Laura algebras and quasi-directed components

Marcelo Lanzilotta and David Smith

ABSTRACT. We introduce a notion of distance between indecomposable modules and deduce from this new characterizations of laura algebras and quasi-directed Auslander-Reiten components. Finally, we show that a convex component is quasi-directed if and only if it is almost directed.

1. Introduction

The aim of the representation theory of Artin algebras is to study an algebra A by means of the category mod$A$ of finitely generated right A-modules, which often turns out to be easier to handle. A key example arises in **[11]** where Happel, Reiten and Smalø show that an algebra A is quasitilted, that is A is the endomorphism ring of a tilting object in a locally finite hereditary abelian category if and only if A has global dimension at most two and any indecomposable A-module either lies in $\mathcal{L}_A$ or in $\mathcal{R}_A$. Recall that $\mathcal{L}_A$ is the full subcategory of mod$A$ consisting of all indecomposable A-modules whose predecessors have projective dimension at most one and $\mathcal{R}_A$ is defined dually.

Following this example, the module category, and particularly $\mathcal{L}_A$ and $\mathcal{R}_A$, gained in importance and new classes of algebras, defined by the homological properties of their indecomposable modules, appeared : the shod algebras **[7]**, the weakly shod algebras **[8]** and the laura algebras **[1, 19]**; each of them generalizing the previous ones. We refer the reader to **[2]** for a complete review on these classes of algebras.

Laura algebras have been introduced independently by Assem and Coelho **[1]** and Reiten and Skowroński **[19]** as a generalization of the representation-finite algebras and the weakly shod algebras. Their nice properties have made them rather interesting and hugely investigated, see, for instance **[1, 19, 28, 2, 3, 10]**. In particular, it is shown in **[28]** that

2000 Mathematics Subject Classification: 16G70, 16G30, 18G05, 16E10.

Key words: laura algebras, quasi-directed Auslander-Reiten components, projectives and injectives

an algebra A is laura if and only if it is quasitilted or its Auslander-Reiten quiver contains a faithful quasi-directed component, which is, in this case, convex. In **[29]**, the second author presented equivalent conditions for an Auslander-Reiten component to be quasi-directed and convex.

In this paper, we introduce a notion of distance between indecomposable modules which allows us to unify the approaches used in **[1]** and **[29]**. We deduce from this new characterizations of laura algebras (see Sec. 3) and convex quasi-directed components (see Sec. 4.1). Afterwards, we show that any morphism in the infinite radical implies the existence of infinitely many modules lying on a cycle and obtain, as a consequence, that any convex component is quasi-directed if and only if it is almost directed (see Sec. 4.2).

2. Preliminaries

2.1. Usual notations. Throughout this paper, all algebras are connected basic Artin algebras. For an algebra A, we denote by $\mathrm{mod}A$ its category of finitely generated right modules and by $\mathrm{ind}A$ a full subcategory of $\mathrm{mod}A$ generated by one representative from each isomorphism class of indecomposable modules. We denote by $\mathrm{rad}(\mathrm{mod}A)$, the *radical* of $\mathrm{mod}A$, that is the ideal generated by all non-isomorphisms between indecomposable modules. The *infinite radical* $\mathrm{rad}^\infty(\mathrm{mod}A)$ of $\mathrm{mod}A$ is the intersection of all powers $\mathrm{rad}^i(\mathrm{mod}A)$, with $i \geq 1$, of $\mathrm{rad}(\mathrm{mod}A)$. For an A-module M, we denote by $\mathrm{pd}_A M$ its projective dimension and by $\mathrm{id}_A M$ its injective dimension.

We denote by $\Gamma(\mathrm{mod}A)$ its Auslander-Reiten quiver and by $\tau_A = \mathrm{DTr}$ the usual Auslander-Reiten translation. By a *component* of $\Gamma(\mathrm{mod}A)$, we mean a connected component of $\Gamma(\mathrm{mod}A)$. A component Γ of $\Gamma(\mathrm{mod}A)$ is *semiregular* if it does not contain a projective module and an injective module, and *non-semiregular* otherwise. Finally, Γ is *faithful* if $\mathrm{ann}(\Gamma)$, that is the intersection of the annihilators of all modules X in Γ, vanishes.

For further details on representation theory of Artin algebras, see **[4, 21]**.

2.2. Walks, paths and hooks. Given X, Y in $\mathrm{ind}A$, a *walk of length* t between X and Y is a sequence :

$$w: \quad X = X_0 \overset{f_1}{\text{———}} X_1 \overset{f_2}{\text{———}} \cdots \overset{f_t}{\text{———}} X_t = Y \ , \quad (t \geq 0)$$

where, for each i, $X_i \in \mathrm{ind}A$ and f_i is a non-zero morphism either from X_{i-1} to X_i or from X_i to X_{i-1}. In this case, we denote by $l(w)$ the length of w. We also denote by $\mathcal{M}^w$ the set of modules in w, that is $\{ X_0, X_1, \ldots, X_t \}$. If each f_i is irreducible, w is a *walk of irreducible morphisms* and we denote by $W(X,Y)$ the set of walks of irreducible morphisms between X and Y. A *path* from X to Y is a walk between X and Y such that each f_i is a morphism from X_{i-1} to X_i :

$$\delta : \qquad X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} X_t = Y \ , \quad (t \geq 0)$$

In this case, we write $\delta : X \rightsquigarrow Y$ and we say that X is a *predecessor* of Y and Y is a *successor* of X. We denote by $\mathcal{M}^{\delta}_{X,Y}$ the set of modules in δ. More generally, we denote by $\mathcal{M}_{X,Y}$ the set of modules M in ind$A$ lying on a path from X to Y. If each f_i is irreducible, δ is a *path of irreducible morphisms* and, in this case, δ is *sectional* if it contains no *hook*, that is a triple (X_{i-1}, X_i, X_{i+1}) such that $\tau_A X_{i+1} = X_{i-1}$. We also denote by $\mathcal{H}^{\delta}_{X,Y}$ the set of all hooks in δ and by $\mathcal{H}_{X,Y}$ the set of all hooks lying on a path from X to Y. A *refinement* of δ is a path $X = X'_0 \xrightarrow{f'_1} X'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_s} X'_s = Y$, with $s \geq t$, together with an order preserving function $\sigma : \{1, \ldots, t-1\} \to \{1, \ldots, s-1\}$ such that $X_i = X'_{\sigma(i)}$ when $1 \leq i \leq t-1$. Finally, a path δ is a *cycle* if $X = Y$ and at least one f_i is not an isomorphism. An A-module X is *directing* if it does not lie on any cycle and a component Γ of $\Gamma(\text{mod}A)$ is *directed* if it contains only directing modules.

2.3. Laura algebras. We recall from [**1**] that an Artin algebra A is called *laura* whenever ind$A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ is a finite set, where, as mentionned in the introduction, $\mathcal{L}_A$ and $\mathcal{R}_A$ are defined as follows :

$$\mathcal{L}_A = \{\ X \in \text{ind}A \mid \text{pd}_A Y \leq 1 \text{ for each predecessor } Y \text{ of } X\ \},$$
$$\mathcal{R}_A = \{\ X \in \text{ind}A \mid \text{id}_A Y \leq 1 \text{ for each successor } Y \text{ of } X\ \}.$$

We recall the following key result:

THEOREM 2.3.1. [**1**] *The following statements are equivalent for an Artin algebra A:*

(a) *A is laura;*
(b) *There are only finitely many modules lying on a path from an injective module to a projective module;*
(c) *There are only finitely many modules lying on a path from a module not in $\mathcal{L}_A$ to a module not in $\mathcal{R}_A$.* □

2.4. Quasi-directed components. We recall that a component Γ of $\Gamma(\text{mod}A)$ is *quasi-directed* [**1, 2, 29**] if it is *generalized standard* [**25**], that is $\text{rad}^{\infty}(X, Y) = 0$ for all X, Y in Γ, and *almost directed*, that is it contains only finitely many non-directing modules. Moreover, Γ is *convex* if any path from X to Y, with X, Y in Γ, contains only modules from Γ.

The following characterizations will play in important role in Sec. 4.

THEOREM 2.4.1. [**29**] *Let A be an Artin algebra and assume that Γ is a semiregular component of $\Gamma(\text{mod}A)$ without projective modules. The following conditions are equivalent :*

(a) Γ *is quasi-directed;*
(b) Γ *is directed;*
(c) *For each* X *in* Γ*, there exists an integer* n_X *such that any cycle from* X *to* X *contains at most* n_X *modules;*
(d) *For each* X *in* Γ*, there exists an integer* m_X *such that any cycle from* X *to* X *contains at most* m_X *distinct hooks;*
(e) *Given* X*,* Y *in* Γ*, there exists an integer* $n_{X,Y}$ *such that any path from* X *to* Y *contains at most* $n_{X,Y}$ *modules;*
(f) *Given* X*,* Y *in* Γ*, there exists an integer* $m_{X,Y}$ *such that any path from* X *to* Y *contains at most* $m_{X,Y}$ *distinct hooks.*

Further, if Γ *contains injective modules, these conditions are equivalent to :*

(g) *For each* X *in* Γ*, there exists an integer* i_X *such that any path from an injective in* Γ *to* X *contains at most* i_X *modules;*
(h) *For each* X *in* Γ*, there exists a integer* j_X *such that any path from an injective in* Γ *to* X *contains at most* j_X *distinct hooks.*

Furthermore, Γ *is convex,* $B = A/\mathrm{ann}(\Gamma)$ *is a tilted algebra and* Γ *is a connecting component of* $\Gamma(\mathrm{mod}B)$*.* □

THEOREM 2.4.2. **[29]** *Let* A *be an Artin algebra and assume that* Γ *is a non-semiregular component of* $\Gamma(\mathrm{mod}A)$*. The following conditions are equivalent :*

(a) Γ *is quasi-directed and convex;*
(b) *Given* X*,* Y *in* Γ*, there exists an integer* $n_{X,Y}$ *such that any path from* X *to* Y *contains at most* $n_{X,Y}$ *modules;*
(c) *Given* X*,* Y *in* Γ*, there exists an integer* $m_{X,Y}$ *such that any path from* X *to* Y *contains at most* $m_{X,Y}$ *distinct hooks;*
(d) *There exists an integer* n_0 *such that any path from an injective in* Γ *to a projective in* Γ *contains at most* n_0 *modules;*
(e) *There exists an integer* m_0 *such that any path from an injective in* Γ *to a projective in* Γ *contains at most* m_0 *distinct hooks.*

Furthermore, $B = A/\mathrm{ann}(\Gamma)$ *is a laura algebra and* Γ *is the unique non-semiregular and faithful component of* $\Gamma(\mathrm{mod}B)$*.* □

3. The main result

This section is devoted to our main Theorem which gives new characterizations of laura algebras by unifying the results stated in subsections 2.3 and 2.4, that is by completing (2.3.1) with the notions of hooks and bounds. The Theorem is the following :

THEOREM 3.0.3. *The following statements are equivalent for an Artin algebra* A*:*

(a) A *is laura;*

(b) *There are only finitely many modules lying on a path from an injective module to a projective module;*
(b') *There are only finitely many modules lying on a path from a module not in $\mathcal{L}_A$ to a module not in $\mathcal{R}_A$;*
(c) *There are only finitely many distinct hooks lying on a path from an injective module to a projective module;*
(c') *There are only finitely many distinct hooks lying on a path from a module not in $\mathcal{L}_A$ to a module not in $\mathcal{R}_A$;*
(d) *There exists an integer n such that any path from an injective module to a projective module contains at most n modules;*
(d') *There exists an integer n' such that any path from a module not in $\mathcal{L}_A$ to a module not in $\mathcal{R}_A$ contains at most n' modules;*
(e) *There exists an integer m such that any path from an injective module to a projective module contains at most m distinct hooks;*
(e') *There exists an integer m' such that any path from a module not in $\mathcal{L}_A$ to a module not in $\mathcal{R}_A$ contains at most m' distinct hooks.*

The rest of this section will be occupied by the proof of this statement. We refer to **[12]** for other characterizations of laura algebras using the Gabriel-Roiter measure.

3.1. A metric for ind$A$. Among other things, we need in the proof of this Theorem to gather the modules lying on a path into a finite neighborhood of a certain module. This is done with the introduction of a *distance* in ind$A$: given two modules X and Y in ind$A$, set

$$\mathrm{d}(X,Y)=\begin{cases}\min\{\ |\mathcal{M}^w|-1\ |\ w\in W(X,Y)\ \},\ \text{if } W(X,Y) \text{ is not empty;}\\ \infty,\ \text{otherwise.}\end{cases}$$

LEMMA 3.1.1. *Let X,Y be modules in* ind$A$ *such that* $\mathrm{d}(X,Y)=n<\infty$. *Then, there exists a walk of irreducible morphisms $X=X_0-X_1-\cdots-X_n=Y$, with $X_i\neq X_j$ when $i\neq j$.*

Proof. If $\mathrm{d}(X,Y)=0$, then $X=Y$ and there is nothing to show. Otherwise, let $\epsilon: X=Y_0\overset{f_1}{-}Y_1\overset{f_2}{-}\cdots\overset{f_t}{-}Y_t=Y$ be a walk in $W(X,Y)$ such that $|\mathcal{M}^\epsilon|=n+1=\mathrm{d}(X,Y)+1$, with $n\geq 1$. Clearly $n\leq t$. If $n<t$, there exist i,j, with $0\leq i<j\leq t$, such that $Y_i=Y_j$. This yields a walk of irreducible morphisms $\epsilon': X=Y_0\overset{f_1}{-}Y_1-\cdots\overset{f_i}{-}Y_i=Y_j\overset{f_{j+1}}{-}\cdots\overset{f_t}{-}Y_t=Y$, such that $n\leq l(\epsilon')=t-(j-i)<t$. Inductively, one gets a walk $X=X_0-X_1-\cdots-X_n=Y$, in $W(X,Y)$, with $X_i\neq X_j$, for $i\neq j$ (because $\mathrm{d}(X,Y)=n$). □

PROPOSITION 3.1.2. (a) *The map* $\mathrm{d}:\mathrm{ind}A\times\mathrm{ind}A\to\mathbb{R}\cup\{\infty\}$ *is a metric;*

(b) *Let* $\widehat{\mathrm{d}} : \mathrm{ind}A \times \mathrm{ind}A \to I\!R \cup \{\infty\}$ *given by*

$$\widehat{\mathrm{d}}(X,Y) = \begin{cases} \min\{\ l(w) \mid w \in W(X,Y)\ \}, \text{ if } W(X,Y) \text{ is not empty;} \\ \infty, \text{ otherwise.} \end{cases}$$

The maps $\mathrm{d}, \widehat{\mathrm{d}}$ *are equal.*

Proof. (a). The proof is easy and omitted.
(b). Let $X, Y \in \mathrm{ind}A$. Since it is obvious that $\mathrm{d}(X,Y) = \widehat{\mathrm{d}}(X,Y) = \infty$ if and only if $W(X,Y)$ is empty, assume that $W(X,Y)$ is not empty. First, if $\widehat{\mathrm{d}}(X,Y) = n < \infty$, then there exists a walk of minimal length $w : X = X_0 - X_1 - \cdots - X_n = Y$ in $W(X,Y)$. The minimality implies that $X_i \neq X_j$ when $i \neq j$. Therefore $\mathrm{d}(X,Y) \leq n = \widehat{\mathrm{d}}(X,Y)$. Conversely, if $\mathrm{d}(X,Y) = n < \infty$, then it follows from (3.1.1), that there exists a walk of irreducible morphisms of length n in $W(X,Y)$, and hence $\widehat{\mathrm{d}}(X,Y) \leq n = \mathrm{d}(X,Y)$. □

DEFINITION 3.1.3. Let $G \subseteq \mathrm{ind}A$ and $n \geq 0$. The *ball of center* G *and radius* n is the set:

$$B(G,n) = \{\ X \in \mathrm{ind}A \mid \text{there exists } M \in G \text{ such that } \mathrm{d}(M,X) \leq n\ \}.$$

REMARK 3.1.4. Since the Auslander-Reiten quiver of an Artin algebra A is locally finite, then $G \subseteq \mathrm{ind}A$ is finite if and only if $B(G,n)$ is finite for all $n \geq 0$.

3.2. The proof of the Theorem. We now start proving Theorem 3.0.3. We first recall the following result obtained in **[27]**, in which we denote (as in the sequel) by $\mathbf{r}$ the rank of the Grothendieck group of A.

LEMMA 3.2.1. *Let* A *be an Artin algebra and* $X_1, X_2, \ldots, X_t$ *be distinct modules in* $\mathrm{ind}A$, *with* $t > \mathbf{r}$. *Then* $\mathrm{Hom}_A(X_i, \tau X_j) \neq 0$, *for some* i, j, *with* $1 \leq i, j \leq t$. □

Our first lemma connects the number of distinct hooks on paths with the number of distinct modules.

LEMMA 3.2.2. *Let* X, Y *be modules in* $\mathrm{ind}A$ *and assume that* $|\mathcal{H}^{\delta}_{X,Y}| \leq b$ *for each path* $\delta : X \rightsquigarrow Y$, *for some* $b \geq 0$. *Then* $|\mathcal{M}^{\delta}_{X,Y}| \leq (b+1)(\mathbf{r}+1) - 1$ *for each path* $\delta : X \rightsquigarrow Y$.

Proof. Assume this is not the case. Then, there exists a path of the form $\delta : X = X_0 \rightsquigarrow X_1 \rightsquigarrow \cdots \rightsquigarrow X_{(b+1)(\mathbf{r}+1)-1} \rightsquigarrow Y$, with $X_i \neq X_j$ when $i \neq j$. This yields a family of subpaths $\{\delta_k\}_{k=1,\ldots,b+1}$, where δ_k is given by $\delta_k : X_{(k-1)(\mathbf{r}+1)} \rightsquigarrow \cdots \rightsquigarrow X_{k\mathbf{r}+(k-1)}$. Since each subpath δ_k contains $\mathbf{r}+1$ distinct X_i's, then (3.2.1) gives, for each k, a path of the form $X_{i_k} \to \tau X_{j_k} \to V_k \to X_{j_k}$. Gluing these paths together gives a path:

$$X = X_0 \rightsquigarrow X_{i_1} \to \tau X_{j_1} \to V_1 \to X_{j_1} \rightsquigarrow \cdots$$
$$\cdots \rightsquigarrow X_{i_{b+1}} \to \tau X_{j_{b+1}} \to V_{b+1} \to X_{j_{b+1}} \rightsquigarrow Y,$$

containing at least $b+1$ distinct hooks, a contradiction to the hypothesis. □

LEMMA 3.2.3. *Let* X, Y *be modules in* ind$A$ *and assume that* $|\mathcal{M}^{\delta}_{X,Y}| \leq b$ *for each path* $\delta : X \rightsquigarrow Y$, *for some* $b \geq 0$. *Then* $\mathcal{M}_{X,Y}$ *is a finite set.*

Proof. Assume $\delta : X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} X_t = Y$ is a path. By [**29**, (1.1)], $f_i \notin \mathrm{rad}^{\infty}(\mathrm{mod}A)$ for each i. Therefore δ can be refined to a path of irreducible morphisms $\delta' : X = X'_0 \to X'_1 \to \cdots \to X'_s = Y$. Since $|\mathcal{M}^{\delta'}_{X,Y}| \leq b$ by hypothesis, we have $|\mathcal{M}^{\delta'_i}_{X,X'_i}| \leq b$ for each subpath $\delta'_i : X = X'_0 \rightsquigarrow X'_i$ of δ', with $1 \leq i \leq s$. This entails that $\mathrm{d}(X, X_i) \leq b$ for each i with $1 \leq i \leq t$. Hence, $\mathcal{M}_{X,Y} \subseteq B(\{X\}, b)$ and $\mathcal{M}_{X,Y}$ is a finite set by (3.1.4). □

As a consequence we obtain the following result. Observe that the equivalence of (c) and (d) has been obtained in [**29**] under the assumption that the given modules X and Y belong to the same component of $\Gamma(\mathrm{mod}A)$; this assumption turns out to be unnecessary.

COROLLARY 3.2.4. *Let* A *be an Artin algebra and* X, Y *be modules in* ind$A$. *The following conditions are equivalent:*

(a) $\mathcal{M}_{X,Y}$ *is finite;*
(b) $\mathcal{H}_{X,Y}$ *is finite;*
(c) *There exists an* n *such that* $|\mathcal{M}^{\delta}_{X,Y}| \leq n$, *for each path* $\delta : X \rightsquigarrow Y$*;*
(d) *There exists an* m *such that* $|\mathcal{H}^{\delta}_{X,Y}| \leq m$, *for each path* $\delta : X \rightsquigarrow Y$.

Proof. Clearly, (a) implies (b) and (b) implies (d). On the other hand, (d) implies (c) by (3.2.2) and (c) implies (a) by (3.2.3). □

As we see, we easily deduce from this corollary the equivalence of the statements (b), (c), (d) and (e) of our main theorem. However, since ind$A$ generally contains infinitely many modules neither in $\mathcal{L}_A$, nor in $\mathcal{R}_A$, the above corollary is useless for proving the equivalence of the other statements since the bounds imposed by (b'), (c'), (d') and (e') may thus tend to infinity. The idea is to impose a finiteness condition on these modules and use the distance introduced in Sec. 3.1. This leads to the following result (compare with [**1**, (1.5)]).

PROPOSITION 3.2.5. (a) *Let* P *be an indecomposable projective* A*-module and* M *be a predecessor of* P. *If* $M \in \mathcal{R}_A$, *then* $M \in B(\{P\}, \mathbf{r})$*;*
(b) *Let* I *be an indecomposable injective* A*-module and* N *be a successor of* I. *If* $N \in \mathcal{L}_A$, *then* $N \in B(\{I\}, \mathbf{r})$.

Proof. We only prove (a) since the proof of (b) is dual.
(a). We first show that any path of irreducible morphisms $\epsilon : N = X_s \to$

$X_{s-1} \to \cdots \to X_0 = P$, with N a successor of M, is sectional. Indeed, if this is not the case there exists a minimal j such that (X_{j+1}, X_j, X_{j-1}) is a hook in ϵ and the subpath $X_j \to X_{j-1} \to \cdots \to X_0 = P$ is sectional. In particular $\mathrm{Hom}_A(X_{j-1}, P) \neq 0$ and so $\mathrm{id}_A X_{j+1} \geq 2$ by [**21**], contradicting $N \in \mathcal{R}_A$. Therefore, any such path is sectional and, in particular, any such path has length at most $\mathbf{r}$ by (3.2.1) and the non-sectionality of cycles [**5, 6**].

Now, let $\delta : M = M_s \stackrel{f_s}{\to} M_{s-1} \stackrel{f_{s-1}}{\to} \cdots \stackrel{f_1}{\to} M_0 = P$ be an arbitrary path from M to P. We claim that δ contains no morphism in $\mathrm{rad}^\infty(\mathrm{mod} A)$, and hence that δ can be refined to a (sectional) path of irreducible morphisms. Indeed, if $f_1 \in \mathrm{rad}^\infty(\mathrm{mod} A)$, then it follows from [**29**, (1.1)] that there exists a path $M \rightsquigarrow M_1 \to N_{\mathbf{r}} \stackrel{g_{\mathbf{r}}}{\to} N_{\mathbf{r}-1} \stackrel{g_{\mathbf{r}-1}}{\to} \cdots \stackrel{g_1}{\to} N_0 \stackrel{g_0}{\to} M_0 = P$, where $g := g_0 g_1 \cdots g_{\mathbf{r}} \neq 0$ and $N_i \ncong N_j$ for any $i \neq j$. By (3.2.1), there exist i, j such that $\mathrm{Hom}_A(N_i, \tau N_j) \neq 0$. This yields a path $M \rightsquigarrow M_1 \rightsquigarrow N_i \to \tau N_j \to V_j \to N_j \stackrel{g_0 g_1 \cdots g_j}{\longrightarrow} M_0 = P$. By [**21**], we have, as before, $\mathrm{id}_A(\tau N_j) \geq 2$, a contradiction since $M \in \mathcal{R}_A$. Therefore, $f_1 \notin \mathrm{rad}^\infty(\mathrm{mod} A)$, and hence can be refined to a (sectional) path of irreducible morphisms.

Inductively, assume that there exists an integer l, with $l \geq 1$, such that the subpath $M_l \stackrel{f_l}{\to} M_{l-1} \stackrel{f_{l-1}}{\to} \cdots \stackrel{f_1}{\to} M_0 = P$ of δ can be refined to a (sectional) path of irreducible morphisms. We show that $f_{l+1} \notin \mathrm{rad}^\infty(\mathrm{mod} A)$. Indeed, if $f_{l+1} \in \mathrm{rad}^\infty(\mathrm{mod} A)$, then it follows from (the proof of) [**29**, (1.1)] that there exists a path of irreducible morphisms $Y_t \to \cdots \to Y_1 \to Y_0 = M_l$ containing at least $\mathbf{r}+1$ distinct modules $Y'_0, Y'_1, \cdots, Y'_{\mathbf{r}}$ and such that $\mathrm{Hom}_A(M_{l+1}, Y_t) \neq 0$. Applying (3.2.1) to these modules entails the existence of a path of irreducible morphisms $\tau Y'_j \to V'_j \to Y'_j \to \cdots \to Y_0 = M_l$, where $\mathrm{Hom}_A(Y'_i, \tau Y'_j) \neq 0$ for some i, j. By the induction hypothesis, we get a path of irreducible morphisms $\tau Y'_j \to V'_j \to Y'_j \to \cdots \to Y_0 = M_l \to \cdots \to M_0 = P$ containing at least one hook. By the first part of the proof, $\tau_A Y'_j \notin \mathcal{R}_A$, which, again, contradicts the fact that $M \in \mathcal{R}_A$. This shows that $f_{l+1} \notin \mathrm{rad}^\infty(\mathrm{mod} A)$. Consequently, the path δ can be refined to a path of irreducible morphisms. Moreover, by the first part of the proof, δ is sectional and, by (3.2.1) and the non-sectionality of cycles, $l(\delta) \leq \mathbf{r}$. Therefore $\mathrm{d}(M, P) \leq \mathbf{r}$ and the proof is complete. □

This yields a new proof of the following corollary, first stated in [**1**], which proof now directly follows from (3.2.5) and (3.1.4).

COROLLARY 3.2.6. (a) *There are only finitely many indecomposable modules in $\mathcal{R}_A$ which are predecessors of a projective module;*
(b) *There are only finitely many indecomposable modules in $\mathcal{L}_A$ which are successors of an injective module.* □

As a consequence, we get the proof of the Theorem 3.0.3.

The proof. The equivalence of (a), (b) and (b') follows from (2.3.1). On the other hand, the equivalence of (b), (c), (d) and (e) follows from (3.2.4) and the fact that there are only finitely many injective modules and projective modules in ind$A$.
For the other statements, we only prove the equivalence of the conditions (c') and (c), the equivalence of the other statements with their corresponding ones are similar. We first prove that (c') implies (c). Indeed, if this is not the case, there exists an infinite family $(\tau M_\lambda, V_\lambda, M_\lambda)_{\lambda\in\Lambda}$ of distinct hooks lying on paths from an injective I to a projective P. By (3.2.6), there exists an infinite subset Λ' of Λ such that $\tau M_\lambda \notin \mathcal{L}_A$ and $M_\lambda \notin \mathcal{R}_A$ for each $\lambda \in \Lambda'$. This yields an infinity of hooks between a module not in $\mathcal{L}_A$ and a module not in $\mathcal{R}_A$, a contradiction to the hypothesis.

Conversely, assume that (c) holds true and that $(\tau M, V, M)$ is a hook lying on a path $L \rightsquigarrow \tau M \to V \to M \rightsquigarrow N$, with $L \notin \mathcal{L}_A$ and $N \notin \mathcal{R}_A$. Since $L \notin \mathcal{L}_A$, there exists a predecessor L' of L such that $\mathrm{pd}_A L' \geq 2$, and hence an indecomposable injective module I such that $\mathrm{Hom}_A(I, \tau L') \neq 0$. This yields a path $\delta : I \to \tau L' \to V' \to L' \rightsquigarrow L$. Dually, there exists a path $\varepsilon : N \rightsquigarrow N' \to W' \to \tau^{-1}N' \rightsquigarrow P$, for some indecomposable projective module P. Gluing these paths together yields a path $I \overset{\delta}{\rightsquigarrow} L \rightsquigarrow \tau M \to V \to M \rightsquigarrow N \overset{\varepsilon}{\rightsquigarrow} P$, showing that the hook $(\tau M, N, M)$ belongs to a finite set, and hence (c') holds true. □

4. Quasi-directed components

In this section, we first deduce new characterizations of quasi-directed and convex components from the results obtained in Sec. 3. Finally, we study the infinite radical of mod$A$ and show that any convex and almost directed component is generalized standard, and hence quasi-directed (4.2.3).

4.1. The immediate consequences. Our first result, which directly follows from (2.4.1) and (3.2.4), characterizes the semiregular quasi-directed components.

COROLLARY 4.1.1. *Let A be an Artin algebra and assume that Γ is a semiregular component of $\Gamma(\mathrm{mod}A)$ without projective modules. The following conditions are equivalent:*

(a) *Γ is quasi-directed;*
(b) *Γ is directed;*
(c) *For each X in Γ, there are only finitely many modules lying on a cycle from X to X;*
(d) *For each X in Γ, there are only finitely many distinct hooks lying on a cycle from X to X;*

(e) *Given* X, Y *in* Γ, *there are only finitely many modules lying on a path from* X *to* Y;
(f) *Given* X, Y *in* Γ, *there are only finitely many distinct hooks lying on a path from* X *to* Y;

Further, if Γ *contains injective modules, these conditions are equivalent to :*
(g) *For each* $X \in \Gamma$, *there are only finitely many modules lying on a path from an injective module in* Γ *to* X;
(h) *For each* $X \in \Gamma$, *there are only finitely many distinct hooks lying on a path from an injective module in* Γ *to* X.

Furthermore, Γ *is convex,* $B = A/\mathrm{ann}(\Gamma)$ *is a tilted algebra and* Γ *is a connecting component of* $\Gamma(\mathrm{mod}B)$. □

Of course, the dual statement on semiregular components without injective modules also holds, we leave the primal-dual translation to the reader.

We deduce from this result a surprising fact, which is that the condition of being generalized standard is unnecessary in the definition of semiregular quasi-directed components.

COROLLARY 4.1.2. *Let* A *be an Artin algebra and assume that* Γ *is a semiregular component of* $\Gamma(\mathrm{mod}A)$. *Then* Γ *is quasi-directed if and only if* Γ *is almost directed.*

Proof. Since the necessity is obvious, assume that Γ is not quasi-directed. By (4.1.1)(c), there exists an X in Γ such that $|\mathcal{M}_{X,X}| = \infty$. It then follows from [**29**, (1.1)] that $|\mathcal{M}_{X,X} \cap \Gamma| = \infty$, saying that Γ contains infinitely many non-directing modules. □

Our classification is completed with the characterization of the non-semiregular quasi-directed and convex components, a corollary which follows directly from (2.4.2) and (3.2.4).

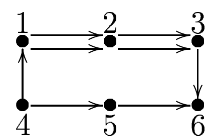
COROLLARY 4.1.3. *Let* A *be an algebra and assume that* Γ *is a non-semiregular component of* $\Gamma(\mathrm{mod}A)$. *The following conditions are equivalent:*

(a) Γ *is quasi-directed and convex;*
(b) *Given* X, Y *in* Γ, *there are only finitely many modules lying on a path from* X *to* Y;
(c) *Given* X, Y *in* Γ, *there are only finitely many distinct hooks lying on a path from* X *to* Y;
(d) *There are only finitely many modules lying on a path from an injective module in* Γ *to a projective module in* Γ;
(e) *There are only finitely many distinct hooks lying on a path from an injective module in* Γ *to a projective module in* Γ.

Furthermore, $B = A/\mathrm{ann}(\Gamma)$ *is a laura algebra and* Γ *is the unique non-semiregular, convex and faithful component of* $\Gamma(\mathrm{mod}B)$. □

We stress that the statement of (4.1.2) does not hold in general for non-semiregular components as shown by the following example. Moreover, this example shows that the conditions (c) and (d) of (4.1.1) cannot be added to the five equivalent conditions of (4.1.3) (while this is trivial for (b) since the Auslander-Reiten quiver of any representation-finite algebra is quasi-directed).

EXAMPLE 4.1.4. Let A be the the radical square zero algebra given by the quiver



Then, if we denote by P_i (or I_i, or S_i) the indecomposable projective (or injective, or simple, respectively) corresponding to the point i in the quiver, the Auslander-Reiten quiver $\Gamma(\mathrm{mod}A)$ of A has the shape shown in Fig. 1, where we identify the two copies of S_2. If Γ is the non-semiregular compo-

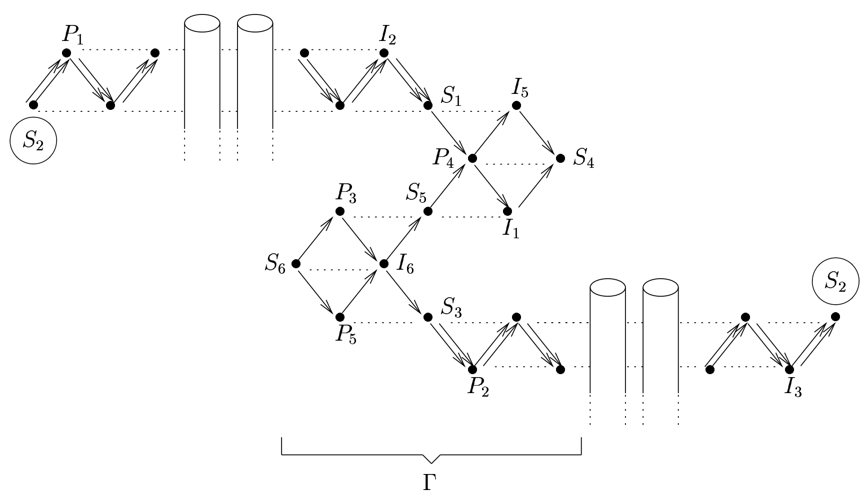


FIGURE 1. $\Gamma(\mathrm{mod}A)$

nent indicated in Fig. 1, it is not hard to see that Γ contains only directing modules and that the sets $\mathcal{M}_{X,X}$ and $\mathcal{H}_{X,X}$ are finite for all X in Γ. However, Γ is not generalized standard since $\mathrm{rad}^{\infty}(P_2, I_2) \neq 0$, nor convex.

In the previous example, Γ is not convex. Our next aim is to show that (4.1.2) holds in general when we assume that Γ is convex. To do this, we need to establish new facts on the infinite radical.

4.2. The infinite radical.

LEMMA 4.2.1. *Let A be a (representation-infinite) tilted Artin algebra and M, N be two modules in* ind$A$. *If* $\text{rad}^{\infty}(M,N) \neq 0$, *then there exist infinitely many non-directing modules lying on a path from M to N.*

Proof. First, if M (or N) does not belong to a connecting component, a postprojective component or a preinjective component, then it follows from [**14**, (3.7)] that M (or N, respectively) belongs to a component obtained from a quasi-serial component (that is a stable tube or a component of the form $\mathbb{Z}A_{\infty}$, [**23**]) by coray insertions or ray insertions, and the result follows trivially. Otherwise, by [**22**, p. 41], we may assume without loss of generality that M belongs to a postprojective component while N belongs to a connecting component Γ, different from the one containing M.

Following [**16**, (1.4)], there exists an infinite path

$$(*) \qquad \cdots \longrightarrow N_i \xrightarrow{h_i} N_{i-1} \xrightarrow{h_{i-i}} \cdots \xrightarrow{h_2} N_1 \xrightarrow{h_1} N_0 = N$$

of irreducible morphisms such that, for each i, there exists k_i in $\text{rad}^{\infty}(M, N_i)$ with $h_1 h_2 \dots h_i k_i \neq 0$. Since Γ is acyclic, there exists $i \geq 1$ such that no projective in Γ is a predecessor (by a path of irreducible morphism) of N_i. Observe that N_j is left stable for all $j \geq i$. Let Γ' be the left stable part of Γ containing N_j, for all $j \geq i$. By [**13**, (3.6)], Γ' is isomorphic to a full subquiver of $\mathbb{Z}\Delta$, where Δ is a finite and acyclic quiver. Let Δ' be the copy of Δ containing N_i. By the Liu-Skowroński criterium [**15, 24**], Δ' is a complete slice of the infinite-representation algebra $B = A/\text{ann}\Delta'$. Therefore, we can assume that $N = N_i$ and that N belongs to a connecting component Γ^N of $\Gamma(\text{mod}B)$ without projective modules. In this case, $\Gamma(\text{mod}B)$ admits a unique postprojective component Γ^M (necessarily different from the one containing N), and this component contains M_B.

Now, let $\mathcal{T}$ be a quasi-serial component of $\Gamma(\text{mod}B)$ such that $\text{Hom}(\Gamma^M, \mathcal{T}) \neq 0$ and $\text{Hom}(\mathcal{T}, \Gamma^N) \neq 0$, and let X be a module in $\mathcal{T}$. Since B is connected, there exists a path $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} P_n \xrightarrow{f_n} X$, where P_i is projective for each i and P_1 belongs to Γ^M. Moreover, we can assume that $P_2 \notin \Gamma^M$. Consequently, $f_1 \in \text{rad}^{\infty}(P_1, P_2)$ and it follows from [**20, 26, 29**], for instance, that for each $t \geq 0$ there exists a path

$$P_1 = X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \cdots \xrightarrow{h_{t-1}} X_t \xrightarrow{h} P_2,$$

where h_i is irreducible for each i and h belongs to $\text{rad}^{\infty}(X_t, P_2)$. Since Γ^M is acyclic, right stable and contains only finitely many τ-orbits, there exists a $t \geq$ such that X_t is a successor of M, inducing a path $M \rightsquigarrow X_t \rightsquigarrow P_2 \rightsquigarrow X$. On the other hand, since Δ' is a complete slice, $\text{rad}^{\infty}(X, T) \neq 0$ for a certain indecomposable module T in Δ' and, dually, there exists a path of the form $X \rightsquigarrow N$. Gluing these paths together yields a path of the form $M \rightsquigarrow X \rightsquigarrow N$. The result then follows from the fact that $\mathcal{T}$ contains infinitely many non-directing modules X. □

The fact that the previous result holds true for tilted algebras implies that it holds true for any Artin algebra.

THEOREM 4.2.2. *Let A be an Artin algebra and M, N be two modules in* $\text{ind}A$. *If* $\text{rad}^\infty(M,N) \neq 0$, *then there exist infinitely many non-directing modules lying on a path from M to N.*

Proof. Let f be a non-zero morphism in $\text{rad}^\infty(M,N)$, and let Γ be the connected component of $\Gamma(\text{mod}A)$ containing N. Following [**16**, (1.4)] and [**29**, (1.1)], there exists an infinite path

$$(*) \qquad \cdots \longrightarrow N_i \xrightarrow{h_i} N_{i-1} \xrightarrow{h_{i-1}} \cdots \xrightarrow{h_2} N_1 \xrightarrow{h_1} N_0 = N$$

such that

(a) For each i, there exists $k_i \in \text{rad}^\infty(M, N_i)$ such that $h_1 h_2 \cdots h_i k_i \neq 0$;
(b) For each i, $N_i \in \Gamma$;
(c) $N_i \neq N_j$ when $i \neq j$.

If $(*)$ contains infinitely many non-directing modules, this ends the proof. Otherwise, we can assume that $(*)$ contains only directing modules. Moreover, it follows from [**18, 27**] that $(*)$ crosses only finitely many τ-orbits. Without loss of generality, we can therefore assume that $(*)$ intersects infinitely many times the τ-orbit of N. In particular, N is left stable and non-periodic.

Let ${}_l\Gamma$ be the left stable part of Γ containing N. Then ${}_l\Gamma$ contains no oriented cycle. Indeed, assume that $\delta \; : \; X = X_0 \to X_1 \to \cdots \to X_n = X$ is a cycle lying in ${}_l\Gamma$. Since ${}_l\Gamma$ is left stable, there exists $i \geq 0$ such that N_i belongs to the τ-orbit of N and is a predecessor of X. Conversely, there exists $r \geq 0$ such that $\tau^r X$ is a predecessor of N_i. But then, since X is a predecessor of $\tau^r X$ by [**9**, (1.4)], we obtain a path $X \rightsquigarrow \tau^r X \rightsquigarrow N_i \rightsquigarrow X$, saying that N_i is not directing, a contradiction.

Moreover, ${}_l\Gamma$ contains only finitely many τ-orbits. Indeed, if this is not the case, then there exists a stable connected component $\mathcal{C}$ of ${}_l\Gamma$ having infinitely many τ-orbits. Let X be a module in $\mathcal{C}$ such that the length of any walk in Γ from a non-stable module to the τ-orbit of X is at least $2\mathbf{r}$, where $\mathbf{r}$ is the rank of the Grothendieck group of A. But then, since X is a predecessor of $\tau^s X$ for all $s \geq 1$ by [**9**, (1.5)], the above argument gives the existence of a N_j which is not directing, a contradiction.

Therefore, by [**13**, (3.6)], ${}_l\Gamma$ is isomorphic to a full subquiver of $\mathbb{Z}\Delta$, which is stable under predecessor, where Δ is a finite and acyclic quiver. Let $\overline{\Delta}$ be a fixed copy of Δ such that no module in $\overline{\Delta}$ is a successor (by a path of irreducible morphisms) of a projective module in Γ. Such a $\overline{\Delta}$ exists by [**17**, (1.2)], for instance. Let $\mathcal{D}$ be the full subquiver of ${}_l\Gamma$ consisting of all predecessors of $\overline{\Delta}$ in ${}_l\Gamma$. Moreover, let T be the direct sum of all modules in $\overline{\Delta}$. We claim that $\mathrm{Hom}_A(T, \tau T) = 0$. Indeed, if this is not the case, then there exist indecomposable direct factors Y and Z of T and a non-zero morphism $s : Y \to \tau Z$. Observe that $s \in \mathrm{rad}^\infty(\mathrm{mod}A)$ since $\mathcal{D}$ is a full subquiver of $\mathbb{Z}\Delta$ which is acyclic and closed under predecessors. By hypothesis, there exists N_i in $(*)$ such that N_i is a predecessor of Y. On the other hand, since $s \in \mathrm{rad}^\infty(\mathrm{mod}A)$, it follows from [**16**, (1.4)] that there exists an infinite path of irreducible morphisms

$$\cdots \longrightarrow Y_j \longrightarrow Y_{j-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = \tau Z$$

in $\mathcal{D}$ such that $\mathrm{Hom}_A(Y, Y_j) \neq 0$ for all $j \geq 0$. Since $\mathcal{D}$ has only finitely many τ-orbits and no oriented cycle, there exists $j \geq 0$ such that Y_j is a predecessor of N_i, giving a path $N_i \rightsquigarrow Y \to Y_j \rightsquigarrow N_i$, a contradiction to the fact that N_i is directing. Consequently, $\mathrm{Hom}_A(T, \tau T) = 0$.

Now, let $B = A/\mathrm{ann}T$. Observe that $\mathrm{ann}T = \mathrm{ann}\mathcal{D}$ by [**24**, (Lemma 3)]. Consequently, T is a faithful B-module and $\mathcal{D}$ consists of B-modules. Since $\mathrm{Hom}_B(T, \tau T) = 0$, B is a tilted algebra by [**15, 24**]. Moreover, T is a slice B-module lying in a connecting component without projective modules.

Let $p \geq 0$ be such that $N_q \in \mathcal{D}$ for all $q \geq p$. We claim that there exists $q \geq p$ such that $\mathrm{Im}(k_q)$ has at least one indecomposable direct factor not in $\mathcal{D}$. Indeed, let l be the length of the A-module M. By [**25**, (2.6)], for instance, there exists $N_q \in \mathcal{D}$ such that any predecessor X of N_q is such that $l(X) > l$. Obviously, $k_q : M \to N_q$ is not surjective, and hence $\mathrm{Im}k_q \neq N_q$. But then, since $l(\mathrm{Im}k_q) \leq l$, it follows from the hypothesis made on N_q that there exists an indecomposable direct factor Q of $\mathrm{Im}k_q$ such that $Q \notin \mathcal{D}$. In particular, Q belongs to a connected component of $\Gamma(\mathrm{mod}B)$ different from the one containing N_q.

Therefore the inclusion $Q \hookrightarrow N_q$ belong to $\mathrm{rad}^\infty(\mathrm{mod}A)$ and it follows from (4.2.1) that there exist infinitely many indecomposable non-directing B-modules (and hence A-modules) X lying on a path $Q \rightsquigarrow X \rightsquigarrow N_q$, giving infinitely many indecomposable modules X lying on a path of the form $M \to Q \rightsquigarrow X \rightsquigarrow N_q \rightsquigarrow N$. This concludes the proof. □

THEOREM 4.2.3. *Let A be an Artin algebra and Γ be a component of $\Gamma(\mathrm{mod}A)$. If Γ is convex, then it is quasi-directed if and only if it is almost directed.*

Proof. The necessity is obvious. For the sufficiency, it remains to show that Γ is generalized standard. Assume to the contrary that $\mathrm{rad}^\infty(M, N) \neq$

0 for some indecomposable A-modules M and N. By (4.2.2), there exist infinitely many non-directing modules X with a path $M \rightsquigarrow X \rightsquigarrow N$, a contradiction since Γ is convex and almost directed. □

We also get the following result.

COROLLARY 4.2.4. *Let A be an Artin algebra. Then A is representation-finite if and only if $\Gamma(\text{mod}A)$ contains only finitely many non-directing modules.*

Proof. Since the necessity is obvious, assume to the contrary that A is representation-infinite. Then $\text{rad}^\infty(\text{mod}A) \neq 0$ and the contradiction follows from (4.2.2). □

ACKNOWLEDGEMENTS. This paper was written while the first author visited the Universities of Sherbrooke and Bishop's under a CRM-ISM post-doctoral fellowship, and while the second author had an NSERC graduate studies fellowship.

References

[1] I. Assem and F. U. Coelho. Two-sided gluings of tilted algebras. *J. Algebra*, 269(2):456–479, 2003.

[2] I. Assem, F. U. Coelho, M. Lanzilotta, D. Smith, and S. Trepode. Algebras determined by their left and right parts. In *Algebraic structures and their representations*, number 376 in Amer. Math. Soc, Contemporary Mathematics, pages xxvi+436 pp., Providence, RI, 2005.

[3] I. Assem, M. Lanzilotta, and M. J. Redondo. Laura skew group algebras. Preprint.

[4] M. Auslander, I. Reiten, and S.O. Smalø. *Representation Theory of Artin Algebras.* Number 36 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997.

[5] R. Bautista and S.O. Smalø. Non-existent cycles. *Comm. Algebra*, 11(6):1755–1767, 1983.

[6] K. Bongartz. On a result of Bautista and Smalø on cycles. *Comm. Algebra*, 11(6):2123–2124, 1983.

[7] F.U. Coelho and M. Lanzilotta. Algebras with small homological dimensions. *Manuscripta Mathematica*, 100(1):1–11, 1999.

[8] F.U. Coelho and M. Lanzilotta. Weakly shod algebras. *J. Algebra*, 265(1):379–403, 2003.

[9] F.U. Coelho and A. Skowroński. On Auslander-Reiten components for quasitilted algebras. *Fund. Math.*, 149(1):67–82, 1996.

[10] J. Dionne and D. Smith. Articulation of algebras and their homological properties. *J. Algebra Appl.*, to appear.

[11] D. Happel, I. Reiten, and S.O. Smalø. Tilting in abelian categories and quasi-tilted algebras. *Mem. Am. Math. Soc.*, 120(575):viii+ 88 pp., 1996.

[12] M. Lanzilotta. The Gabriel-Roiter measure of a laura algebra. In preparation.

[13] S. Liu. Semi-stable components of an Auslander-Reiten quiver. *J. London Math. Soc. (2)*, 47(3):405–416, 1993.

[14] S. Liu. The connected components of the Auslander-Reiten quiver of a tilted algebra. *J. Algebra*, 161(2):505–523, 1993.

[15] S. Liu. Tilted algebras and generalized standard Auslander-Reiten components. *Arch. Math. (Basel)*, 61(1):12–19, 1993.

[16] S. Liu. On short cycles in a module category. *J. London Math. Soc. (2)*, 51(1):62–74, 1995.

[17] S. Liu. Preprojective modules and Auslander-Reiten components. *Comm. Algebra*, 31(12):6051–6061, 2003.

[18] L.G. Peng and J. Xiao. On the number of DTr-orbits containing directing modules. *Proc. Amer. Math. Soc.*, 118(3):753–756, 1993.

[19] I. Reiten and A. Skowroński. Generalized double tilted algebras. *J. Math. Soc. Japan*, 56(1):269–288, 2004.

[20] C.M. Ringel. *Report on the Brauer-Thrall conjecture*, pages 104–135. Number 831 in Lecture Notes in Math. Springer, Berlin, 1980.

[21] C.M. Ringel. *Tame algebras and integral quadratic forms.* Number 1099 in Lecture Notes in Mathematics. Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.

[22] C.M. Ringel. *Representation theory of finite-dimensional algebras.* London Mathematical Society Lecture Note Series, 116. Cambridge University Press, Cambridge-New York, 1986.

[23] C.M. Ringel. The Regular Components of the Auslander-Reiten Quiver of a Tilted Algebra. *Chin. Ann. of Math.*, 9B(1):1–18, 1988.

[24] A. Skowroński. Generalized standard Auslander-Reiten components without oriented cycles. *Osaka J. Math.*, 30(3):515–527, 1993.

[25] A. Skowroński. Generalized standard Auslander-Reiten components. *J. Math. Soc. Japan*, 46(3):517–543, 1994.

[26] A. Skowroński. Minimal representation-infinite artin algebras. *Math. Proc. Cam. Phi. Soc.*, 116:229–243, 1994.

[27] A. Skowroński. Regular Auslander-Reiten components containing directing modules. *Proc. Amer. Math. Soc.*, 120(1):19–26, 1994.

[28] A. Skowroński. On artin algebras with almost all indecomposable modules of projective or injective dimension at most one. *Cent. Eur. J. Math.*, 1(1):108–122, 2003.

[29] D. Smith. On generalized standard Auslander-Reiten components having only finitely many non-directing modules. *J. Algebra*, 279(2):493–513, 2004.

M. Lanzilotta; Centro de Matemática (CMAT), Iguá 4225, Universidad de la República, CP 11400, Montevideo, Uruguay.
E-mail address: *marclan@cmat.edu.uy*

D. Smith; Département de Mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada, J1K 2R1.
E-mail address: *david.smith@usherbrooke.ca*