique non-semiregular component of $\Gamma(\text{mod}B)$ for all $S \in S_{BC} \cup S_{CB}$ by (4.5) and [**2**, (4.6)] and (ii) holds. Otherwise, B is quasi-tilted and it follows from (4.5) and [**14**, (6.5)] that Γ_B^S is a connecting component of $\Gamma(\text{mod}B)$ for each $S \in S_{BC} \cup S_{CB}$. It remains to show that those components coincide. Since this is trivial when $S_{BC} = \emptyset$, $S_{CB} = \emptyset$ or B is representation-finite, it suffices to show that B is not concealed whenever $S_{BC} \neq \emptyset$, $S_{CB} \neq \emptyset$ and B is representation-infinite. Suppose to the contrary that B is concealed. In particular $\Gamma(\text{mod}B)$ admits a postprojective connecting component $\Gamma_p (\supseteq S_{CB})$ and a preinjective connecting component $\Gamma_i (\supseteq S_{BC})$, with $\Gamma_p \neq \Gamma_i$. Now, choose $S_2 \in S_{CB}$ and let (I_2, f) be its injective envelope. Clearly $f \in \text{rad}_B^\infty(S_2, I_2)$ and then it follows from [**20**, (2.1)] that, for all $t \geq 1$, there exists a path

$$S_2 \xrightarrow{h_t} J_t \xrightarrow{g_t} J_{t-1} \xrightarrow{g_{t-1}} \cdots \xrightarrow{g_2} J_1 \xrightarrow{g_1} J_0 = I_2$$

in ind$B$, where $g_t, g_{t-1}, \ldots, g_1$ are irreducible morphisms and $h_t \in \text{rad}_B^\infty(S_2, J_t)$. Therefore, since Γ_i contains only finitely many τ_B-orbits and no cycles, there exists a $t' \geq 1$ such that $J_{t'}$ is a predecessor of S_1, for some S_1 in S_{BC}. Now, applying [**20**, (2.1)] to $h_{t'}$, this yields, for all $s \geq 1$, a path of the form

$$S_2 = T_0 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \cdots \xrightarrow{f_s} T_s \xrightarrow{k_s} J_{t'} \rightsquigarrow S_1$$

and so, by (3.7), a path in ind$A$

$$I \longrightarrow S_2 = T_0 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \cdots \xrightarrow{f_s} T_s \xrightarrow{k_s} J_{t'} \rightsquigarrow S_1 \longrightarrow P$$

where I is an indecomposable injective A-module and P an indecomposable projective A-module. But then, since Γ_p has only finitely many τ_B-orbits and s can be arbitrarily large, this gives arbitrarily many modules in ind$A$ which, by (4.2) do not belong to $\mathcal{L}_A \cup \mathcal{R}_A$, a contradiction. Therefore, B is not concealed and the necessity is proven.

Sufficiency. Assume that B and C satisfy the conditions of the statement, and that $M \in \text{ind}A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$. Moreover, assume, without loss of generality, that $M \in \text{ind}B$. Then, there are three possible situations:
First, if $M \in \mathcal{L}_B \setminus \mathcal{R}_B$, then there exists a projective B-module and a non sectionally refinable path $M \rightsquigarrow P$ in ind$B$. On the other hand, since $M \in \mathcal{L}_B$ but $M \notin \mathcal{L}_A$, there exists a non sectionally refinable path $I \rightsquigarrow M$ in ind$A$, where I is an injective A-module. Applying (3.11) this gives a path $S_2 \rightsquigarrow M$ in ind$B$ for some $S_2 \in S_{CB}$, and so a path $S_2 \rightsquigarrow M \rightsquigarrow P$ in ind$B$. This means, in the notation of (4.4), that $M \in \mathcal{M}_B^{S_2,P}$. Secondly, if $M \in \mathcal{R}_B \setminus \mathcal{L}_B$, then dually $M \in \mathcal{M}_B^{I,S_1}$, where I is an injective B-module and $S_1 \in S_{BC}$. Thirdly, it is possible that $M \notin \mathcal{L}_B \cup \mathcal{R}_B$.
Thus, since the situation is similar when $M \in \text{ind}C$, we obtain, by (4.4) and the assumption that B and C are laura algebras, that ind$A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ is contained in a finite union of finite sets, saying that A is laura. □

4.3. Left and right glued articulated algebras. We now investigate the case where an articulated algebra is left or right glued. We recall (see [**2**, (2.2)]) that an algebra A is **left glued** if and only if ind$A \setminus \mathcal{R}_A$ is finite. We characterize dually the **right glued** algebras. In particular, any left or right glued algebra is laura. Moreover, a connected component Γ of $\Gamma(\text{mod}A)$ is called a **π-component** [**10**] if (i) almost all modules in Γ lie in the τ-orbit of a projective module; and (ii) at most finitely many modules in Γ belong to cycles. Then, it follows from [**1**] that if A is a left glued algebra, then a component Γ of $\Gamma(\text{mod}A)$ is a π-component if and only if it contains projective modules. We refer the reader to [**1**] for more details on those families of algebras.

Corollary 4.7. *Let A be an articulated algebra along two algebras B and C. Then A is left glued if and only if:*

(i) *B and C are left glued algebras;*
(ii) *$S_{BC} \cup S_{CB} \subseteq \Gamma_B \cap \Gamma_C$, where Γ_B and Γ_C are π-components of $\Gamma(\text{mod}B)$ and $\Gamma(\text{mod}C)$ respectively.*

Proof : *Necessity.* We only prove the statement for B since the proof for C is similar. Assume A is left glued and let M be such that $M \in \text{ind}B \setminus \mathcal{R}_B$. Since $\mathcal{R}_A \subseteq \mathcal{R}_B \cup \mathcal{R}_C$, this means that $M \notin \mathcal{R}_A$ or $M \in \mathcal{R}_C$, that is $M \in \text{ind}A \setminus \mathcal{R}_A$, or $M \in \text{ind}B \cap \text{ind}C = S_{BC} \cup S_{CB}$. Hence, ind$B \setminus \mathcal{R}_B \subseteq (\text{ind}A \setminus \mathcal{R}_A) \cup (S_{BC} \cup S_{CB})$, and so B is left glued. Now, by (4.6), $S_{BC} \cup S_{CB} \subseteq \Gamma_B$, where Γ_B is a non-semiregular or a connecting component of $\Gamma(\text{mod}B)$. We claim that Γ_B is a π-component of $\Gamma(\text{mod}B)$, that is Γ_B contains projective B-modules. Indeed, let $S \in S_{BC} \cup S_{CB}$. First, if $S \in S_{CB}$, then S is a simple projective B-module and there is nothing to show since $\Gamma_B = \Gamma_B^S$. Otherwise, assume $S \in S_{BC}$ but $\Gamma_B (= \Gamma_B^S)$ contains no projective B-modules. Then, for all $t \geq 1$, there is a path of irreducible morphisms of the form

$$\tau_A^t S \to * \to \tau_A^{t-1} S \to \ldots \to \tau_A S \to * \to S$$

where $\tau_A^i S \neq \tau_A^j S$ whenever $i \neq j$ since S is not periodic. But then, since this path can be right-extended to a path from S to a projective A-module by (3.7), this implies that $\tau_A^i S \notin \mathcal{R}_A$ for all i such that $1 \leq i \leq t$. Since t can be arbitrarily large, this contradicts the fact that A is left glued. Therefore, Γ_B contains projective modules and hence is a π-component of $\Gamma(\text{mod}B)$. This proves the necessity.

Sufficiency. First observe that A is a laura algebra by (4.6) since Γ_B and Γ_C are non-semiregular or connecting components of $\Gamma(\mathrm{mod}B)$ and $\Gamma(\mathrm{mod}C)$ respectively by [**1**]. We now prove that A is left glued by showing that the set $\mathrm{ind}A \setminus \mathcal{R}_A$ lies in a finite union of finite sets. So, assume $M \in \mathrm{ind}A \setminus \mathcal{R}_A$ and, without loss of generality, that $M \in \mathrm{ind}B$. If $M \notin \mathcal{R}_B$, then $M \in \mathrm{ind}B \setminus \mathcal{R}_B$ which is a finite set. Otherwise, $M \in \mathcal{R}_B$ and, since $M \notin \mathcal{R}_A$ by assumption, there exists by the dual of (4.2) a non sectionally refinable path $(*) : M \rightsquigarrow P$ in $\mathrm{ind}A$, where P is a projective A-module, which is not a path in $\mathrm{ind}B$ (otherwise this path would be sectionally refinable since $M \in \mathcal{R}_B$). Following (3.11), this path can be refined to a path of the form $M \underset{B}{\rightsquigarrow} S_1 \rightsquigarrow P$, where $S_1 \in \mathcal{S}_{BC}$ and the path $M \underset{B}{\rightsquigarrow} S_1$ is a path in $\mathrm{ind}B$. Moreover, let P' be an indecomposable direct summand of the projective cover of M_B. Using the fact that any π-component is closed under predecessors by [**9**, (1.2)] and [**23**, (1.1)], this implies that $P' \in \Gamma_B (= \Gamma_B^{S_1})$. This means $M \in \mathcal{M}_B^{P',S_1}$, which is a finite set by (4.4). Since the situation is similar when $M \in \mathrm{ind}C$, this gives that $\mathrm{ind}A \setminus \mathcal{R}_A$ is contained in a finite union of finite sets, and therefore A is left glued. □

4.4. Weakly shod articulated algebras. Another specific family of laura algebras is the one of weakly shod algebras introduced in [**13**] (see also [**12**]). Recall that, following [**2**, (4.8)] and [**13**, (2.5)], an algebra A is **weakly shod** if A is a laura algebra such that any indecomposable module lying in a non-semiregular component of $\Gamma(\mathrm{mod}A)$ (which is unique if it exists) is directing.

This leads to the following characterization of articulated weakly shod algebras.

COROLLARY 4.8. *Let A be an articulated algebra along two algebras B and C. Then A is weakly shod if and only if:*

(i) *B and C are weakly shod algebras;*
(ii) *$\mathcal{S}_{BC} \cup \mathcal{S}_{CB} \subseteq \Gamma_B \cap \Gamma_C$, where Γ_B and Γ_C are non-semiregular or connecting components of $\Gamma(\mathrm{mod}B)$ and $\Gamma(\mathrm{mod}C)$ respectively;*
(iii) *Each module in $\mathcal{S}_{BC} \cup \mathcal{S}_{CB}$ is directing.*

Proof : *Necessity.* We only prove the statement for B since the proof for C is similar. Assume A weakly shod. First, by (4.6), the condition (ii) holds and B is a laura algebra, and therefore weakly shod since any non-semiregular component of $\Gamma(\mathrm{mod}B)$ is embedded in a non-semiregular component of $\Gamma(\mathrm{mod}A)$. This means that (i) also holds. Finally, (iii) clearly holds since if $S \in \mathcal{S}_{BC} \cup \mathcal{S}_{CB}$, then Γ_A^S is non-semiregular by (3.7).

Sufficiency. Since A is a laura algebra by (4.6), let $\Gamma_A = \Gamma_B \cup \Gamma_C$ be the unique non-semiregular component of $\Gamma(\mathrm{mod}A)$ and assume it contains a module X lying on a cycle δ in $\mathrm{ind}A$. Without loss of generality, assume $X \in \Gamma_B$. Since B is weakly shod and Γ_B is a connecting or a non-semiregular component of $\Gamma(\mathrm{mod}B)$ (that is a directed component), the cycle δ is not a cycle in $\mathrm{ind}B$. Consequently, (3.11) yields that δ can be refined to a path of the form $X \rightsquigarrow S_1 \rightsquigarrow S_2 \rightsquigarrow X$ for some $S_1 \in \mathcal{S}_{BC}$ and $S_2 \in \mathcal{S}_{CB}$, a contradiction to (iii). Since the same contradiction holds when X is in Γ_C, we infer that A is weakly shod. □

4.5. Shod articulated algebras. The next step in the study of laura algebras is the investigation of shod algebras, introduced in [**11**] as a generalization of quasi-tilted algebras. Recall that an algebra A is called **shod** if it satisfies one of the following equivalent conditions :

(i) $\mathcal{L}_A \cup \mathcal{R}_A = \text{ind}A$;
(ii) Any path from an indecomposable injective module to an indecomposable projective module can be refined to a path of irreducible maps, any such refinement has at most two hooks, and, in case there are two, they are consecutive.

Although this definition shows that shod algebras appear in a very natural way, condition (ii) also underlines that shod algebras can be hard to manage in an elegant manner. For this reason, we say that a path $(*) : X \rightsquigarrow Y$ in ind$A$ is **shod** if the condition (ii) holds for $(*)$, that is if $(*)$ can be refined to a path of irreducible maps, any such refinement has at most two hooks, and, in case there are two, they are consecutive. In particular, this means that an algebra A is shod if and only if each path in ind$A$ from an injective module to a projective module is shod. Then we get the following characterization of shod articulated algebras.

Proposition 4.9. *Let A be an articulated algebra along two algebras B and C. Then A is shod if and only if:*

(i) *B and C are shod algebras;*
(ii) *Each $S_1 \in S_{BC}$ satisfies at least one of the following conditions:*
 (a) *$S_1 \in \mathcal{L}_B$ and any path $S_1 \rightsquigarrow P_C$ or $S_1 \rightsquigarrow S_2$, where P_C is a projective C-module and $S_2 \in S_{CB}$ is shod. Moreover, in the latter case, $S_2 \in \mathcal{R}_B$ and there is no path from S_2 to a module in S_{BC};*
 (b) *$S_1 \in \mathcal{R}_C$, any path $I_B \rightsquigarrow S_1$, where I_B is an injective B-module, is shod and there is no path from S_1 to a module in S_{CB}.*
(iii) *Each $S_2 \in S_{BC}$ satisfies at least one of the following conditions:*
 (a) *$S_2 \in \mathcal{L}_C$ and any path $S_2 \rightsquigarrow P_B$ or $S_2 \rightsquigarrow S_1$, where P_B is a projective B-module and $S_1 \in S_{BC}$ is shod. Moreover, in the latter case, $S_1 \in \mathcal{R}_C$ and there is no path from S_1 to a module in S_{CB};*
 (b) *$S_2 \in \mathcal{R}_B$, any path $I_C \rightsquigarrow S_2$, where I_C is an injective C-module, is shod and there is no path from S_2 to a module in S_{BC}.*

Proof : *Necessity.* (i). Since any path in ind$B$ is a path in ind$A$, then it is easily seen that B is shod. Similarly, C is shod.
We now only prove (ii) since the proof of (iii) is similar.
Let $S_1 \in S_{BC}$. We first show that $S_1 \in \mathcal{L}_B \cup \mathcal{R}_C$. In fact, if $S_1 \notin \mathcal{L}_B \cup \mathcal{R}_C$, then it follows from (4.2) and its dual that there exist non sectionally refinable paths $I_B \overset{\phi_B}{\rightsquigarrow} S_1$ and $S_1 \overset{\phi_C}{\rightsquigarrow} P_C$ in ind$A$, and hence a path $I_B \overset{\phi_B}{\rightsquigarrow} S_1 \overset{\phi_C}{\rightsquigarrow} P_C$ in ind$A$ which is not shod. Since this path can be extended to a path in ind$A$ from an injective A-module to a projective A-module by (3.8) and its dual, this contradicts the fact that A is shod. Consequently, $S_1 \in \mathcal{L}_B \cup \mathcal{R}_C$.
We claim that if $S_1 \in \mathcal{L}_B$, then S_1 satisfies the condition (ii)(a). Indeed, assume $S_1 \in \mathcal{L}_B$ and let $(*)$ be a path of the form $S_1 \rightsquigarrow P_C$, where P_C is a projective C-module, or of the form $S_1 \rightsquigarrow S_2$, where $S_2 \in S_{CB}$. In both cases, the path $(*)$ is shod since it can be extended by (3.7) and (3.8) to a path from an injective A-module to a projective A-module and A is shod. Moreover, in the latter case, if $S_2 \notin \mathcal{R}_B$, then by (4.2) there exists a non sectionally refinable path $S_2 \rightsquigarrow P_B$ in ind$B$, where P_B is a projective B-module. This yields a non shod path $S_1 \overset{(*)}{\rightsquigarrow} S_2 \rightsquigarrow P_B$ in ind$A$

(since $(*)$ cannot be sectional) which is a contradiction by (3.7). Therefore S_1 satisfies condition (ii)(a).
There remains to show that if $S_1 \in \mathcal{R}_C \setminus \mathcal{L}_B$, then S_1 satisfies the condition (ii)(b). The first part easily follows from (3.7) and the fact that A is shod. On the other hand, assume there exists a path $S_1 \rightsquigarrow S_2$ in ind$A$, where $S_2 \in \mathcal{S}_{CB}$. Since $S_1 \notin \mathcal{L}_B$ by assumption, there exists a non sectionally refinable path $I_B \rightsquigarrow S_1$ in ind$B$ (and hence in ind$A$ by (4.1)). This yields a non shod path $I_B \rightsquigarrow S_1 \rightsquigarrow S_2$ in ind$A$, which, again, yields a contradiction to (3.7). This proves the necessity.

Sufficiency. We prove that A is shod by showing that any path in ind$A$ from an injective A-module to a projective A-module is shod. So, let $(*) : I_A \rightsquigarrow P_A$ be such a path. If any refinement of $(*)$ either lies in ind$B$ or in ind$C$, then $(*)$ is shod since B and C are shod algebras. Otherwise, there are two cases to consider.
If $I_A, P_A \in$ ind$B$, then any refinement of $(*)$ either lies in ind$B$ or is of the form $I_A \overset{\phi_1}{\rightsquigarrow} S_1 \overset{\phi_2}{\rightsquigarrow} S_2 \overset{\phi_3}{\rightsquigarrow} P_A$ for some $S_1 \in \mathcal{S}_{BC}$ and $S_2 \in \mathcal{S}_{CB}$, where we can assume that ϕ_1 and ϕ_3 are paths in ind$B$. Since S_1 clearly does not satisfy condition (ii)(a), it follows from the assumption that it satisfies condition (ii)(b). This implies that ϕ_1 and ϕ_2 are sectionally refinable while ϕ_2 is shod and hence so is $(*)$. Similarly, $(*)$ is shod when $I_A, P_A \in$ ind$C$.
We now study the case where I_A and P_A do not simultaneously belong to ind$B$ or ind$C$. Without loss of generality, we can assume $I_A \in$ ind$B$ and $P_A \in$ ind$C$. In this case, any refinement of $(*)$ is of the form $I_A \overset{\psi_1}{\rightsquigarrow} S_1 \overset{\psi_2}{\rightsquigarrow} P_A$ for some $S_1 \in \mathcal{S}_{BC}$ and where we can assume that ψ_1 is a path in ind$B$. Then, if S_1 satisfies the condition (ii)(a), ψ_1 is sectionally refinable (since $S_1 \in \mathcal{L}_B$) and ψ_2 is shod, and hence so is $(*)$. Otherwise, if S_1 satisfies the condition (ii)(b), then ψ_1 is shod by assumption while any refinement of ψ_2 is a path ind$C$ (if this is not the case, there would exist a path from S_1 to a module in $\mathcal{S}_{CB}$). But then, since $S_1 \in \mathcal{R}_C$, ψ_2 is sectionally refinable in ind$A$, and then the composed path $(*)$ is shod. Consequently, A is shod. □

Corollary 4.10. *Let A be an articulated algebra along two algebras B and C.*

(i) *If $\mathcal{S}_{BC} \nsubseteq \mathcal{L}_B \cup \mathcal{R}_C$, then A is not shod.*
(ii) *If $\mathcal{S}_{CB} \nsubseteq \mathcal{L}_C \cup \mathcal{R}_B$, then A is not shod.* □

4.6. Quasi-tilted and tilted articulated algebras.

We finish our study of laura articulated algebras with the class of quasi-tilted algebras introduced by Happel-Reiten-Smalø [**15**]. In particular, we see that an articulated algebra is quasi-tilted if and only if it is tilted. Moreover, in this case, the algebra is left and right supported (see [**3**]).

Theorem 4.11. *Let A be an articulated algebra along two algebras B and C. Then A is quasi-tilted if and only if:*

(i) *B and C are quasi-tilted;*
(ii) *$\mathcal{S}_{BC} \subseteq \mathcal{L}_B \cap \mathcal{R}_C$ and $\mathcal{S}_{CB} \subseteq \mathcal{L}_C \cap \mathcal{R}_B$;*
(iii) *There is no non-trivial path in* ind$A$ *between two modules in $\mathcal{S}_{BC} \cup \mathcal{S}_{CB}$.*

Proof : *Necessity.* (i). It is an easy consequence of (4.3) and (3.3) that any indecomposable projective B-module (or C-module) belongs to $\mathcal{L}_B$ (or $\mathcal{L}_C$, respectively). By [**15**, (II.1.14)], B and C are quasi-tilted algebras.
(ii). We only prove $\mathcal{S}_{BC} \subseteq \mathcal{L}_B \cap \mathcal{R}_C$ since the proof of $\mathcal{S}_{CB} \subseteq \mathcal{L}_C \cap \mathcal{R}_B$ is similar.

Let $S \in \mathcal{S}_{BC}$. If $S \notin \mathcal{L}_B$, then there is a non sectionally refinable path $I \underset{B}{\rightsquigarrow} S$ in ind$B$, where I is an injective B-module. In particular $I \notin \mathcal{S}_{BC} \cup \mathcal{S}_{CB}$, and then I is an injective A-module by (3.3). Since this path can be extended from I to a projective A-module by (3.7), this contradicts the fact that A is quasi-tilted by [**15**, (II.1.14)]. Thus $S \in \mathcal{L}_B$ and, dually, $S \in \mathcal{R}_C$.
(iii). Clearly, it suffices to show that there is no path in ind$A$ of the form $S_1 \rightsquigarrow S_2$ or $S_2 \rightsquigarrow S_1$, where $S_1 \in \mathcal{S}_{BC}$ and $S_2 \in \mathcal{S}_{CB}$. Assume to the contrary that $(*)$ is such a path. By Schur's lemma, $(*)$ is not sectionally refinable. Since this path can be extended to a path from an injective A-module to a projective A-module by (3.7), this contradicts the fact that A is quasi-tilted by [**15**, (II.1.14)]. Hence no such path exists and the necessity is proven.

Sufficiency. Assume $(*) : I_A \rightsquigarrow P_A$ is a path in ind$A$ from an injective to a projective. First, if I_A and P_A simultaneously belong to ind$B$ (or ind$C$), then it follows from (iv) that any refinement of $(*)$ is a path in ind$B$ (or ind$C$). Therefore, this path is sectionally refinable since B (or C, respectively) is quasi-tilted. On the other hand, if $I_A \in$ ind$B$ and $P_A \in$ ind$C$, then any irreducible refinement of $(*)$ is of the form $I_A \overset{\phi_B}{\rightsquigarrow} S_1 \overset{\phi_C}{\rightsquigarrow} P_A$, where ϕ_B is a path in ind$B$, ϕ_C is a path in ind$C$ and $S_1 \in \mathcal{S}_{BC}$. But then, since $S_1 \in \mathcal{L}_B \cap \mathcal{R}_C$ by assumption, the paths ϕ_B and ϕ_C are both sectionally refinable by (4.2), and hence so is $(*)$. Since the situation is similar if $I_A \in$ ind$C$ and $P_A \in$ ind$B$, then $(*)$ is sectionally refinable. This implies that A is quasi-tilted by [**15**, (II.1.14)]. □

We get the following results as easy corollaries of the above theorem and [**23**, (3.8)] (see also [**3**]). We refer the reader to [**4**] for more details on left and right supported algebras.

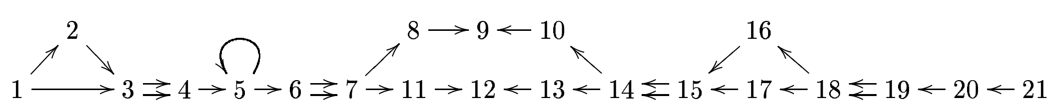
Corollary 4.12. *Let A be a quasi-tilted articulated algebra along two algebras B and C.*

(i) *If $\mathcal{S}_{BC} \neq \emptyset$, then B is a left supported tilted algebra, C is a right supported tilted algebra and A is a left and right supported algebra which is not concealed.*
(ii) *If $\mathcal{S}_{CB} \neq \emptyset$, then B is a right supported tilted algebra, C is a left supported tilted algebra and A is a left and right supported algebra which is not concealed.*
(iii) *Moreover, if $\mathcal{S}_{BC} \neq \emptyset$ and $\mathcal{S}_{CB} \neq \emptyset$, then B and C are not concealed.* □

Corollary 4.13. *Let A be an articulated algebra along two algebras B and C. Then A is quasi-tilted if and only if A is tilted not concealed. Moreover, in this case, A is left and right supported.* □

Corollary 4.14. *Let A be an articulated algebra along two algebras B and C. Then A is quasi-tilted if and only if A is tilted and $\Gamma(\text{mod}A)$ has a unique non-semiregular connecting component.* □

Example 4.15. Let A be the radical square zero algebra given by the quiver



Then $\Gamma(\text{mod}A)$ has the form illustrated in Fig.2: where we identify the two copies of S_2, S_5 and S_{10}, respectively. We have the following:

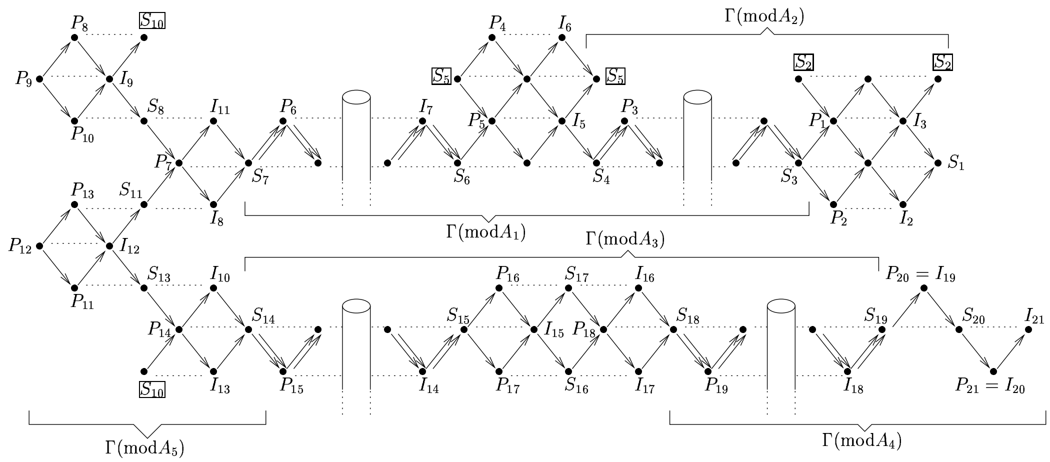


FIGURE 2

(a) Let A_1 be the full subcategory generated by the set of points $\{3,4,5,6,7\}$, whose Auslander-Reiten quiver is shown in Fig.2. It is easily seen that $A_1 = (B_1, C_1)$, where B_1 and C_1 are respectively the full subcategories generated by the sets of points $\{3,4\}$ and $\{4,5,6,7\}$. Here A is laura but not weakly shod (since the unique non-semiregular component contains cycles).

(b) Let A_2 be the full subcategory generated by the set of points $\{1,2,3,4\}$, whose Auslander-Reiten quiver is explicited in Fig.2. Then $A_2 = (B_2, C_2)$, where B_2 and C_2 are respectively the full subcategories generated by the sets of points $\{1,2,3\}$ and $\{3,4\}$. Clearly, A_2 is right glued but not weakly shod.

(c) Let A_3 be the full subcategory generated by the set of points $\{14,15,16,17,18,19\}$, whose Auslander-Reiten quiver is shown in Fig.2. Then $A_3 = (B_3, C_3)$, where B_3 and C_3 are respectively the full subcategories generated by the sets of points $\{14,15\}$ and $\{15,16,17,18,19\}$. It is easily seen that A_3 is weakly shod, but not shod since the path $I_{14} \to S_{15} \to P_{17} \to I_{15} \to S_{16} \to P_{18} \to I_{17} \to S_{18} \to P_{19}$ is not shod (see Section 4.5).

(d) Let A_4 be the full subcategory generated by the set of points $\{18,19,20,21\}$, whose Auslander-Reiten quiver is explicited in Fig.2. It is easily seen that $A_4 = (B_4, C_4)$, where B_4 and C_4 are respectively the full subcategories generated by the sets of points $\{18,19\}$ and $\{19,20,21\}$, which is shod, but not quasi-tilted by [**15**, (II.1.14)] since the path $I_{18} \to S_{19} \to P_{20} \to S_{20} \to P_{21}$ is not sectionally refinable.

(e) Let A_5 be the full subcategory generated by the set of points $\{7,8,9,10,11,12,13,14\}$, whose Auslander-Reiten quiver is shown in Fig.2. Then $A_5 = (B_5, C_5)$, where B_5 and C_5 are respectively the full subcategories generated by the sets of points $\{7,8,9,10,11\}$ and $\{10,11,12,13,14\}$. In particular, the articulation (B_5, C_5) is not unidirectional. Then it is an easy checking that A_5 is quasi-tilted.

4.7. Global dimension of articulated algebras. The shape of the Auslander-Reiten quiver of an articulated algebra (see Section 3) also gives information on the global dimension gl.dim.A of an articulated algebra A.

We recall that a module I is called **hereditary injective** if every quotient of I^r, for some $r > 0$, is an injective module.

Proposition 4.16. *Let A be an articulated algebra along two algebras B and C.*

(i) $\sup\{\text{gl.dim.}B, \text{gl.dim.}C\} \leq \text{gl.dim.}A$;
(ii) *If $A = (B, C)$ is unidirectional, with $S_{CB} = \emptyset$, then there exists a ring isomorphism $A \cong \begin{pmatrix} B & 0 \\ M & C \end{pmatrix}$, with M_B an hereditary injective B-module, and* $\text{gl.dim.}A \leq \text{gl.dim.}B + \text{gl.dim.}C$.

Proof : (i). Let M be an indecomposable A-module and assume, without loss of generality, that M belongs to ind$B$. We claim that $\text{dp}_B M \leq \text{dp}_A M$. Of course, there is nothing to prove if $\text{dp}_A M = \infty$ or if M is a projective B-module. Otherwise, assume $\text{dp}_A M = n < \infty$ and let

$$(**) \qquad 0 \longrightarrow P_n \xrightarrow{d_{n-1}} P_{n-1} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

be a minimal projective resolution of M in mod$A$, and take $K_i = \text{Ker}d_i$ for all $i \geq 0$. Clearly, if P_i is a B-module for all $i \geq 0$, then $(**)$ is a projective resolution of M in mod$B$ and $\text{dp}_A M \leq \text{dp}_B M$. Otherwise, let l be the smallest integer i such that P_{i+1} is not a B-module : P_0 is a B-module by (3.3) and (3.11). The minimality of $(**)$ implies that $K_l = S_l \oplus L_l$, where $S_l \in \text{add}(S_{CB})$ and $L_l \in \text{mod}B \setminus \text{mod}C$. Moreover, $\text{dp}_A S_l = n - l$ by dimension shifting. Therefore, if

$$0 \longrightarrow Q_{n-l} \xrightarrow{d'_{n-1}} Q_{n-l-1} \xrightarrow{d'_{n-2}} \cdots \xrightarrow{d'_1} Q_0 \xrightarrow{d'_0} S_l \longrightarrow 0$$

is a minimal projective resolution of S_l in mod$A$, we obtain an exact sequence

$$(*) \qquad 0 \longrightarrow \frac{P_n}{Q_{n-l}} \xrightarrow{d''_{n-1}} \frac{P_{n-1}}{Q_{n-l-1}} \xrightarrow{d''_{n-2}} \cdots \xrightarrow{d_1} \frac{P_{l+1}}{Q_1} \xrightarrow{d''_l}$$
$$\frac{P_l \oplus L_1}{Q_0} \xrightarrow{d''_{l-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

where the second line is an exact sequence in mod$B$ in which all the modules (apart from M) are projective in mod$B$. Applying repeatedly this process to $(*)$, one obtains a projective resolution of M in mod$B$ of length smaller or equal to $n = \text{dp}_A M$. Therefore, $\text{dp}_B M \leq \text{dp}_A M$. Since the same argument holds in case M belongs to ind$C$, this shows that $\sup\{\text{gl.dim.}B, \text{gl.dim.}C\} \leq \text{gl.dim.}A$.

(ii). Assume that the articulation $A = (B, C)$ is unidirectional with $S_{CB} = \emptyset$. By (3.1) and (3.11), ind$B$ is a full subcategory of ind$A$ which is closed under predecessors. Therefore, since the inclusion functor mod$B \hookrightarrow$ mod$A$ is exact, it follows from [**5**, (2.5)] that there exists a ring isomorphism $A \cong \begin{pmatrix} B & 0 \\ M & C \end{pmatrix}$, with M_B an hereditary injective B-module, and gl.dim.B $= \sup\{\text{pd}_A X \mid X \in \text{mod}B\}$. To finish the proof, there remains to show that, for any indecomposable C-module X which is not a B-module, we have $\text{dp}_A X \leq \text{dp}_C X + \text{gl.dim.}B$. So let X be such a module and

$$(**) \qquad 0 \longrightarrow P_n \xrightarrow{d_{n-1}} P_{n-1} \xrightarrow{d_{n-2}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \longrightarrow 0$$

be a minimal projective resolution of X in mod$C$ (of course one can assume that $\text{dp}_C X = n < \infty$ since otherwise there is nothing to prove), and take $K_i = \text{Ker}d_i$ for all $i \geq 0$. Clearly, if P_i is an A-module for all i, then $(**)$ is a projective resolution of

X in mod$A$ and $\mathrm{dp}_A X = \mathrm{dp}_C X \leq \mathrm{dp}_C X + \mathrm{gl.dim.}B$. Otherwise, let l be the smallest integer i such that P_{i+1} is not a projective A-module : P_0 is a projective A-module since $X \notin S_{BC}$ by assumption. The minimality of $(**)$ implies that $K_l = S_l \oplus L_l$, where $S_l \in \mathrm{add}(S_{BC})$ and $L_l \in \mathrm{mod}C \setminus \mathrm{mod}B$. Moreover, $\mathrm{dp}_A S_l = m \leq \mathrm{gl.dim.}B$. Therefore, if

$$0 \longrightarrow Q_m \xrightarrow{d'_{m-1}} Q_{m-1} \xrightarrow{d'_{m-2}} \cdots \xrightarrow{d'_1} Q_0 \xrightarrow{d'_0} S_l \longrightarrow 0$$

is a minimal projective resolution of S_l in mod$A$, we obtain an exact sequence

$$(*) \quad 0 \longrightarrow Q_m \xrightarrow{d'_{l+m}} \cdots \longrightarrow Q_{n-l+2} \xrightarrow{d'_{n+2}} P_n \oplus Q_{n-l+1} \xrightarrow{d'_{n+1}}$$
$$P_{n-1} \oplus Q_{n-l} \xrightarrow{d'_n} \cdots \xrightarrow{d'_{l+2}} \frac{P_{l+1} \oplus Q_0}{S_l} \xrightarrow{d'_l} P_l \xrightarrow{d_{l-1}} \cdots$$
$$\cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \longrightarrow 0$$

of length t, with $t \leq l + \mathrm{gl.dim.}B$, and where $Q_m, \ldots, Q_{n-l+2}, \ldots, \frac{P_{l+1} \oplus Q_0}{S_l}, P_l, \ldots, P_0$ are all projective A-modules. Applying repeatedly this process to $(*)$, one obtains a projective resolution of X in mod$A$ of length smaller or equal to $\mathrm{dp}_C X + \mathrm{gl.dim.}B \leq \mathrm{gl.dim.}C + \mathrm{gl.dim.}B$. □

The bounds given in the above proposition are the best possible. Indeed, if we refer to (4.15), it is easily seen that $\sup\{\mathrm{gl.dim.}B_5, \mathrm{gl.dim.}C_5\} = 2 = \mathrm{gl.dim.}A_5$ while $\mathrm{gl.dim.}A_4 = 3 = \mathrm{gl.dim.}B_4 + \mathrm{gl.dim.}C_4$. This also shows that the converse of (ii) does not hold since the articulation $A_4 = (B_4, C_4)$ is not unidirectional. Finally, (ii) fails to be true in general : indeed, let A be the radical square algebra given by the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ then $A = (B, C)$, where B and C are the algebras given by the quivers $1 \xrightarrow{\alpha} 2$ and $1 \xleftarrow[\beta]{} 2$ respectively. Here, $\mathrm{gl.dim.}B + \mathrm{gl.dim.}C = 2$, while $\mathrm{gl.dim.}A = \infty$.

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