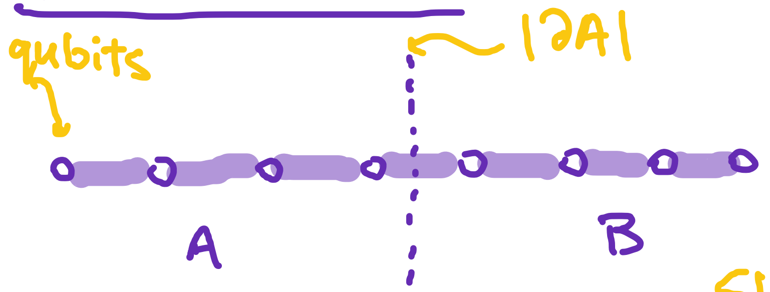
Area law in 1D



$S(|\psi_{AB}\rangle) \sim |\partial A| \sim O(1)$ in 1D

Schmidt decomposition:

dimensions of Hilbert spaces

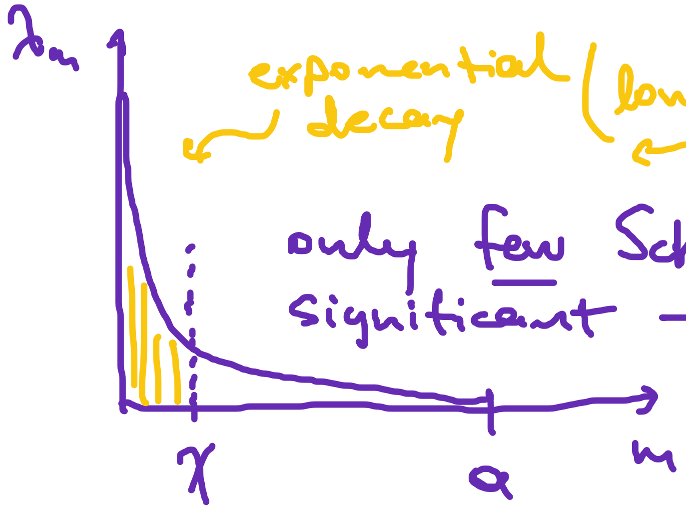
$a := \min(D_A, D_B)$

$$|\psi_{AB}\rangle = \sum_{i,k} C_{ik} |i\rangle_A |k\rangle_B = \sum_{m=1}^{a} \sqrt{\lambda_m} |\phi_m\rangle_A |\varphi_m\rangle_B$$

Shannon entropy of entanglement spectrum

$$\rightarrow S(|\psi_{AB}\rangle) = -\sum_m \lambda_m \log \lambda_m \sim O(1) \quad \text{cf. } O(n) \text{ (maximum)}$$

How should the spectrum $\{\lambda_m\}$ look like if $S \sim O(1)$?



exponential decay

(low entropy $\leftrightarrow$ more certainty)

only few Schmidt values significant $\rightarrow$ truncate the rest

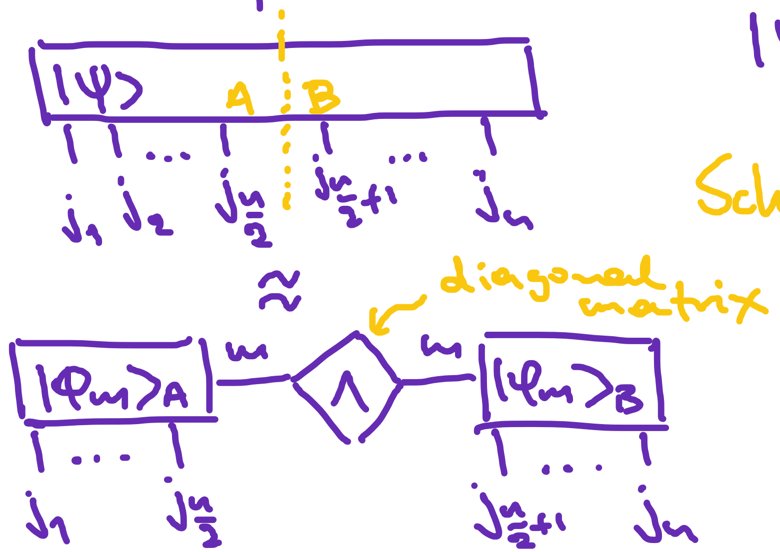
$$|\psi_{AB}\rangle \approx \sum_{m=1}^{\chi} \sqrt{\lambda_m} |\phi_m\rangle_A |\varphi_m\rangle_B \equiv |\psi_{AB}^{\text{trunc.}}(\chi)\rangle$$

More formally:

$|\psi\rangle$ obeys 1D area law $\iff \exists \chi : \| |\psi\rangle - |\psi_{\text{trunc.}}(\chi)\rangle \| < \varepsilon \quad \forall \varepsilon > 0$

for any n !

Matrix product states



Consider an n-qubit state

$$|\psi\rangle = \sum_{j_1=0,1}\sum_{j_2=0,1}\cdots\sum_{j_n=0,1}\psi_{j_1 j_2\cdots j_n}\,|j_1 j_2\cdots j_n\rangle$$

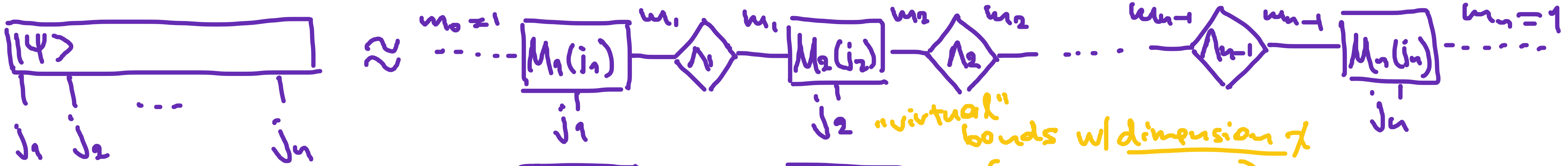
Schmidt decomposition:

$$\approx \sum_{m=1}^{\chi}\sqrt{\lambda_m}\,|\phi_m\rangle_A\,|\varphi_m\rangle_B\,,$$

$$|\phi_m\rangle_A = \sum_{j_1\cdots j_{\frac{n}{2}}}\phi^{(m)}_{j_1\cdots j_{\frac{n}{2}}}\,|j_1\cdots j_{\frac{n}{2}}\rangle$$

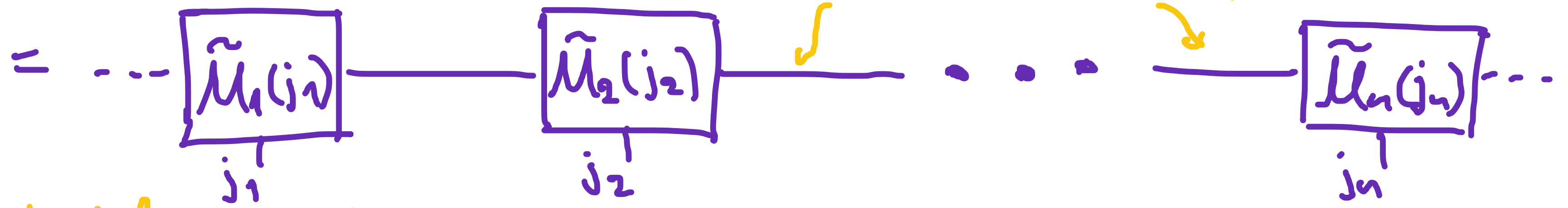
$$|\varphi_m\rangle_B = \sum_{j_{\frac{n}{2}+1}\cdots j_n}\varphi^{(m)}_{j_{\frac{n}{2}+1}\cdots j_n}\,|j_{\frac{n}{2}+1}\cdots j_n\rangle$$

Now let's Schmidt-decompose $|\phi_m\rangle_A$, $|\varphi_m\rangle_B$ and so on...



$$\Lambda_i = \sqrt{\Lambda_i}\cdot\sqrt{\Lambda_i} \Rightarrow$$

$$\tilde{M}_i(j_i) = \sqrt{\Lambda_{i-1}}\,M_i(j_i)\,\sqrt{\Lambda_i}$$



$$|\psi\rangle \approx |\psi_{MPS}\rangle = \underbrace{\sum_{j_1\cdots j_n}}_{\text{physical}}\ \underbrace{\sum_{m_1\cdots m_{n-1}}}_{\text{"virtual"}}\left[\tilde{M}_1(j_1)\right]_{m_0 m_1}\left[\tilde{M}_2(j_2)\right]_{m_1 m_2}\cdots\left[\tilde{M}_n(j_n)\right]_{m_{n-1} m_n}|j_1\cdots j_n\rangle$$

$$|\psi_{MPS}\rangle = \sum_{j_1 \cdots j_n} \tilde{M}_1(j_1)\tilde{M}_2(j_2)\cdots\tilde{M}_n(j_n)|j_1 j_2 \cdots j_n\rangle$$ ~ thus matrix product

Suppose <u>all</u> $m_i \in [1, \chi]$ Size of $|\psi_{MPS}\rangle$? i.e. # of complex numbers

$\text{size}(|\psi_{MPS}\rangle) =$ 2 matrices per qubit (1 per qubit state)
$\times\, \chi^2$ elements per matrix

$2n\chi^2$ complex numbers ~ $O(n)$

$\text{size}(|\psi\rangle) = O(2^n)$ $\longrightarrow$ <u>exponential</u> <u>size</u> <u>compression</u> <u>for finite</u> χ

Since we started by Schmidt decomposition that reveals the entanglement spectrum clearly entanglement entropy determines efficacy of MPS representation

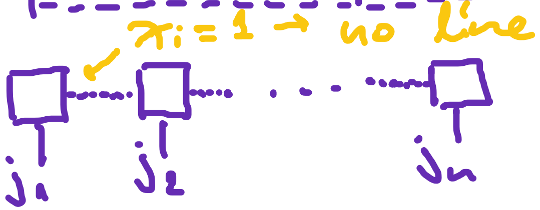
$$\chi \propto 2^{S(|\psi_{AB}\rangle)} \Longrightarrow \begin{cases} \bullet\ \text{1D area law: } \chi \sim O(1) \ \checkmark \\ \bullet\ \text{things in between: } \ldots \text{ maybe} \\ \bullet\ \text{volume law: } \chi \sim 2^{O(n)} \ \times \end{cases}$$

<u>Main idea behind MPS methods:</u>

start with simple MPS (e.g. product state); "evolve" to build desired state (e.g. GSs, excited states, dynamics) $\rightarrow$ efficient if entanglement <u>low</u> throughout evolution

Examples

- **Product state:** $|\psi\rangle = |\varphi_1(j_1)\rangle \otimes |\varphi_2(j_2)\rangle \otimes \dots \otimes |\varphi_n(j_n)\rangle$



$\tilde{M}_i(j_i) = \varphi_i(j_i)$ i.e., product of 1x1 matrices

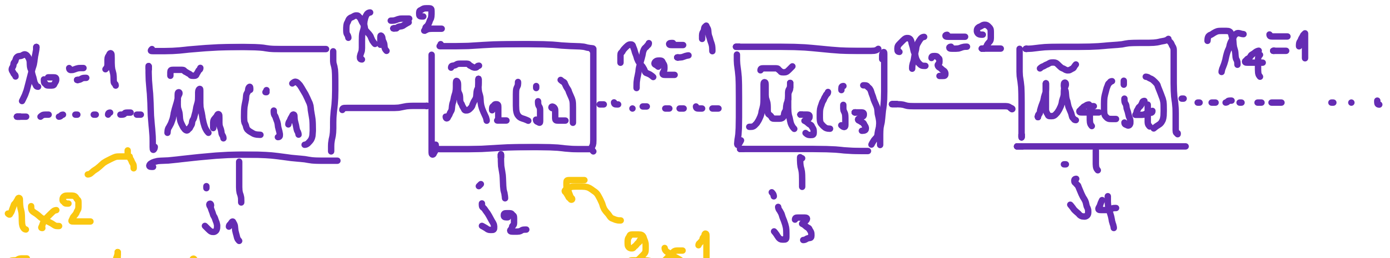
$|{-}{-}\dots{-}\rangle = \frac{1}{2^{n/2}}\left(|0\rangle - |1\rangle\right)^{\otimes n} \Rightarrow \tilde{M}_i(0) = \frac{1}{\sqrt{2}} = -\tilde{M}_i(1)$

Néel state: $|\psi\rangle = |0101\dots01\rangle = |0\rangle\otimes|1\rangle\otimes\dots\otimes|1\rangle$ written in qubit language

$\tilde{M}_{2k-1}(0) = \tilde{M}_{2k}(1) = 1 \;,\; \tilde{M}_{2k-1}(1) = \tilde{M}_{2k}(0) = 0 \;,\; k = 1,\dots,\frac{n}{2}$

- dimerized chain (1D VBS if you wanna be fancy....)

$$|\psi\rangle = \left[\frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)\right] \otimes \dots \otimes \left[\frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)\right] = \frac{1}{2^{n/4}}\left(|10\rangle - |01\rangle\right)^{\otimes \frac{n}{2}}$$



$\tilde{M}_{2k-1}(1) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \; \tilde{M}_{2k-1}(1) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\tilde{M}_{2k}(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; \tilde{M}_{2k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Single dimer:

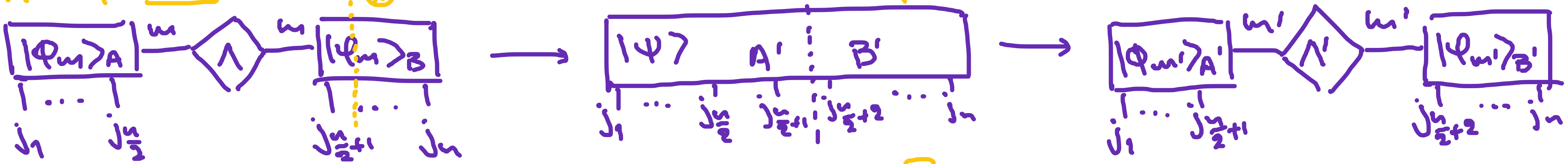
$$|\psi_{MPS}\rangle = \sum_{j_1 j_2} M_1(j_1) M_2(j_2) |j_1 j_2\rangle = M_1(1)M_2(1)|11\rangle + M_1(1)M_2(0)|10\rangle + M_1(0)M_2(1)|01\rangle + M_1(0)M_2(0)|00\rangle$$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}|11\rangle + \begin{pmatrix} 1/\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}|10\rangle + \begin{pmatrix} 0 & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}|01\rangle + \begin{pmatrix} 0 & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}|00\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)$$

Canonical form: permits switching between different Schmidt decomps. efficiently (i.e. different partition A-B)

→ Very efficient methods for evaluating observables, time evolution

How to NOT do it: ⊛ Builds full state!



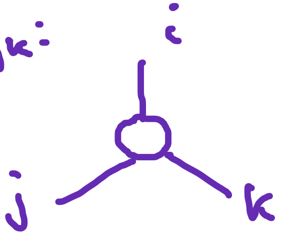
⊛ why not decompose $|\psi_m\rangle_B$?

→ Answer: states for A part not orthonormal – not a Schmidt basis

But first: • Index grouping & tensor reshape:

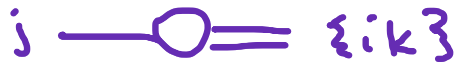
Tensor T_{ijk}:

$i = 1, \dots, d_i$
$j = 1, \dots, d_j$
$k = 1, \dots, d_k$



⟷

Matrix $T_{j\{ik\}}$ ← multi-index



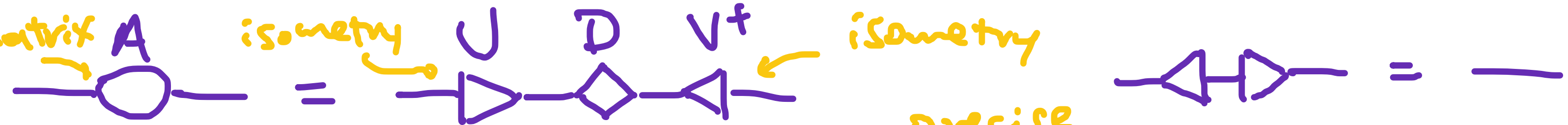
$j = 1, \dots, d_j$
$\{ik\} = 1, \dots, d_i \cdot d_k$

in code:

numpy.reshape

Tensor decomposition: matricization + matrix decomposition

• compact SVD in blobs:



Next few derivations in blobs, for two reasons: (i) precise equations unnecessarily complicated
(ii) this can only be truly learned by doing