Tensor networks since Aristotle

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IRL school 2022

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Introduction

Tensor networks

- mathematical representation of <u>interdependent degrees of freedom</u>
- classification of numerical techniques

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- mathematical representation of interdependent degrees of freedom
- classification of numerical techniques

Main idea:

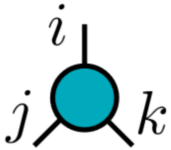
- make correlations (entanglement) explicit
- use resources proportional to relevant correlations

What is a tensor?

- a multidimensional array

  e.g.:

T_{ijk} :



$$T_{ijk} = z \in \mathbb{C}$$

$$i = 1, \ldots, d_i \, , \;\; j = 1, \ldots, d_j \, , \;\; k = 1, \ldots, d_k$$

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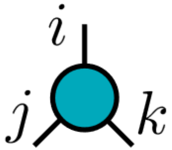
  T_{ijk} : 

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- matrix: rank-2 tensor T_{ij} : 

- vector: rank-1 tensor T_i : 

- scalar: rank-0 tensor T : 

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"To-do" list of tensor multiplications - contractions

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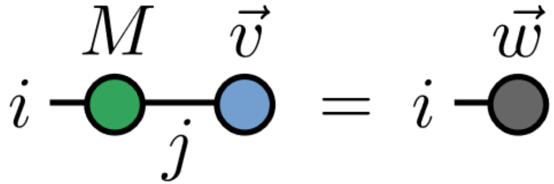
Examples

- matrix-vector multiplication:

$$\vec{w} = M\vec{v} \; : \; w_i = \sum_j M_{ij} v_j$$



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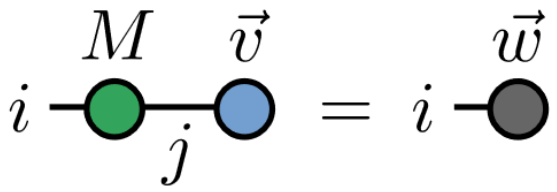
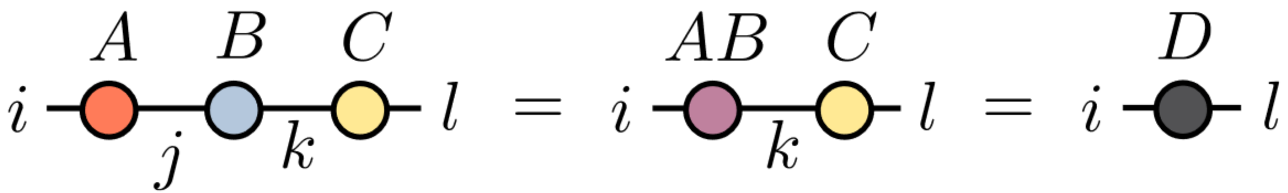


- matrix product:

$$D = ABC \; : \; D_{il} = \sum_{j,k} A_{ij} B_{jk} C_{kl}$$



Tensor networks for many-body states

- single spin-1/2 (or qubit)
  Hilbert space $\mathcal{H};\ \dim(\mathcal{H}) = 2$

- *n* spins-1/2 (or qubits)
  Hilbert space $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$
  $$D := \dim(\mathcal{H}^{\otimes n}) = 2^n$$
  quantum state: $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_D \end{pmatrix}$

  # of elements to store: $O(2^n)$

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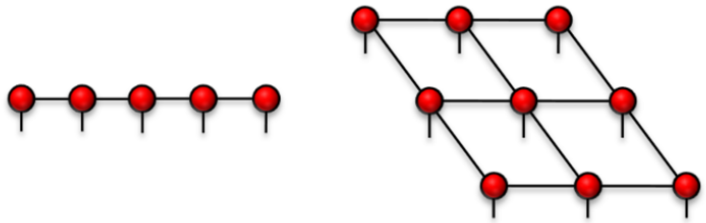

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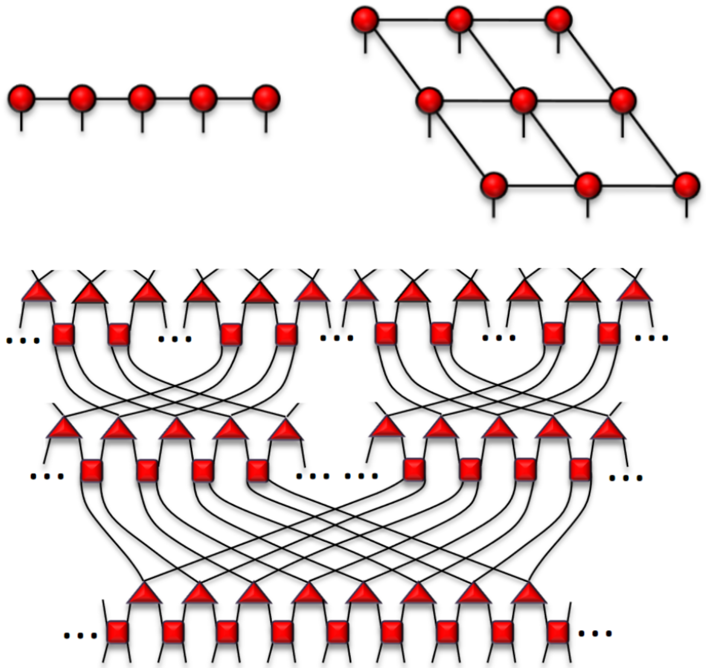


# of elements to store: $O(\chi^\alpha n)$

Course outline

- Lecture 1
    a. Bird's-eye view: ~2.5 millennia of tensor networks
    b. Elementary theory of entanglement

- Lecture 2
    a. Matrix-product states & operators
    b. Eigenstate-finding; time evolution

- Lecture 3
    a. Logic as tensor networks
    b. Everything as tensor networks

- Lecture 4
    a. Tensor network contraction & examples
    b. Summary & outlook

Lecture 1

History

- **1992-today** - DMRG algorithm for 1D quantum magnetism; matrix product states
  - many extensions (2D, dynamics, finite-T, open systems, ...)
- **1999-today** - quantum chemistry
- **2003-today** - quantum computation as TNs; quantum information
- **2004** - higher dimensional TN states; strongly correlated electrons in 2D
- **2007** - systems at / close to criticality; tensor renormalization group

  2007-today - TN coarse graining methods
- **2012-today** - TNs for holographic duality -> quantum gravity
  - TNs for lattice gauge theories -> high-energy physics
- **2013-today** - TN methods in data science (compression, reconstruction, ...)
- **2016-today** - TN methods for machine learning
- **2020s** - broad use in artificial intelligence
  - state-of-the-art for simulation of quantum computation

Prehistory

- Kontsevich (1994); Bar-Natan (1995) - knot invariants
- Penrose (1971) - graph invariants; 4-color problem
- Baxter (1968) - partition function of dimer model
- Kramers & Wannier (1941) - partition function of Ising model
- Sylvester (1878) - invariants of binary forms

- Chrysippus (3rd century BCE) - propositional inference
- Aristotle (4th century BCE) - syllogistic inference

Highlights

Quantum magnetism

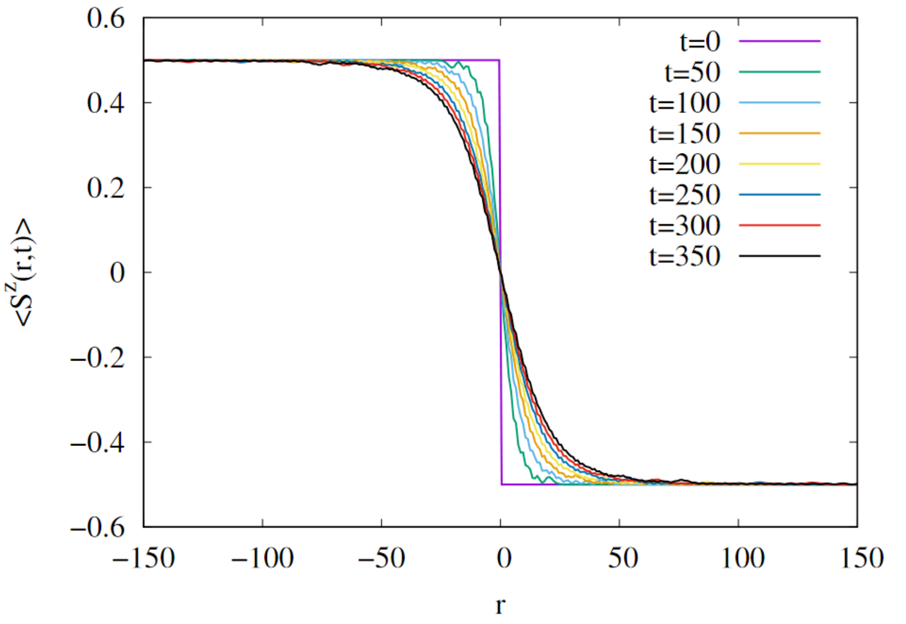
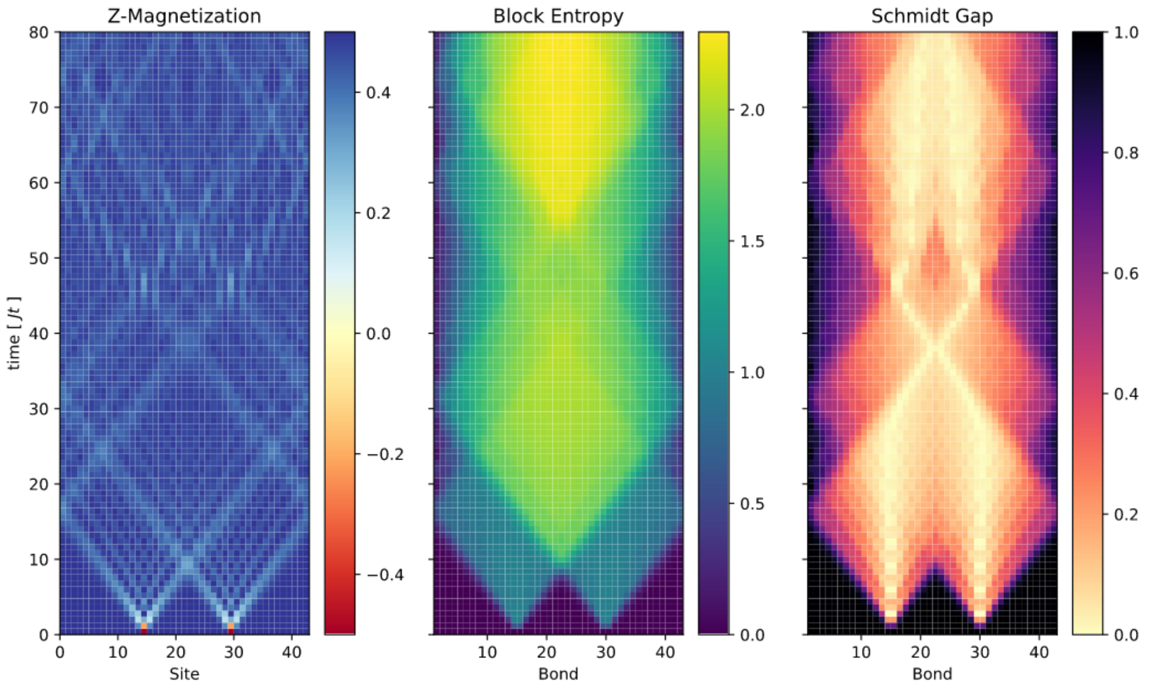


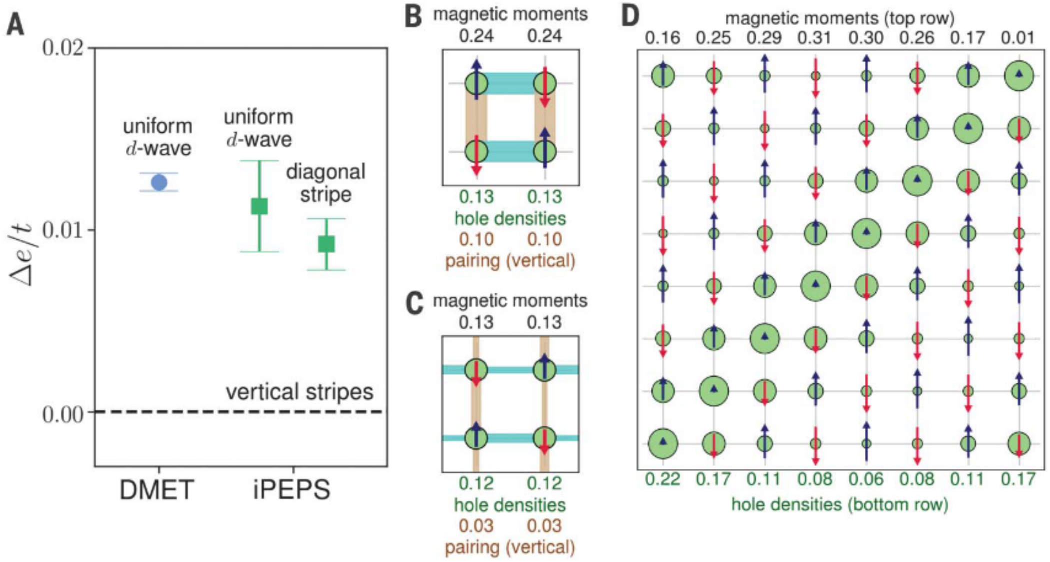
FIG. 1: Magnetization profiles at different times. Simulation parameters: maximum bond dimension $\chi = 2000$, Trotter step $\tau = 0.3$, and system size $L = 800$ sites (only 300 sites in the center are shown here).

Misguich, Mallick, Krapivsky (2017)

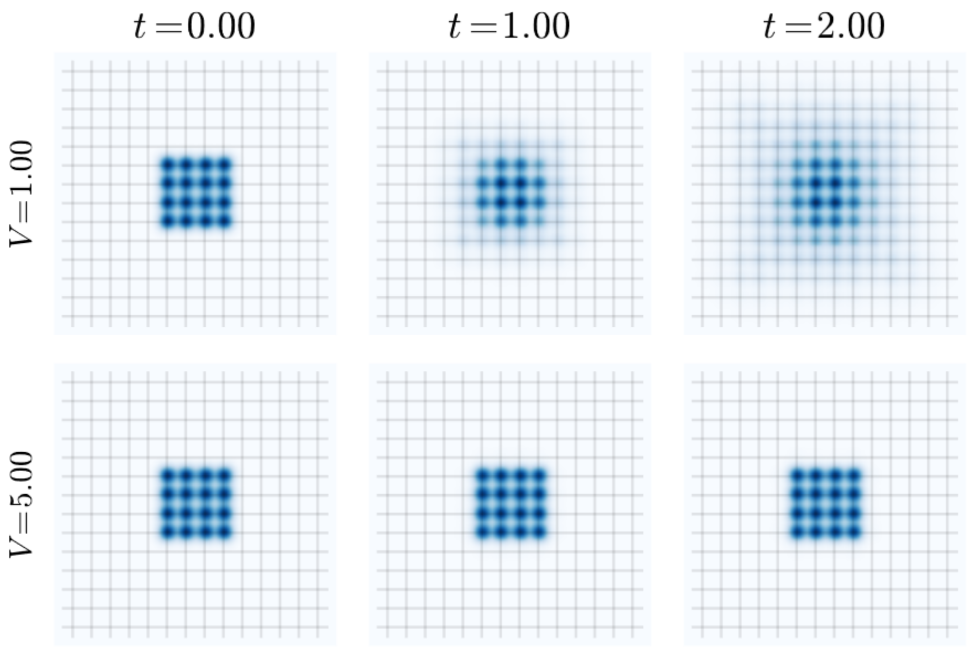


Johnnie Gray, *quimb* library
https://github.com/jcmgray/quimb

Hubbard & Mott physics



Zheng *et al.* (2017)



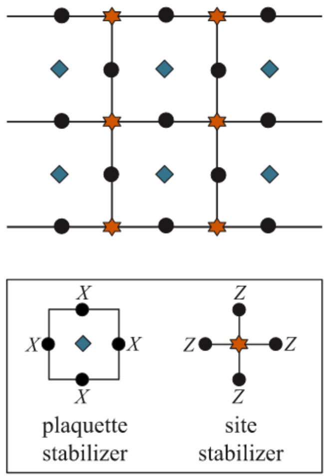
Zaletel *et al.* (2015)

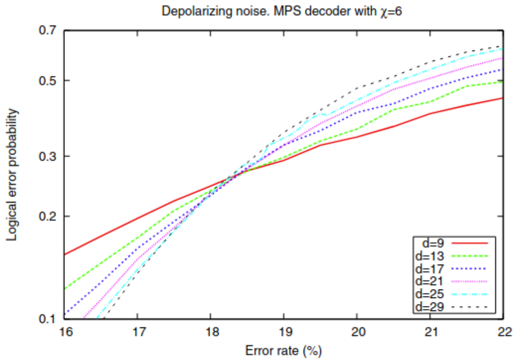
Combinatorial optimization

PHYSICAL REVIEW A **90**, 032326 (2014)

Efficient algorithms for maximum likelihood decoding in the surface code

Sergey Bravyi, Martin Suchara, and Alexander Vargo
IBM Watson Research Center, Yorktown Heights, New York 10598, USA

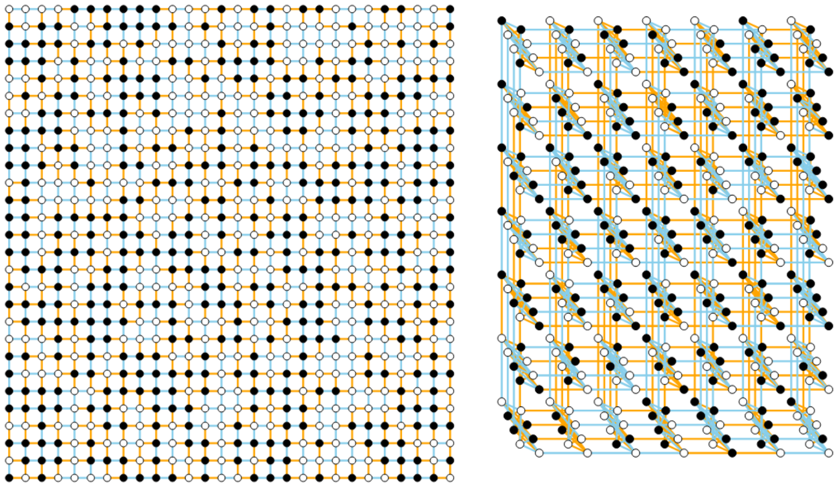




PHYSICAL REVIEW LETTERS **126**, 090506 (2021)

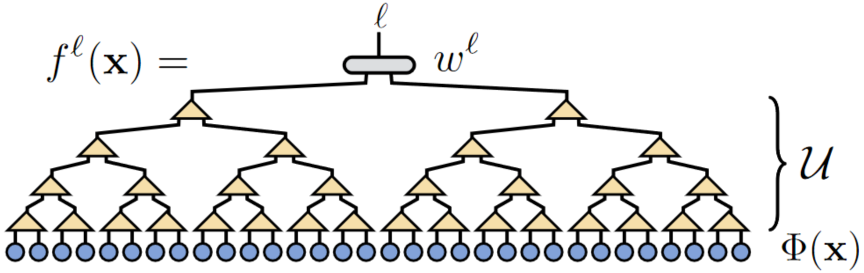
Tropical Tensor Network for Ground States of Spin Glasses

Jin-Guo Liu,[1,2,3,*] Lei Wang,[1,4,†] and Pan Zhang[5,6,7,‡]



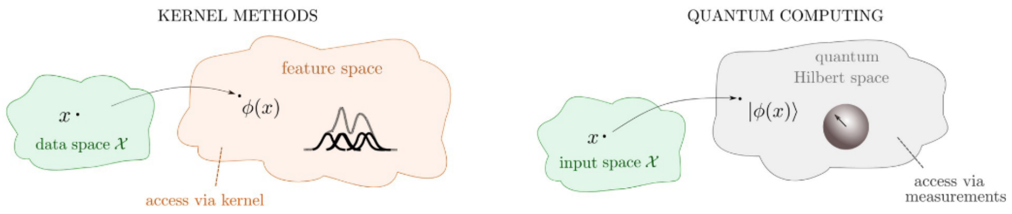
Machine learning

Supervised learning:

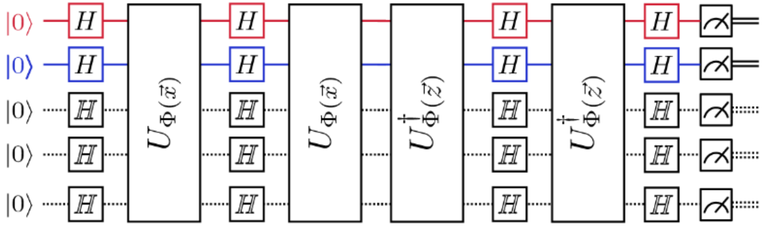


Stoudenmire & Schwab (2016)
Stoudenmire (2018)

Quantum kernel learning:

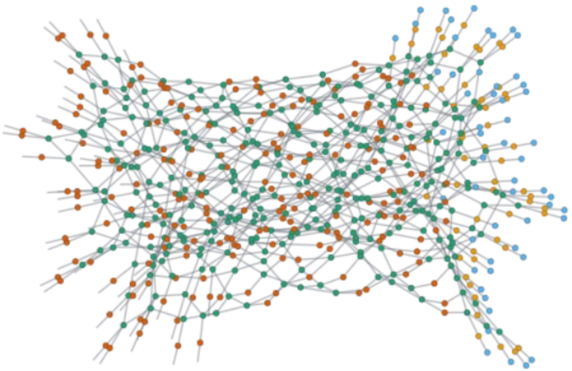


Schuld, 2021



Havlicek *et al.*, Nature 2019

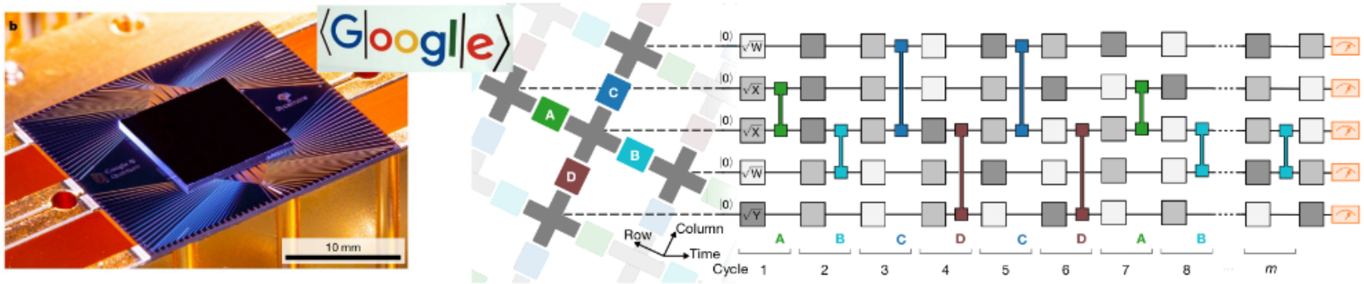
TN:

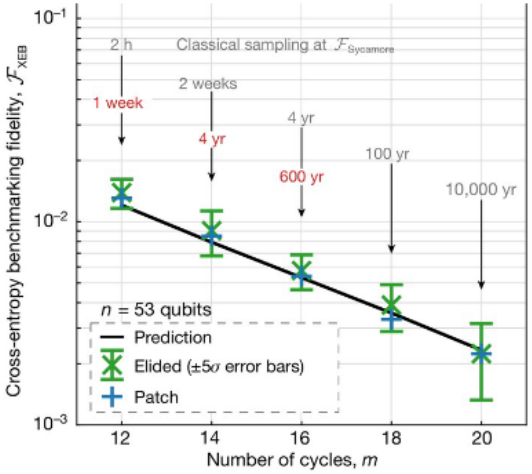


Near-term quantum computing

Quantum supremacy using a programmable superconducting processor

Arute *et al.*, Nature (2019)





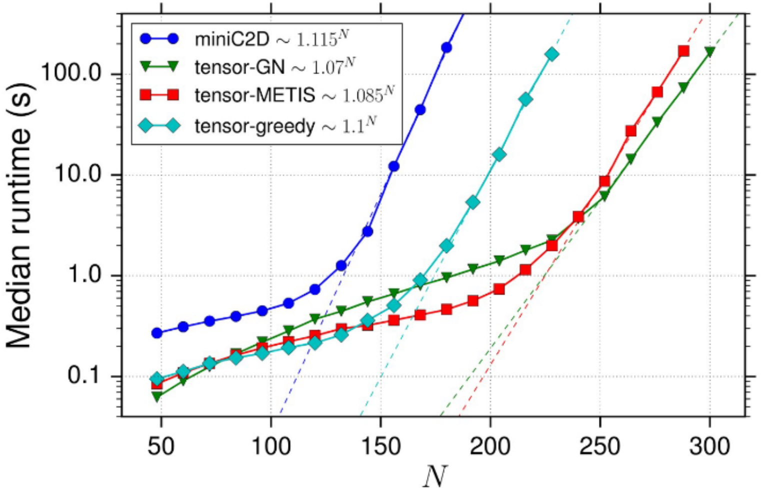
<u>**Tensor network simulation:**</u>

- Gray & Kourtis (2020) ~195 days @ 281 petaFLOPs (Summit) (*est.*)
- Yong *et al.* (2021) ~300s @ 1.2 exaFLOPs (*est.*)
- Pan, Chen, Zhang (2021) ~few dozen s @ exaFLOPs → <u>15 hours using 512 GPUs</u>

Computation at large

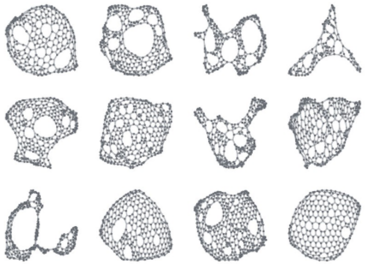
Model counting (AI):

Kourtis *et al.* (2018)



MC 2020 (Model Counting 2020)

The *1st International Competition on Model Counting (MC 2020)* is a competition to deepen the relationship between latest theoretical and practical development on the various model counting problems and their practical applications. It targets the problem of counting the number of models of a Boolean formula.



2020 champion: 69/100 instances solved
Gray & Kourtis: 99/100 instances solved

Gray & Kourtis (2021)

ELSEVIER

Advances in Engineering Software
Volume 159, September 2021, 103033

Fast evaluation of finite element weak forms using python tensor contraction packages

Robert Cimrman

ELSEVIER

Structural Safety
Volume 75, November 2018, Pages 110-118

Quantum-inspired Boolean states for bounding engineering network reliability assessment

Leonardo Dueñas-Osorio [a], Moshe Vardi [b], Javier Rojo [c]

Optimizing radiotherapy plans for cancer treatment with Tensor Networks

Samuele Cavinato[7,1,2], Timo Felser[1,3,4,5], Marco Fusella[2], Marta Paiusco[2] and Simone Montangero[1,3,6]

Published 16 June 2021 • © 2021 Institute of Physics and Engineering in Medicine
Physics in Medicine & Biology, Volume 66, Number 12

Code

Meta

Tensor Comprehensions

Google

Tensor Network

NVIDIA

cuTENSOR

Bibliography

<u>Physics</u>

- Schollwöck (2011) - arXiv:1008.3477
- Hauschild & Pollmann (2018) - arXiv:1805.00055
- Orús (2019) - arXiv:1812.04011
- Baker, Derosiers, Tremblay, Thompson (2021) - arXiv:1911.11566

<u>Quantum info, math, CS:</u>

- Bridgeman & Chubb (2017) - arXiv:1603.03039
- Cichocki *et al.* (2016-2017) - arXiv:1609.00893, arxiv:1708.09165
- Biamonte (2019) - arXiv:1912.10049

Theory of entanglement