Entanglement

Entanglement is a phenomenon – how to quantify?

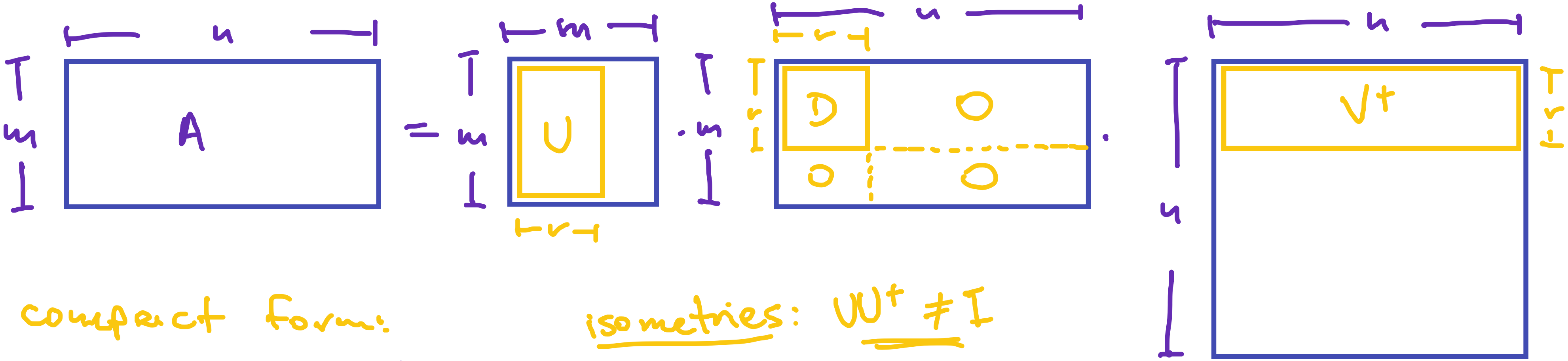
entanglement entropy ← Schmidt decomposition ← singular value decomposition (SVD)

SVD: for every matrix $A \in \mathbb{C}^{m\times n}$ we can write:

$$A = \tilde{U}\tilde{D}\tilde{V}^\dagger \quad ; \quad \tilde{U}^\dagger \tilde{U} = I_{m\times m}, \quad \tilde{V}^\dagger\tilde{V} = I_{n\times n} \quad ; \quad \tilde{D} \in \mathbb{R}_{\geq 0}^{m\times n} \text{ diagonal}$$

$$\tilde{D}_{ii} > 0 \text{ for } i \in [1,r] \text{ where } r = \mathrm{rank}(A) \quad ; \quad r \leq \min(m,n)$$

Visually: (suppose $m<n$) $\tilde{U}$ $\tilde{D}$ $\tilde{V}^\dagger$



compact form:

isometries: $UU^\dagger \neq I$

$$\Rightarrow \quad A = UDV^\dagger \quad ; \quad U^\dagger U = V^\dagger V = I_{r\times r}$$

$$D \in \mathbb{R}_{\geq 0}^{r\times r} \text{ diagonal}$$

Existence Why does the SVD exist for any complex matrix?

$$A \in \mathbb{C}^{m\times n} \Rightarrow \begin{array}{l} A^\dagger A \in \mathbb{C}^{n\times n} \text{ hermitian} \Rightarrow A^\dagger A = \tilde V \tilde D^\dagger \tilde U^\dagger \tilde U \tilde D \tilde V^\dagger = \tilde V \underbrace{\tilde D^\dagger \tilde D}_{\text{diagonal}} \tilde V^\dagger \\ A A^\dagger \in \mathbb{C}^{m\times m} \text{ hermitian} \Rightarrow A A^\dagger = \tilde U \tilde D \tilde V^\dagger \tilde V \tilde D^\dagger \tilde U^\dagger = \tilde U \underline{\tilde D \tilde D^\dagger} \tilde U^\dagger \end{array}$$

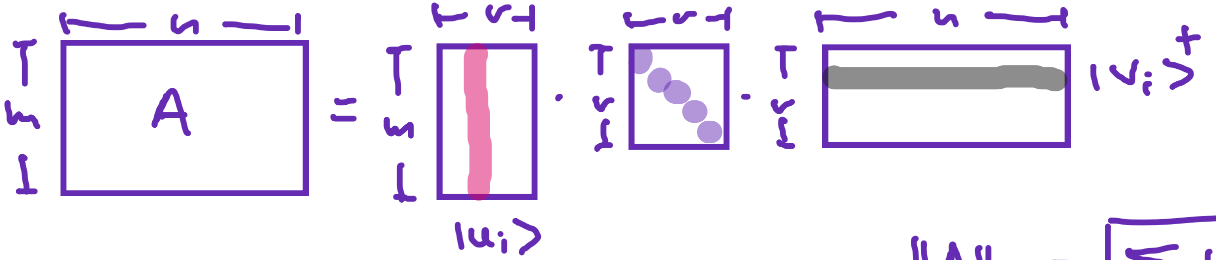
SVD of A $\Leftrightarrow$ eigendecomposition of $A^\dagger A$ / $AA^\dagger$

$$\tilde D \tilde D^\dagger = \mathrm{diag}(D_{11}^2, D_{22}^2, \dots, D_{rr}^2, \overbrace{0, \dots, 0}^{m-r})$$ and similar for $\tilde D^\dagger \tilde D$

Why are $D_{ii} \in \mathbb{R}_{\geq 0}$? $\longrightarrow$ $A^\dagger A$, $AA^\dagger$ <u>positive semi-definite</u> – <u>exercise</u>!

Low-rank approximation Can we <u>compress</u> A by ignoring unimportant features?

$$A = U \cdot D \cdot V^\dagger$$



$$A = \sum_{i=1}^{r} D_{ii} |u_i\rangle |v_i\rangle^\dagger = \sum_{i=1}^{r} A^{(i)}$$

Assume $D_{11} \geq D_{22} \geq \dots \geq D_{rr}$

$$\|A\|_F = \sqrt{\sum_{ij} |A_{ij}|^2} = \sqrt{\mathrm{Tr}(A^\dagger A)} = \sqrt{\mathrm{Tr}(\tilde V \tilde D^\dagger \tilde D \tilde V^\dagger)} = \sqrt{\sum_{i=1}^{r} D_{ii}^2}$$

What if I stop at $k < r$?

Truncation error: $\sqrt{\sum_{i=k+1}^{r} D_{ii}}$ $\sim$ ignore <u>smallest</u> singular values

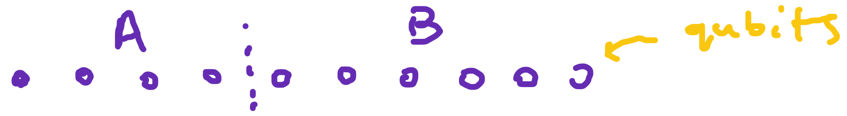
$\Downarrow$

<u>best</u> low-rank approximation of A

Eckart-Young <u>thm.</u>

Schmidt decomposition

Consider bipartite system of quantum degrees of freedom (qubits) in state $|\Psi_{AB}\rangle$



vector, Hilbert space

$$|\Psi_{AB}\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$\dim(\mathcal{H}_{AB}) \equiv D$

$\dim(\mathcal{H}_{A|B}) \equiv D_{A|B}$

$$|\Psi_{AB}\rangle = \sum_{n=1}^{\chi} \sqrt{\lambda_n}\, |\varphi_n\rangle_A \otimes |\Phi_n\rangle_B \; ; \quad \lambda_n \in \mathbb{R}_{\geq 0}, \quad \sum_n \lambda_n = 1 \text{ - Schmidt values}$$

$\{|\varphi_n\rangle_A\}, \{|\Phi_n\rangle_B\}$: orthonormal basis of $\mathcal{H}_{A|B}$: $\langle \varphi_n | \varphi_m \rangle_A = \langle \Phi_n | \Phi_m \rangle_B = \delta_{nm}$

Why? Rewrite: occupation number basis for $\mathcal{H}_{AB}$ / $\mathcal{H}_{A|B}$

$$|\Psi_{AB}\rangle = \sum_{k=1}^{D} a_k |k\rangle = \sum_{p=1}^{D_A} \sum_{q=1}^{D_B} A_{pq} |p\rangle_A \otimes |q\rangle_B \, , \quad A \in \mathbb{C}^{D_A \times D_B}$$

unitary transform

$$= \sum_{p,q} \sum_{n=1}^{r} A^{(n)}_{pq} |p\rangle_A \otimes |q\rangle_B = \sum_{n=1}^{r} \underbrace{D_{nn}}_{\sqrt{\lambda_n}} \Big(\underbrace{\sum_p U_{pn} |p\rangle_A}_{|\varphi_n\rangle_A} \Big) \otimes \Big(\underbrace{\sum_q V^{\dagger}_{qn} |q\rangle_B}_{|\Phi_n\rangle_B} \Big)$$

Examples

- product state: $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ — already in Schmidt form! $\lambda_1 = 1$, $\lambda_i = 0 \;\forall i > 1$
- Bell pair: $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$

$$A = \begin{array}{c c} & \begin{array}{cc} |0\rangle_B & |1\rangle_B \end{array} \\ \begin{array}{c} |0\rangle_A \\ |1\rangle_A \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

already diagonal

Schmidt bases: $\{|0\rangle_A, |1\rangle_A\}$, $\{|0\rangle_B, |1\rangle_B\}$

Schmidt values: $\lambda_1 = \lambda_2 = 1/2$

Entanglement

State $|\psi_{AB}\rangle$ that has Schmidt rank $= 1$ is not entangled w.r.t to bipartition A-B

Any state with Schmidt rank > 1 is entangled — " —

How to quantify entanglement?

Entanglement entropy

$$\boxed{S(|\psi_{AB}\rangle) = -\sum_n \lambda_n \log \lambda_n}$$

Shannon entropy of the entanglement spectrum $\{\lambda_n\}$

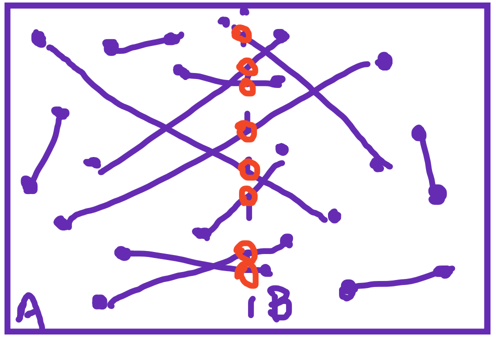
E.g.: Bell pair: $\lambda_1 = \lambda_2 = \frac{1}{2}$

$$\rightarrow S(|\psi_{AB}\rangle) = -\sum_{n=1}^{2} \frac{1}{2} \log \frac{1}{2} = \underbrace{\log 2}_{1 \text{ ebit}}$$

Laws of entanglement

How much entanglement do commonly encountered systems have?

- "Bell pair gas" Put finite density of Bell pairs in a box, then shake.



$S(|\psi_{AB}\rangle) \sim ? \rightarrow$ # Bell pairs across bipartition

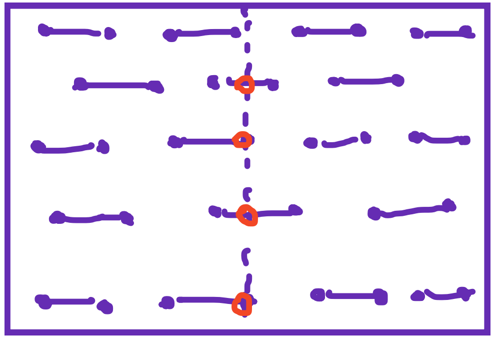
If $O(n)$ particles $\Rightarrow$ $O(n)$ Bell pairs in total

$\Rightarrow S(|\psi_{AB}\rangle) \sim n \rightarrow$ volume law of entanglement

almost all states in a Hilbert space are volume-law states

NB.: the dimensions of space or the boundary don't matter (Page, 1993)

- ordered Bell pairs (e.g. valence-bond solid)



$S(|\psi_{AB}\rangle) \sim$ "area" of boundary (cf. volume of box

E.g.: In 2D, finite density $\Rightarrow n \sim L^2$, L: side of box

# of Bell pairs across bipartition $\sim L$

$\Rightarrow S(|\psi_{AB}\rangle) \sim \sqrt{n} \rightarrow$ area law of entanglement

In general: $S \sim |\partial A| \sim n^{\frac{D-1}{D}}$ in D dimensions

therefore exceptional (Hastings, 2007)

In 1D: $S \sim O(1)$; Ground states of gapped 1D Hamiltonians are area-law